K-THEORY AND THE ANTI-AUTOMORPHISM OF THE STEENROD ALGEBRA

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Abstract. We give simple proofs of some relations in the Steenrod algebra involving the powers $\mathcal{P}^i$ and their duals $\chi \mathcal{P}^i$ and show how these relations arise from $K$-theory.

1. Introduction

Barratt and Miller in [3] derived some relations in the Steenrod algebra $\mathcal{A}(p)$ using expressions for the Adem relations due to Bullett and Macdonald [4]. In Section 2 we reformulate and give a simple proof of the former in the spirit of the latter. These relations arise naturally from properties of the Adams operations in complex $K$-theory, as was explained in [7, 6] for the prime 2; alternative and shorter $K$-theoretic derivations valid for all primes are given in Section 3.

2. Results of Barratt and Miller

We use standard notation for the mod $p$ Steenrod algebra $\mathcal{A}(p)$ with the convention that $\mathcal{P}^i$ denotes $Sq^i$ when $p = 2$. For each non-negative integer $n$, let

$$\phi(n) = \min \{k \geq 0 \mid k + \nu_p(k!) \geq n\}.$$  

Another, more illuminating description of $\phi(n)$ is given in Proposition 2.6.

Theorem 2.1. The polynomial

$$F_n(T) = \sum_{i=0}^{n} \mathcal{P}^{n-i} \chi(\mathcal{P}^i) T^i \in \mathcal{A}(p)[T]$$

is divisible by $(T - 1)^{\phi(n)}$.

This result can be restated in many ways. Let $T^m F_n(T)$, for $m \in \mathbb{Z}$, be expressed as a formal power series in $S = T - 1$. By considering the coefficient of $S^j$ with $0 \leq j < \phi(n)$, it follows that

$$\sum_{i=0}^{n} \binom{m+i}{j} \mathcal{P}^{n-i} \chi(\mathcal{P}^i) = 0 \quad \text{for} \quad 0 \leq j < \phi(n).$$

Looking at $T^n F(T^{-1})$ instead of $F(T)$ (or applying the anti-automorphism $\chi$) we obtain

$$\sum_{i=0}^{n} \binom{m+i}{j} \mathcal{P}^i \chi(\mathcal{P}^{n-i}) = 0 \quad \text{for} \quad 0 \leq j < \phi(n).$$

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This is Theorem 1 of [3]. (The condition $0 < j < \phi(n)$ is equivalent to $(p-1)n > jp - \alpha(j)$ in the notation of [3] and setting $(N,K,L) = (n,j,m-1,j)$ gives $(m+j) = (-1)^L (K,L).$

It will be convenient to write $\gamma(r) = (p^r - 1)/(p-1)$.

**Corollary 2.4.** Let $n \geq \gamma(r+1) - r$ and $0 \leq k < p^r$. Then

$$\sum_{j \geq 0} p^{k+p^r j} \chi(p^{n-k-p^r j}) = 0.$$ \hfill (2.5)

This is Theorem 3.2 of [3]. The condition $n \geq \gamma(r+1) - r$ is equivalent to $p^r \leq \phi(n)$; see Proposition 2.6. From Theorem 2.1, $F_n(T)$ is divisible by $T^{p^r-1} = \gamma(r+1)$. The result follows as a polynomial $\sum a_i T^i$ is divisible by $T^{N-1}$ if and only if $\sum a_{k+Nj} = 0$ for each $k$, $0 \leq k < N$.

From Corollary 2.4, one obtains Straffin’s formula [10]:

$$\sum_{j=0}^n p^r \chi(p^{n-(p^r-j)}) = 0.$$ \hfill (2.5)

In the next proposition we collect some properties of $\phi(n)$. (We are grateful to the referee for drawing our attention to the characterization (d) of $\phi$.)

**Proposition 2.6.** The function $\phi$ has the following properties.

(a) $\phi(\gamma(r)) = p^r - 1$, for $r \geq 1$.

(b) $\phi(i+j) \leq \phi(i) + \phi(j)$ for $i, j \geq 0$.

(c) Let $\gamma(r) < n < \gamma(r+1)$. Then $\phi(n) = \phi(n - \gamma(r)) + \phi(\gamma(r))$.

(d) Let $\psi(n)$, defined for non-negative integers $n$, satisfy: $\psi(0) = 1$; $\psi(i+j) \leq \psi(i) + \psi(j)$ for $i, j \geq 0$; $\psi(\gamma(r)) \leq p^r - 1$ for $r \geq 1$. Then $\psi(n) \leq \phi(n)$ for all $n \geq 0$.

**Proof.** Recall that $\nu_p(k!) = (k - \alpha_p(k))/(p-1)$, where $\alpha_p(k)$ is the sum of the coefficients in the $p$-adic expansion of $k$. One finds that $p^r - 1 + \nu_p(p^{r-1}) = \gamma(r)$, and this establishes (a).

Part (b) follows from the inequality $k + \nu_p(k!) + l + \nu_p(l!) \leq (k+l) + \nu_p((k+l)!)$.

In (c) we have $p^r - 1 < \phi(n) \leq p^r$. Consider $k = \phi(n) - p^r$. Now

$$\alpha_p(k) = \alpha_p(\phi(n)) + \begin{cases} -1 & \text{if } \phi(n) < p^r, \\ 0 & \text{if } \phi(n) = p^r, \\ 1 & \text{if } \phi(n) = p^r. \end{cases}$$

and

$$k + \nu_p(k!) = \phi(n) + \nu_p(\phi(n)!): \gamma(r) - \begin{cases} 0 & \text{if } \phi(n) < p^r, \\ 2 & \text{if } \phi(n) = p^r, \end{cases}$$

In both cases we have $k \geq \phi(n - \gamma(r))$, since $n < \gamma(r+1)$ when $\phi(n) = p^r$. The reverse inequality is supplied by (b).

Part (d) is then an immediate consequence of (a)-(c). \hfill $\square$

The properties of $\psi$ postulated in (d) above are precisely those required of $\phi$ in the proof of Theorem 2.1 which follows.

**Proof of Theorem 2.1.** Both the hypothesis and the conclusion of the theorem are multiplicative: $\phi$, as we have noted above, is subadditive and the $F_n(T)$ satisfy a Cartan formula. For let $P = \sum_{i \geq 0} \rho^i w^i \in A(p)[[w]]$ be the total Steenrod power and $P_T^{-1} = \sum_{i \geq 0} \chi(p^i - 1) w^i$ in $A(p)[[w]]$. So $P \cdot P_T^{-1} = \sum_{n \geq 0} F_n(T) w^n$. As both $P$ and its inverse are multiplicative, $P \cdot P_T^{-1}$.
It is, therefore, sufficient to verify the assertion in the theorem on a generator \( e \in H^2(\mathbb{C}P^\infty; \mathbb{F}_p) \) for \( p \) odd or \( e \in H^1(\mathbb{R}P^\infty; \mathbb{F}_2) \) for \( p = 2 \). (See, for example, Chapters I and VI of [9].)

A short calculation shows that

\[
(P \cdot P_T^{-1})e = \sum_{r \geq 0} (-1)^r e^{pr}(1 + we^{p-1})^{pr} (wT)^{\gamma(r)}.
\]

So

\[
F_n(T)e = \begin{cases} 
  e, & n = 0, \\
  (-1)^r e^{pr} (T - 1)^{pr-1} T^{\gamma(r-1)}, & n = \gamma(r), r \geq 1, \\
  0, & \text{otherwise}.
\end{cases}
\]

Since \( \phi(\gamma(r)) = p^{r-1} \), this completes the proof. \( \square \)

The proof of Theorem 1 in [3] uses an auxiliary result which can be established by similar reasoning.

**Proposition 2.7.** The polynomial

\[
G_n(T) = \sum_{i=0}^n \chi(P^{n-i}) P^iT^i \in A(p)[T]
\]

is of the form \( (T - 1)^{r} g(T^p) \), where \( r \equiv n \pmod{p} \).

Again statements are multiplicative and it suffices to check the result on the class \( e \). One verifies that

\[
G_n(T)e = \begin{cases} 
  e, & n = 0, \\
  (-1)^r e^{pr} (1 - T), & n = \gamma(r), r \geq 1, \\
  0, & \text{otherwise}.
\end{cases}
\]

So, setting \( S = T - 1 \) as above and expanding \( T^{pm} G_n(T) \), it follows that

\[
\sum_{j=0}^n (i+jm) \chi(P^{n-i}) P^i = 0, \quad \text{when } j \not\equiv n \pmod{p}.
\]

(2.8)

The basic technique in these proofs can be found already in work of Atiyah and Hirzebruch [2]. Working in the category of finite CW-complexes we set

\[
V^* = H^*(-; \mathbb{F}_p) \otimes \mathbb{F}_p[w],
\]

where \( w \) has degree \(-1\) if \( p = 2 \) and \(-2(p-1)\) if \( p \) is odd, and equip the cohomology theory \( V^* \) with the obvious product.

Any automorphism \( A \) of this multiplicative theory which fixes \( w \) is of the form

\[
A = \sum_{i \geq 0} \alpha_i w^i, \quad \text{with } \alpha_i \in A(p), \text{ and } \alpha_0 = 1.
\]

Now \( A(e) = \sum_{r \geq 0} a_r e^{pr} w^{\gamma(r)}, a_r \in \mathbb{F}_p \), with \( a_0 = 1 \). (As we have restricted attention to finite complexes, we should work with the skeleta of the infinite-dimensional projective space.) We associate to \( A \) the formal power series

\[
\xi_A(X) = \sum_{r \geq 0} a_r w^{\gamma(r)} X^{pr} \in \mathbb{F}_p[w][[X]], \quad a_0 = 1,
\]

and notice that \( \xi_{AB} = \xi_B \circ \xi_A \).
Theorem 2.10 (Atiyah-Hirzebruch). The map \( A \mapsto \xi_A(X) \) gives an anti-isomorphism between the group of automorphisms of the multiplicative cohomology theory \( \mathcal{V}^*(-) \) which fix \( w \) and the group of formal power series of the form

\[
\sum_{r \geq 0} a_r u^r \in \mathbb{F}_p[[u]][[X]], \quad a_0 = 1.
\]

The basic example of such an automorphism is the total Steenrod power \( P \). More generally, given a commutative \( \mathbb{F}_p \)-algebra \( R \), one can consider \( R \)-automorphisms of the cohomology theory \( \mathcal{V}^*(-) \otimes R \). Theorem 2.10 generalizes immediately. We have been working with the polynomial ring \( R = \mathbb{F}_p[T] \).

For reference we write

\[
\pi(X) = \xi_P(X), \quad \pi_T(X) = \xi_{P_T}(X).
\]

3. Connections with Adams operations in \( K \)-theory

The arithmetic result used to translate statements about complex \( K \)-theory with \( p \)-local coefficients into statements about cohomology operations with \( \mathbb{F}_p \)-coefficients is the following.

Lemma 3.1. Let \( f(T) \in \mathbb{Z}_p[T] \) be a polynomial such that \( p^n \) divides \( f(k^{p^{-1}}) \) for every integer \( k \) prime to \( p \). Then the \( \mod p \) reduction of \( f(T) \) in \( \mathbb{F}_p[T] \) is divisible by \( (T - 1)^{\phi(n)} \).

Proof. We recall that when \( g(U) \in \mathbb{Q}[U] \) is a rational polynomial of degree \( d \) which has the property that \( g(l) \in \mathbb{Z}_p \) for every \( l \in \mathbb{Z} \), then for some \( b_i \in \mathbb{Z}_p \)

\[
g(U) = \sum_{i=0}^{d} b_i \binom{U}{i}.
\]

Given (non-zero) \( f(T) \) of degree \( d \), we define \( g(U) = p^{-n} f(1 + pU) \). If \( k \) is prime to \( p \), then \( k^{p^{-1}} = 1 + pl \) for some \( l \in \mathbb{Z} \). On the other hand, for any integer \( l \) there is some \( k \) prime to \( p \) with \( k^{p^{-1}} \equiv 1 + pl \) (mod \( p^n \)). So \( g(l) \in \mathbb{Z}_p \) for all \( l \in \mathbb{Z} \).

Hence

\[
f(T) = \sum_{i=0}^{d} p^n b_i \binom{(T - 1)/p}{i}
\]

for some \( b_i \in \mathbb{Z}_p \).

By comparing coefficients of \( T^d \), we see that \( p^{n-d-v_p(d)} b_d \) is equal to the leading coefficient \( a_d \in \mathbb{Z}_p \) of \( f(T) \). So

\[
p^n b_d \binom{(T - 1)/p}{d} \in \mathbb{Z}_p[T]
\]

and has \( \mod p \) reduction \( a_d(T - 1)^d \). If \( d < \phi(n) \), then \( a_d \) is divisible by \( p \). Otherwise, there is no restriction on \( a_d \mod p \). The proof is completed by induction on \( d \).

We note that for each integer \( n \geq 0 \) there is a polynomial \( f(T) \) satisfying the hypotheses of Lemma 3.1 with \( \mod p \) reduction equal to \( (T - 1)^{\phi(n)} \), namely

\[
f(T) = p^k k! \binom{(T - 1)/p}{k},
\]

where \( k = \phi(n) \). So the exponent \( \phi(n) \) in Lemma 3.1 cannot be improved.
Until the final paragraph of this section we will now assume that \( p \) is an odd prime. We give two closely related ways of seeing that the relations of Theorem 2.1 arise from standard properties of Adams operations.

The first follows the approach of [7, 6]. For a finite CW-complex \( Z \) without homology \( p \)-torsion, it was shown in [8], by pulling the unstable Adams operations apart, that there exist homomorphisms \( S^i : H^*(Z;\mathbb{Z}(p)) \to H^{*+2(p-1)i}(Z;\mathbb{Z}(p)) \), with \( S^0 \) the identity, satisfying certain properties. (These operations are not natural: their construction depends upon a choice of isomorphism between \( K \)-theory and cohomology.) When reduced mod \( p \), \( S^i \) coincides with the Steenrod power \( P^i \).

Let us introduce

\[
\Phi_n(T) = \sum_{i=0}^{n} S^{n-i} \chi(S^i) T^i,
\]

where \( \chi(S^i) \) is defined inductively by \( \sum_{i=0}^{n} S^{n-i} \chi(S^i) = 0 \) if \( n > 0 \) and is the identity homomorphism if \( n = 0 \). Corollary 2.12 of [8] asserts that the homomorphism \( \Phi_n(k^{p-1}) \) is divisible by \( p^n \), as a linear operator, for each integer \( k \) prime to \( p \).

Now Lemma 3.1 extends immediately to the polynomial \( f(T) = \Phi_n(T) \) with values in the endomorphism ring of \( H^*(Z;\mathbb{Z}(p)) \). Hence \( F_n(T) \) is divisible by \( (T-1)^{p(n)} \) as an endomorphism of \( H^*(Z;\mathbb{F}_p) \). Since relations among the Steenrod powers are detected by torsion-free spaces, Theorem 2.1 follows.

The second approach is closer to that in Section 2. Let \( \ell^* \) be the Adams summand of \( p \)-local connective complex \( K \)-theory. Again restricting attention to finite complexes, we define a multiplicative cohomology theory \( W^* \) by

\[
W^*(-) = H^*(-;\mathbb{Z}(p)) \otimes \mathbb{Z}(p)[w] = \bigoplus_{j \geq 0} H^{*+2(p-1)j}(-;\mathbb{Z}(p)) w^j,
\]

where \( w \) has degree \(-2(p-1)\). Both the Adams integral Chern character

\[
\text{ch} : \ell^*(-) \to W^*(-)
\]

and the Thom homomorphism \( \dim : \ell^*(-) \to H^*(-;\mathbb{Z}(p)) \) are multiplicative transformations. We have \( \text{ch}(v) = pw \), where \( v \) is a Bott generator of \( \ell^{-2(p-1)} \).

According to Adams there is the commutative diagram:

\[
\begin{array}{ccc}
\ell^*(-) & \xrightarrow{\text{ch}} & W^*(-) \\
\downarrow \text{dim} & & \downarrow \text{mod } p \\
H^*(-;\mathbb{Z}(p)) & \xrightarrow{\text{mod } p} & H^*(-;\mathbb{F}_p) \\
\downarrow \text{p-1} & & \downarrow \text{p-1} \\
W^*(-) & \xrightarrow{\dim} & V^*(-)
\end{array}
\]

(See [1] and, for example, [5].)

We choose a complex orientation for \( \ell^* \) lifting the standard orientation on integral cohomology. Let \( e_\ell(\lambda) \in \ell^2(Z) \) be the associated Euler class of a complex line bundle \( \lambda \) over a finite complex \( Z \) and \( e(\lambda) \in H^2(Z;\mathbb{Z}(p)) \) be the standard Euler class.

By looking at the Hopf line bundle over \( \mathbb{C}P^\infty \) we see that

\[
\text{ch}(e_\ell(\lambda)) = \sigma^{-1}(e(\lambda)),
\]

for some formal power series \( \sigma^{-1}(X) = \sum t_n w^n X^{(p-1)n+1} \), in \( \mathbb{Z}(p)[w][[X]] \), with \( t_0 = 1 \). The inverse formal power series \( \sigma(X) \) has a similar form \( \sum s_n w^n X^{(p-1)n+1} \)

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with $s_0 = 1$. We shall also need $\sigma_T(X) = \sum s_n(wT)^n X^{(p-1)n+1}$. From the commutative diagram above one deduces that $\sigma$ and $\sigma_T^{-1}$ lift the power series $\pi$ and $\tau_T^{-1}$ in $F_p[w][[X]]$ corresponding to the total Steenrod power and its inverse, (2.11).

The Adams operations $\psi^k$ are defined on $\ell^*(-)$ for $k$ prime to $p$. We have

$$\text{ch} \circ \psi^k = \psi^k \circ \text{ch},$$

where the corresponding operator $\psi^k_W$ on $W^*(-)$ is the identity on $H^*(-; \mathbb{Z}(p))$ and $\psi^k_W(w) = k^{p-1}w$.

In $\ell^*$-theory we can write $\psi^k(e_\ell(\lambda)) = \Psi^k(e_\ell(\lambda))$, where

$$\Psi^k(X) = \sum c_n v^n X^{(p-1)n+1} \in \mathbb{Z}(p)[v][[X]], \quad c_0 = 1.$$

The relations in Theorem 2.1 arise from the equality

$$\text{ch}(\psi^k(e_\ell(\lambda))) = \psi^k_W(\text{ch}(e_\ell(\lambda))).$$

This gives

$$\Psi^k_W(\sigma^{-1}(-)) = \sigma_{k^{-1}}(\sigma(X)),$$

where $\Psi^k_W$ is obtained from $\Psi^k$ by substituting $pw$ for $v$. But by considering $\mathbb{CP}^\infty$, this is a general equality in $\mathbb{Z}(p)[w][[X]]$ where $e$ is replaced by $X$, and so

$$\Psi^k_W(X) = \sigma_{k^{-1}}(\sigma(X)).$$

We consider the coefficient of $w^n X^{n(p-1)+1}$ in $\sigma_T^{-1}(\sigma(X))$, which is a polynomial $f_n(T) \in \mathbb{Z}(p)[T]$. Now $f_n(k^{p-1})$, the coefficient of $w^n X^{n(p-1)+1}$ in $\Psi^k_W(X)$, is divisible by $p^n$. The reduction of $f_n(T)$ mod $p$ is the coefficient of $w^n X^{n(p-1)+1}$ in $\pi_T^{-1}(\pi(X))$. So Lemma 3.1 applies to give a non-computational proof of the essential step in the proof of Theorem 2.1 that $F_\ell(T)e$ is divisible by $(T - 1)^\phi(n)$.

When $p = 2$, $\mathbb{P}^1$ in this section must be replaced by $S^2\mathbb{P}^1$ and the cohomology theories $V^*(-)$ and $W^*(-)$ re-defined with the class $w$ of degree $-2$. Then the results above are valid. The connection with Theorem 2.1 can be made, as in [6], by noting that a homogeneous polynomial in the $S^2\mathbb{P}^1$ vanishes on the cohomology of a product of complex projective spaces if and only if the same polynomial in the $S^2\mathbb{P}^1$ vanishes on the corresponding product of real projective spaces.

**Remarks on formal groups.** The formal power series occurring in Theorem 2.10 as the automorphisms of the cohomology theory $V^*(-)$ are the strict automorphisms of the additive formal group law $F$ over $\mathbb{Z}(p)[w]$: $F(X, Y) = X + Y$. (The degree of $X$ is 1 if $p = 2$ and 2 if $p$ is odd.) The ring of endomorphisms of $F$ is the ring $F_p[[F]]$ of formal power series in the Frobenius $F$, $F(X) = X^p$. The automorphism group is the group of units in this ring with constant term 1 and is free abelian over the $(p$-adic integers) $\mathbb{Z}_p$ of rank $p - 1$. More generally, if we work over a commutative $F_p$-algebra $R$, the endomorphism ring is the twisted formal power series ring $R[[F]]$, with $F\tau = r^p F$ for $r \in R$. The calculations in the proof of Theorem 2.1 are carried out in this ring, with $R = F_p[T]$.

In Section 3, the power series $\sigma$ and $\sigma^{-1}$ are isomorphisms between the additive formal group law $F$ over $\mathbb{Z}(p)[w]$ and the group law $F_\ell$ coming from the chosen complex orientation of $\ell^*$:

$$F(\sigma(X), \sigma(Y)) = \sigma(X) + \sigma(Y) = \sigma(F_\ell(X, Y)).$$
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REFERENCES


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