

K-THEORY AND THE ANTI-AUTOMORPHISM OF THE STEENROD ALGEBRA

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ABSTRACT. We give simple proofs of some relations in the Steenrod algebra involving the powers \mathcal{P}^i and their duals $\chi\mathcal{P}^i$ and show how these relations arise from K -theory.

1. INTRODUCTION

Barratt and Miller in [3] derived some relations in the Steenrod algebra $\mathcal{A}(p)$ using expressions for the Adem relations due to Bullett and Macdonald [4]. In Section 2 we reformulate and give a simple proof of the former in the spirit of the latter. These relations arise naturally from properties of the Adams operations in complex K -theory, as was explained in [7, 6] for the prime 2; alternative and shorter K -theoretic derivations valid for all primes are given in Section 3.

2. RESULTS OF BARRATT AND MILLER

We use standard notation for the mod p Steenrod algebra $\mathcal{A}(p)$ with the convention that \mathcal{P}^i denotes Sq^i when $p = 2$. For each non-negative integer n , let

$$\phi(n) = \min\{k \geq 0 \mid k + \nu_p(k!) \geq n\}.$$

Another, more illuminating description of $\phi(n)$ is given in Proposition 2.6.

Theorem 2.1. *The polynomial*

$$F_n(T) = \sum_{i=0}^n \mathcal{P}^{n-i} \chi(\mathcal{P}^i) T^i \in \mathcal{A}(p)[T]$$

is divisible by $(T - 1)^{\phi(n)}$.

This result can be restated in many ways. Let $T^m F_n(T)$, for $m \in \mathbb{Z}$, be expressed as a formal power series in $S = T - 1$. By considering the coefficient of S^j with $0 \leq j < \phi(n)$, it follows that

$$(2.2) \quad \sum_{i=0}^n \binom{m+i}{j} \mathcal{P}^{n-i} \chi(\mathcal{P}^i) = 0 \quad \text{for } 0 \leq j < \phi(n).$$

Looking at $T^n F(T^{-1})$ instead of $F(T)$ (or applying the anti-automorphism χ) we obtain

$$(2.3) \quad \sum_{i=0}^n \binom{m+i}{j} \mathcal{P}^i \chi(\mathcal{P}^{n-i}) = 0 \quad \text{for } 0 \leq j < \phi(n).$$

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This is Theorem 1 of [3]. (The condition $0 \leq j < \phi(n)$ is equivalent to $(p - 1)n > jp - \alpha(j)$ in the notation of [3] and setting $(N, K, L) = (n, j - m - 1, j)$ gives $\binom{m+i}{j} = (-1)^L \binom{K-i}{L}$.)

It will be convenient to write $\gamma(r) = (p^r - 1)/(p - 1)$.

Corollary 2.4. *Let $n \geq \gamma(r + 1) - r$ and $0 \leq k < p^r$. Then*

$$\sum_{j \geq 0} \mathcal{P}^{k+p^r j} \chi(\mathcal{P}^{n-k-p^r j}) = 0.$$

This is Theorem 3.2 of [3]. The condition $n \geq \gamma(r + 1) - r$ is equivalent to $p^r \leq \phi(n)$; see Proposition 2.6. From Theorem 2.1, $F_n(T)$ is divisible by $T^{p^r} - 1 = (T - 1)^{p^r}$ and the result follows as a polynomial $\sum a_i T^i$ is divisible by $T^N - 1$ if and only if $\sum a_{k+Nj} = 0$ for each $k, 0 \leq k < N$.

From Corollary 2.4, one obtains Straffin’s formula [10]:

$$(2.5) \quad \sum_{j=0}^p \mathcal{P}^{p^r j} \chi(\mathcal{P}^{p^r(p-j)}) = 0.$$

In the next proposition we collect some properties of $\phi(n)$. (We are grateful to the referee for drawing our attention to the characterization (d) of ϕ .)

Proposition 2.6. *The function ϕ has the following properties.*

- (a) $\phi(\gamma(r)) = p^{r-1}$, for $r \geq 1$.
- (b) $\phi(i + j) \leq \phi(i) + \phi(j)$ for $i, j \geq 0$.
- (c) Let $\gamma(r) < n < \gamma(r + 1)$. Then $\phi(n) = \phi(n - \gamma(r)) + \phi(\gamma(r))$.
- (d) Let $\psi(n)$, defined for non-negative integers n , satisfy: $\psi(0) = 1$; $\psi(i + j) \leq \psi(i) + \psi(j)$ for $i, j \geq 0$; $\psi(\gamma(r)) \leq p^{r-1}$ for $r \geq 1$. Then $\psi(n) \leq \phi(n)$ for all $n \geq 0$.

Proof. Recall that $\nu_p(k!) = (k - \alpha_p(k))/(p - 1)$, where $\alpha_p(k)$ is the sum of the coefficients in the p -adic expansion of k . One finds that $p^{r-1} + \nu_p(p^{r-1}!) = \gamma(r)$, and this establishes (a).

Part (b) follows from the inequality $k + \nu_p(k!) + l + \nu_p(l!) \leq (k + l) + \nu_p((k + l)!)$.

In (c) we have $p^{r-1} < \phi(n) \leq p^r$. Consider $k = \phi(n) - p^{r-1}$. Now

$$\alpha_p(k) = \alpha_p(\phi(n)) + \begin{cases} -1 & \text{if } \phi(n) < p^r, \\ p - 2 & \text{if } \phi(n) = p^r \end{cases}$$

and

$$k + \nu_p(k!) = \phi(n) + \nu_p(\phi(n)!) - \gamma(r) - \begin{cases} 0 & \text{if } \phi(n) < p^r, \\ 1 & \text{if } \phi(n) = p^r. \end{cases}$$

In both cases we have $k \geq \phi(n - \gamma(r))$, since $n < \gamma(r + 1)$ when $\phi(n) = p^r$. The reverse inequality is supplied by (b).

Part (d) is then an immediate consequence of (a)-(c). □

The properties of ψ postulated in (d) above are precisely those required of ϕ in the proof of Theorem 2.1 which follows.

Proof of Theorem 2.1. Both the hypothesis and the conclusion of the theorem are multiplicative: ϕ , as we have noted above, is subadditive and the $F_n(T)$ satisfy a Cartan formula. For let $P = \sum_{i \geq 0} \mathcal{P}^i w^i \in \mathcal{A}(p)[[w]]$ be the total Steenrod power and $P_T^{-1} = \sum_{i \geq 0} \chi(\mathcal{P}^i)(wT)^i \in \mathcal{A}(p)[T][[w]]$. So $P \cdot P_T^{-1} = \sum_{n \geq 0} F_n(T)w^n$. As both P and its inverse are multiplicative on cohomology classes, so is $P \cdot P_T^{-1}$.

It is, therefore, sufficient to verify the assertion in the theorem on a generator $e \in H^2(\mathbb{C}P^\infty; \mathbb{F}_p)$ for p odd or $e \in H^1(\mathbb{R}P^\infty; \mathbb{F}_2)$ for $p = 2$. (See, for example, Chapters I and VI of [9].)

A short calculation shows that

$$(P \cdot P_T^{-1})e = \sum_{r \geq 0} (-1)^r e^{p^r} (1 + we^{p-1})^{p^r} (wT)^{\gamma(r)}.$$

So

$$F_n(T)e = \begin{cases} e, & n = 0, \\ (-1)^r e^{p^r} (T - 1)^{p^r-1} T^{\gamma(r-1)}, & n = \gamma(r), r \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\phi(\gamma(r)) = p^{r-1}$, this completes the proof. □

The proof of Theorem 1 in [3] uses an auxiliary result which can be established by similar reasoning.

Proposition 2.7. *The polynomial*

$$G_n(T) = \sum_{i=0}^n \chi(\mathcal{P}^{n-i}) \mathcal{P}^i T^i \in \mathcal{A}(p)[T]$$

is of the form $(T - 1)^r g(T^p)$, where $r \equiv n \pmod{p}$.

Again statements are multiplicative and it suffices to check the result on the class e . One verifies that

$$G_n(T)e = \begin{cases} e, & n = 0, \\ (-1)^r e^{p^r} (1 - T), & n = \gamma(r), r \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

So, setting $S = T - 1$ as above and expanding $T^{pm} G_n(T)$, it follows that

$$(2.8) \quad \sum_{i=0}^n \binom{i+pm}{j} \chi(\mathcal{P}^{n-i}) \mathcal{P}^i = 0, \quad \text{when } j \not\equiv n \pmod{p}.$$

The basic technique in these proofs can be found already in work of Atiyah and Hirzebruch [2]. Working in the category of finite CW-complexes we set

$$(2.9) \quad \mathcal{V}^*(-) = H^*(-; \mathbb{F}_p) \otimes \mathbb{F}_p[w],$$

where w has degree -1 if $p = 2$ and $-2(p-1)$ if p is odd, and equip the cohomology theory \mathcal{V}^* with the obvious product.

Any automorphism A of this multiplicative theory which fixes w is of the form

$$A = \sum_{i \geq 0} \alpha_i w^i, \quad \text{with } \alpha_i \in \mathcal{A}(p), \text{ and } \alpha_0 = 1.$$

Now $A(e) = \sum_{r \geq 0} a_r e^{p^r} w^{\gamma(r)}$, $a_r \in \mathbb{F}_p$, with $a_0 = 1$. (As we have restricted attention to finite complexes, we should work with the skeleta of the infinite-dimensional projective space.) We associate to A the formal power series

$$\xi_A(X) = \sum_{r \geq 0} a_r w^{\gamma(r)} X^{p^r} \in \mathbb{F}_p[w][[X]], \quad a_0 = 1,$$

and notice that $\xi_{AB} = \xi_B \circ \xi_A$.

Theorem 2.10 (Atiyah-Hirzebruch). *The map $A \mapsto \xi_A(X)$ gives an anti-isomorphism between the group of automorphisms of the multiplicative cohomology theory $\mathcal{V}^*(-)$ which fix w and the group of formal power series of the form*

$$\sum_{r \geq 0} a_r w^{\gamma(r)} X^{p^r} \in \mathbb{F}_p[w][[X]], \quad a_0 = 1.$$

The basic example of such an automorphism is the total Steenrod power P . More generally, given a commutative \mathbb{F}_p -algebra R , one can consider R -automorphisms of the cohomology theory $\mathcal{V}^*(-) \otimes R$. Theorem 2.10 generalizes immediately. We have been working with the polynomial ring $R = \mathbb{F}_p[T]$.

For reference we write

$$(2.11) \quad \pi(X) = \xi_P(X), \quad \pi_T(X) = \xi_{P_T}(X).$$

3. CONNECTIONS WITH ADAMS OPERATIONS IN K -THEORY

The arithmetic result used to translate statements about complex K -theory with p -local coefficients into statements about cohomology operations with \mathbb{F}_p -coefficients is the following.

Lemma 3.1. *Let $f(T) \in \mathbb{Z}_{(p)}[T]$ be a polynomial such that p^n divides $f(k^{p-1})$ for every integer k prime to p . Then the mod p reduction of $f(T)$ in $\mathbb{F}_p[T]$ is divisible by $(T - 1)^{\phi(n)}$.*

Proof. We recall that when $g(U) \in \mathbb{Q}[U]$ is a rational polynomial of degree d which has the property that $g(l) \in \mathbb{Z}_{(p)}$ for every $l \in \mathbb{Z}$, then for some $b_i \in \mathbb{Z}_{(p)}$

$$g(U) = \sum_{i=0}^d b_i \binom{U}{i}.$$

Given (non-zero) $f(T)$ of degree d , we define $g(U) = p^{-n} f(1 + pU)$. If k is prime to p , then $k^{p-1} = 1 + pl$ for some $l \in \mathbb{Z}$. On the other hand, for any integer l there is some k prime to p with $k^{p-1} \equiv 1 + pl \pmod{p^n}$. So $g(l) \in \mathbb{Z}_{(p)}$ for all $l \in \mathbb{Z}$. Hence

$$f(T) = \sum_{i=0}^d p^n b_i \binom{(T-1)/p}{i}$$

for some $b_i \in \mathbb{Z}_{(p)}$.

By comparing coefficients of T^d , we see that $p^{n-d-\nu_p(d!)} b_d$ is equal to the leading coefficient $a_d \in \mathbb{Z}_{(p)}$ of $f(T)$. So

$$p^n b_d \binom{(T-1)/p}{d} \in \mathbb{Z}_{(p)}[T]$$

and has mod p reduction $a_d(T - 1)^d$. If $d < \phi(n)$, then a_d is divisible by p . Otherwise, there is no restriction on $a_d \pmod{p}$. The proof is completed by induction on d . □

We note that for each integer $n \geq 0$ there is a polynomial $f(T)$ satisfying the hypotheses of Lemma 3.1 with mod p reduction equal to $(T - 1)^{\phi(n)}$, namely

$$f(T) = p^k k! \binom{(T-1)/p}{k},$$

where $k = \phi(n)$. So the exponent $\phi(n)$ in Lemma 3.1 cannot be improved.

Until the final paragraph of this section we will now assume that p is an odd prime. We give two closely related ways of seeing that the relations of Theorem 2.1 arise from standard properties of Adams operations.

The first follows the approach of [7, 6]. For a finite CW-complex Z without homology p -torsion, it was shown in [8], by pulling the unstable Adams operations apart, that there exist homomorphisms $S^i : H^*(Z; \mathbb{Z}_{(p)}) \rightarrow H^{*+2(p-1)i}(Z; \mathbb{Z}_{(p)})$, with S^0 the identity, satisfying certain properties. (These operations are not natural: their construction depends upon a choice of isomorphism between K -theory and cohomology.) When reduced mod p , S^i coincides with the Steenrod power \mathcal{P}^i . Let us introduce

$$\Phi_n(T) = \sum_{i=0}^n S^{n-i} \chi(S^i) T^i,$$

where $\chi(S^i)$ is defined inductively by $\sum_{i=0}^n S^{n-i} \chi(S^i) = 0$ if $n > 0$ and is the identity homomorphism if $n = 0$. Corollary 2.12 of [8] asserts that the homomorphism $\Phi_n(k^{p-1})$ is divisible by p^n , as a linear operator, for each integer k prime to p .

Now Lemma 3.1 extends immediately to the polynomial $f(T) = \Phi_n(T)$ with values in the endomorphism ring of $H^*(Z; \mathbb{Z}_{(p)})$. Hence $F_n(T)$ is divisible by $(T-1)^{\phi(n)}$ as an endomorphism of $H^*(Z; \mathbb{F}_p)$. Since relations among the Steenrod powers are detected by torsion-free spaces, Theorem 2.1 follows.

The second approach is closer to that in Section 2. Let ℓ^* be the Adams summand of p -local connective complex K -theory. Again restricting attention to finite complexes, we define a multiplicative cohomology theory \mathcal{W}^* by

$$\mathcal{W}^*(-) = H^*(-; \mathbb{Z}_{(p)}) \otimes \mathbb{Z}_{(p)}[w] = \bigoplus_{j \geq 0} H^{*+2(p-1)j}(-; \mathbb{Z}_{(p)}) w^j,$$

where w has degree $-2(p-1)$. Both the Adams integral Chern character

$$\text{ch} : \ell^*(-) \rightarrow \mathcal{W}^*(-)$$

and the Thom homomorphism $\dim : \ell^*(-) \rightarrow H^*(-; \mathbb{Z}_{(p)})$ are multiplicative transformations. We have $\text{ch}(v) = pw$, where v is a Bott generator of $\ell^{-2(p-1)}$.

According to Adams there is the commutative diagram:

$$\begin{array}{ccc} \ell^*(-) & \xrightarrow{\text{ch}} & \mathcal{W}^*(-) \\ \dim \downarrow & & \downarrow \text{mod } p \\ H^*(-; \mathbb{Z}_{(p)}) & & \\ \text{mod } p \downarrow & & \\ H^*(-; \mathbb{F}_p) & \xrightarrow{\mathcal{P}^{-1}} & \mathcal{V}^*(-) \end{array}$$

(See [1] and, for example, [5].)

We choose a complex orientation for ℓ^* lifting the standard orientation on integral cohomology. Let $e_\ell(\lambda) \in \ell^2(Z)$ be the associated Euler class of a complex line bundle λ over a finite complex Z and $e(\lambda) \in H^2(Z; \mathbb{Z}_{(p)})$ be the standard Euler class.

By looking at the Hopf line bundle over $\mathbb{C}P^\infty$ we see that

$$\text{ch}(e_\ell(\lambda)) = \sigma^{-1}(e(\lambda)),$$

for some formal power series $\sigma^{-1}(X) = \sum t_n w^n X^{(p-1)n+1}$, in $\mathbb{Z}_{(p)}[w][[X]]$, with $t_0 = 1$. The inverse formal power series $\sigma(X)$ has a similar form $\sum s_n w^n X^{(p-1)n+1}$

with $s_0 = 1$. We shall also need $\sigma_T(X) = \sum s_n(wT)^n X^{(p-1)n+1}$. From the commutative diagram above one deduces that σ and σ_T^{-1} lift the power series π and π_T^{-1} in $\mathbb{F}_p[w][[X]]$ corresponding to the total Steenrod power and its inverse, (2.11).

The Adams operations ψ^k are defined on $\ell^*(-)$ for k prime to p . We have

$$\text{ch} \circ \psi^k = \psi_{\mathcal{W}}^k \circ \text{ch},$$

where the corresponding operator $\psi_{\mathcal{W}}^k$ on $\mathcal{W}^*(-)$ is the identity on $H^*(-; \mathbb{Z}_{(p)})$ and $\psi_{\mathcal{W}}^k(w) = k^{p-1}w$.

In ℓ^* -theory we can write $\psi^k(e_\ell(\lambda)) = \Psi^k(e_\ell(\lambda))$, where

$$\Psi^k(X) = \sum c_n v^n X^{(p-1)n+1} \in \mathbb{Z}_{(p)}[v][[X]], \quad c_0 = 1.$$

The relations in Theorem 2.1 arise from the equality

$$\text{ch}(\psi^k(e_\ell(\lambda))) = \psi_{\mathcal{W}}^k(\text{ch}(e_\ell(\lambda))).$$

This gives

$$\Psi_{\mathcal{W}}^k(\sigma^{-1}(e)) = \sigma_{k^{p-1}}^{-1}(e),$$

where $\Psi_{\mathcal{W}}^k$ is obtained from Ψ^k by substituting pw for v . But by considering $\mathbb{C}P^\infty$, this is a general equality in $\mathbb{Z}_{(p)}[w][[X]]$ when e is replaced by X , and so

$$\Psi_{\mathcal{W}}^k(X) = \sigma_{k^{p-1}}^{-1}(\sigma(X)).$$

We consider the coefficient of $w^n X^{n(p-1)+1}$ in $\sigma_T^{-1}(\sigma(X))$, which is a polynomial $f_n(T) \in \mathbb{Z}_{(p)}[T]$. Now $f_n(k^{p-1})$, the coefficient of $w^n X^{n(p-1)+1}$ in $\Psi_{\mathcal{W}}^k(X)$, is divisible by p^n . The reduction of $f_n(T)$ mod p is the coefficient of $w^n X^{n(p-1)+1}$ in $\pi_T^{-1}(\pi(X))$. So Lemma 3.1 applies to give a non-computational proof of the essential step in the proof of Theorem 2.1 that $F_n(T)e$ is divisible by $(T-1)^{\phi(n)}$.

When $p = 2$, \mathcal{P}^i in this section must be replaced by Sq^{2^i} and the cohomology theories $\mathcal{V}^*(-)$ and $\mathcal{W}^*(-)$ re-defined with the class w of degree -2 . Then the results above are valid. The connection with Theorem 2.1 can be made, as in [6], by noting that a homogeneous polynomial in the Sq^{2^i} vanishes on the cohomology of a product of complex projective spaces if and only if the same polynomial in the Sq^i vanishes on the corresponding product of real projective spaces.

Remarks on formal groups. The formal power series occurring in Theorem 2.10 as the automorphisms of the cohomology theory $\mathcal{V}^*(-)$ are the strict automorphisms of the additive formal group law F over $\mathbb{F}_p[w]$: $F(X, Y) = X + Y$. (The degree of X is 1 if $p = 2$ and 2 if p is odd.) The ring of endomorphisms of F is the ring $\mathbb{F}_p[[\mathcal{F}]]$ of formal power series in the Frobenius \mathcal{F} , $\mathcal{F}(X) = X^p$. The automorphism group is the group of units in this ring with constant term 1 and is free abelian over (the p -adic integers) \mathbb{Z}_p of rank $p - 1$. More generally, if we work over a commutative \mathbb{F}_p -algebra R , the endomorphism ring is the twisted formal power series ring $R[[\mathcal{F}]]$, with $\mathcal{F}r = r^p\mathcal{F}$ for $r \in R$. The calculations in the proof of Theorem 2.1 are carried out in this ring, with $R = \mathbb{F}_p[T]$.

In Section 3, the power series σ and σ^{-1} are isomorphisms between the additive formal group law F over $\mathbb{Z}_{(p)}[w]$ and the group law F_ℓ coming from the chosen complex orientation of ℓ^* :

$$F(\sigma(X), \sigma(Y)) = \sigma(X) + \sigma(Y) = \sigma(F_\ell(X, Y)).$$

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