

## COHOMOLOGY OF GROUPS WITH METACYCLIC SYLOW $p$ -SUBGROUPS

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ABSTRACT. We determine the cohomology algebras  $H^*(G; \mathbf{F}_p)$  for all groups  $G$  with a metacyclic Sylow  $p$ -subgroup. The complete  $p$ -local stable decomposition of the classifying space  $BG$  is also determined.

### 1. INTRODUCTION AND STATEMENT OF RESULTS

Let  $P$  be a non-abelian metacyclic  $p$ -group of odd order and  $G$  a finite group with  $P$  as a Sylow  $p$ -subgroup. In this note we classify all possible mod- $p$  cohomology algebras  $H^*(G)$  and determine complete  $p$ -local stable splittings for the classifying spaces  $BG$ . Much of the topological part of this work was done by the first author in [D]; recent results on Swan groups [MP] have enabled us to show that in all cases  $H^*(G)$  is given by a ring of invariants. Similar but less complete information for metacyclic 2-groups was obtained in [D1, MP2, M].

A metacyclic  $p$ -group is a  $p$ -group  $P$  which is an extension of a cyclic group by a cyclic group. Following [D] we say that  $P$  is *split* if  $P$  can be expressed by some split extension. We recall that up to isomorphism any non-abelian metacyclic  $p$ -group can be expressed as

$$P = P(p^m, p^n, p^l + 1, p^q) = \langle x, y \mid x^{p^m} = 1, y^{p^n} = x^{p^q}, yxy^{-1} = x^{p^l+1} \rangle$$

for positive integers  $m, n, l, q$  satisfying  $l, q \leq m$ ,  $(p^l+1)^{p^n} \equiv 1 \pmod{p^m}$ ,  $(p^l+1)p^q \equiv p^q \pmod{p^m}$ ,  $n+l \geq m$  and  $q+l \geq m$ . In these terms  $P$  splits unless  $m \neq q$  and  $l < q < n$  [D, Thm. 3.1].

Let  $W_G(P) = N_G(P)/P \cdot C_G(P)$ ; then  $W_G(P) \leq \text{Out}(P)$ . If  $P$  is split, then  $\text{Out}(P) \cong O_p \text{Out}(P) \rtimes \mathbf{Z}/(p-1)$  where  $O_p \text{Out}(P)$  is a Sylow  $p$ -subgroup [D, Prop. 3.2]. Therefore  $W_G(P) = \mathbf{Z}/d$  where  $d$  is a divisor of  $p-1$ . If  $P$  is non-split,  $\text{Out}(P)$  is a  $p$ -group and so  $W_G(P) = 1$ . We denote by  $\mathbf{F}_p[\cdot]$  and  $E[\cdot]$  the polynomial and exterior algebras over  $\mathbf{F}_p$ .

**Theorem 1.1.** *As an algebra,  $H^*(G)$  has one of the following forms:*

- (1) *If  $P$  is split and  $l \neq m - n$ , then*

$$H^*(G) \cong H^*(P)^{W_G(P)} = \mathbf{F}_p[u_d, v] \otimes E[a_d, b]$$

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where  $|u_d| = 2d, |v| = 2, |a_d| = 2d - 1, |b| = 1$ .

(2) If  $P$  is split and  $l = m - n$ , then

$$H^*(G) \cong H^*(P)^{W_G(P)} = \mathbf{F}_p[v, z] \otimes E[b, \alpha_{2i-1}, i = 1, \dots, p]/R$$

where the relations  $R$  are given by

$$\alpha_{2i-1}\alpha_{2j-1} = 0, \quad 1 \leq i, j \leq p,$$

$$\alpha_{2i-1}v = 0, \quad 1 \leq i \leq p - 1,$$

and  $|b| = 1, |v| = 2, |z| = 2p, |\alpha_{2i-1}| = 2i - 1 + 2pd(i)$ , where  $0 \leq d(i) < d$  is the residue of  $-i \pmod d$ .

(3) If  $P$  is non-split, then  $H^*(G) = H^*(P)$  is isomorphic to the algebra of (1) with  $d = 1$  if  $m = l + q$  and to that of (2) with  $d = 1$  if  $m < l + q$ .

Generators for these cohomology groups are specified explicitly in the proof.

*Remark 1.2.* Groups exemplifying the cases above are easily given by  $G = P \rtimes \mathbf{Z}/d$ .

R. Lyons has suggested other, more natural examples, which occur as automorphism groups of Chevalley groups. For example, let  $\mathbf{F}_q$  be a finite field of characteristic different from  $p$  such that the Sylow  $p$ -subgroup of  $PSL_2(\mathbf{F}_q)$  has order  $p$ , i.e.,  $q^2 - 1$  is divisible by  $p$  but not by  $p^2$ . Then the Sylow  $p$ -subgroup of  $H = PSL_2(\mathbf{F}_{q^p})$  is cyclic of order  $p^2$ . Let  $\phi$  be the Frobenius automorphism of  $\mathbf{F}_{q^p}$  of order  $p$ . Then it is easy to see that  $\phi$  fixes a cyclic subgroup  $C \leq H$  of order  $p^a + 1$  which contains one such Sylow  $p$ -subgroup. Thus  $G = \text{Aut}(H) = PSL_2(\mathbf{F}_q) \rtimes \mathbf{Z}/p\langle\phi\rangle$  has  $P = M_3(p)$  as a Sylow  $p$ -subgroup. Furthermore  $N_H(C)$  is a dihedral group [Hu, II, 8.4 Satz] containing the permutation matrix of order two. Since this matrix is fixed by  $\phi$ ,  $W_G(P) = \mathbf{Z}/2$  and  $H^*(G)$  is of type (2) in Theorem 1.1 with  $d = 2$ .

The group cohomology of a group  $G$  is the cohomology of the classifying space  $BG$  of  $G$ . The space  $BG$  is stably homotopy equivalent to a wedge product of indecomposable spectra,

$$BG \simeq X_1 \vee X_2 \vee \dots \vee X_n.$$

A complete stable decomposition of  $BG$  is a splitting into indecomposable spectra. The decomposition is unique up to stable homotopy type and ordering. If  $G$  is a  $p$ -group, then all of these spectra are  $p$ -local. Otherwise, if  $P$  is a Sylow  $p$ -subgroup of  $G$ , then a simple transfer argument shows the  $p$ -localization of  $BG$  is a stable summand of  $BP$ ,

$$BP \simeq BG_p \vee Y,$$

where  $BG_p$  is the  $p$ -localization of  $BG$ . Hence  $BG_p$  consists of some, but possibly not all, of the summands of  $BP$ . Note  $H^*(BG_p; \mathbf{F}_p) = H^*(BG; \mathbf{F}_p)$ .

Each indecomposable spectrum  $X$  of  $BP$  corresponds up to conjugacy to a primitive idempotent  $e$  in the ring of stable self-maps  $\{BP, BP\}$ . The spectrum  $X$  is the infinite mapping telescope or homotopy colimit of  $e$ ,

$$X \simeq eBP = \text{Tel}(BP \xrightarrow{e} BP) = \text{hocolim}(BP \xrightarrow{e} BP \xrightarrow{e} \dots).$$

For more information see either [BF] or [MP1].

For the remainder of the paper all spectra are localized at the prime  $p$ . If  $P$  is a Swan group, then  $BG \simeq BN_G(P) \simeq B(P \rtimes W_G(P))$ . Thus determining the stable homotopy type of  $BG$  involves determining which summands have their cohomology left invariant by the action of the Weyl group of  $G$ .

$Z_p \text{Out}(P) \subseteq \{BP, BP\}$  is a subring, in fact a retract. Therefore, certain indecomposable summands of  $BP$  correspond to simple modules of the outer automorphism group  $\text{Out}(P)$ . A summand corresponding to a simple  $\text{Out}(P)$ -module is said to *originate* in  $BP$ . A summand originating in  $BP$  does not occur as the summand of the classifying space of any proper subgroup of  $P$ .

In this paragraph we introduce some notation for Theorem 1.3 below.  $L(2, k)$  originates in  $B(\mathbf{Z}/p \times \mathbf{Z}/p)$  and corresponds to  $St \otimes (\det)^k$  where  $St$  is the Steinberg module for  $\mathbf{F}_p GL_2(\mathbf{F}_p)$  and  $\det$  is the determinant module. It is well known that the group ring  $\mathbf{F}_p[\mathbf{Z}/(p-1)]$  has a complete set of orthogonal primitive idempotents  $e_0, \dots, e_{p-2}$  [D]. Lifting these idempotents to  $Z_p[\mathbf{Z}/(p-1)]$  determines a complete stable splitting of

$$B\mathbf{Z}/p^n \simeq \bigvee_{i=0}^{p-2} L(1, n, i),$$

where  $L(1, n, i)$  originates in  $B\mathbf{Z}/p^n$ . For more information on these summands see [HK] and [D].

If  $P$  is a split metacyclic  $p$ -group, then since  $W_G(P)$  is a  $p'$ -group, we have  $W_G(P) \leq \mathbf{Z}/(p-1)$ . Thus the primitive idempotents  $e_0, \dots, e_{p-2}$  above determine a stable splitting of  $BG$ . If  $P$  is non-split, then  $BP$  is stably indecomposable [D, Thm. 1.3].

Among the split metacyclic groups there is one which plays a special role, the extra-special modular group  $M_3(p) = P(p^2, p, p+1, 1)$ . It is characterized by its order and exponent which are  $p^3$  and  $p^2$  respectively.

**Theorem 1.3.** (1) *If  $P$  is split and  $P \neq M_3(p)$ , then*

$$e_0BP = X_0 \vee B(\mathbf{Z}/p^n), \quad e_iBP = X_i, \quad l = m - n, \quad 1 \leq i \leq p - 2.$$

$$e_0BP = X_0 \vee B(\mathbf{Z}/p^n) \vee L(1, n, 0), \quad e_iBP = X_i \vee L(1, n, i), \quad l \neq m - n, \quad 1 \leq i \leq p - 2.$$

(2) *If  $P = M_3(p)$ , then*

$$e_0BP = X_0 \vee \bigvee_{i=0}^{p-2} L(2, i) \vee L(1, 1, i), \quad e_iBP = X_i, \quad 1 \leq i \leq p - 2,$$

where  $X_i$  originates in  $BP$ .

(3) *In both cases this yields a complete stable decomposition of  $BP$  and*

$$BG \simeq \bigvee_{i=0}^{(p-1/d)-1} e_{id}BP.$$

**Corollary 1.4.** *Localized at  $p$ , the complete stable decomposition of  $BG$  is given by one of the following:*

- (1) *If  $P$  is non-split, then  $BG \simeq BP$ .*
- (2) *If  $P$  is split and  $P \neq M_3(p)$ , then*

$$BG \simeq \bigvee_{i=0}^{(p-1/d)-1} X_{id} \vee \bigvee_{j=0}^{p-2} L(1, n, j), \quad l = m - n.$$

$$BG \simeq \bigvee_{i=0}^{(p-1/d)-1} X_{id} \vee L(1, n, id) \vee \bigvee_{j=0}^{p-2} L(1, n, j), \quad l \neq m - n.$$

- (3) *If  $P = M_3(p)$ , then*

$$BG \simeq \bigvee_{i=0}^{(p-1/d)-1} X_{id} \vee \bigvee_{i=0}^{p-2} L(2, i) \vee L(1, 1, i).$$

If  $P$  is split and  $P \neq M_3(p)$ , then Theorem 1.3 and Corollary 1.4 were proved by the first author [D, Thm. 1.3].

Throughout we assume  $P$  is a non-abelian metacyclic  $p$ -group and  $p$  is an odd prime. All cohomology is taken with simple coefficients in  $\mathbf{F}_p$  and all spaces are considered stably, localized at  $p$ .

## 2. PROOFS

The classifying space  $BP$  is indecomposable if and only if  $P$  is non-split [D, Thm. 1.1]. Thus, since  $BG$  is a summand in  $BP$ , we have  $BG \simeq BP$  and  $H^*(G) \cong H^*(P)$ . In this case the cohomology algebras are given by [Hb, Thm. B if  $m < l + q$  and Thm. E if  $m = l + q$ ]. This completes the proof of both theorems for the non-split case.

Before turning to the proof of Theorem 1.1 we recall the notion of a Swan group [MP]. A  $p$ -group  $P$  is called a *Swan group* if the cohomology of any group  $G$  with  $P$  as a Sylow  $p$ -subgroup is given by its invariants, i.e.,

$$\text{res} : H^*(G; \mathbf{F}_p) \xrightarrow{\cong} H^*(P; \mathbf{F}_p)^{W_G(P)}.$$

The following result of Dietz and Glauberman [MP] is fundamental to our classification.

**Theorem 2.1.** *If  $P$  is a metacyclic group of odd order, then  $P$  is a Swan group.*

*Proof of Theorem 1.1 for  $P$  split.* Let  $\Phi(P) = \langle x^p, y^p \rangle$  be the Frattini subgroup. Then  $P/\Phi(P) \cong \mathbf{Z}/p \times \mathbf{Z}/p = \langle \bar{x}, \bar{y} \rangle$ . Thus quotienting by  $\Phi(P)$  induces a homomorphism  $\text{Out}(P) \xrightarrow{\pi} \text{Aut}(P/\Phi(P)) = GL_2(\mathbf{F}_p)$ . By [D, Prop. 3.2] if  $P$  is split, then  $\text{Out}(P) \cong O_p \text{Out}(P) \rtimes \mathbf{Z}/(p-1)$ ; moreover,

$$(1) \quad \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \in \text{Im}(\pi).$$

Now consider the extension

$$1 \rightarrow N \rightarrow P \rightarrow K \rightarrow 1$$

where  $N \cong \mathbf{Z}/p^m = \langle x \rangle$  and  $K \cong \mathbf{Z}/p^n = \langle \bar{y} \rangle$ . The Lyndon-Hochschild-Serre spectral sequence has  $E_2 = \mathbf{F}_p[u, v] \otimes E[a, b]$  where  $H^*(N) = \mathbf{F}_p[u] \otimes E[a]$ ,  $H^*(K) = \mathbf{F}_p[v] \otimes E[b]$ ,  $|a| = |b| = 1$ ,  $|u| = |v| = 2$ . Explicitly,  $a, b$  are given as canonical homomorphisms dual to  $x, y$  respectively, and  $u = \beta_n(a)$ ,  $v = \beta_m(b)$  are their respective Bocksteins.

If  $l \neq m - n$ , then this spectral sequence collapses at  $E_2$  [Dh, Thm.1]. Since  $P$  is a Swan group, we need only compute invariants. Let  $\zeta$  be a generator of  $\mathbf{Z}/(p-1) \leq \text{Out}(P)$  so that  $\gamma = \zeta^{p-1/d}$  generates  $W_G(P) = \mathbf{Z}/d$ . By (1)  $\gamma^*(a) = c \cdot a$  where  $c^d = 1$  is a primitive  $d$ -th root of unity;  $\gamma^*(u) = c \cdot u$  by application of the Bockstein. Similarly  $\gamma^*$  is trivial on  $v, b$ . Computing we find  $\gamma^*(u^k v^l a^\epsilon b^\delta) = u^k v^l a^\epsilon b^\delta$  iff  $k + \epsilon \equiv 0 \pmod d$ . Theorem 1.1 (1) follows with  $u_d = u^d$ ,  $a_d = u^{d-1}a$ .

If  $l = m - n$  the spectral sequence collapses at  $E_3$  [Dh, Thm.2] and we have  $E_3 = \mathbf{F}_p[z, v] \otimes E[b, \xi_{2i-1}, i = 1, \dots, p]/R$  where  $z = u^p$ ,  $\xi_{2i-1} = au^{i-1}$ . Relations are given by

$$\begin{aligned} \xi_{2i-1}\xi_{2j-1} &= 0, \quad 1 \leq i, j \leq p, \\ \xi_{2i-1}v &= 0, \quad 1 \leq i \leq p-1. \end{aligned}$$

In this case,  $\gamma^*(z) = c \cdot z$ ,  $\gamma^*(\xi_{2i-1}) = c^i \cdot \xi_{2i-1}$ . For  $d > 1$  we have invariants  $z^d$  and  $\alpha_{2i-1} = \xi_{2i-1}z^{d(i)}$ , where  $0 \leq d(i) < d$  is the residue of  $-i \pmod d$ . The result follows. For  $d = 1$ ,  $H^*(G) = H^*(P)$  and the result holds setting  $\alpha_{2i-1} = \xi_{2i-1}$  since  $d(i) = 0$  in this case.  $\square$

*Proof of Theorem 1.3 for  $P = M_3(p)$ .* In this case we have [D, Thm. 1.1 (3)]

$$BP \simeq \bigvee_{i=0}^{p-2} X_i \vee \bigvee_{k=0}^{p-2} L(2, k) \vee \bigvee_{k=0}^{p-2} L(1, 1, k).$$

Since  $P$  is a Swan group, we may assume  $G = N_G(P)$  and  $C_G(P) < P$ , i.e.,  $P \triangleleft G$  and  $G = P \rtimes C$ , where  $C \leq \mathbf{Z}/(p-1)$ . From (1) in the proof of Theorem 1.1 it is clear that the subgroup  $\langle x \rangle \rtimes C$  is normal in  $G = P \rtimes C$ . Therefore,  $\mathbf{Z}/p\langle y \rangle$  is a retract of  $G$ ; hence,  $B\mathbf{Z}/p$  is a summand of  $BG$  for every  $G$  with  $M_3(p)$  as a Sylow  $p$ -subgroup. Thus

$$B\mathbf{Z}/p = \bigvee_{k=0}^{p-2} L(1, 1, k)$$

is a summand of  $e_0BP = B(P \rtimes \mathbf{Z}/(p-1))$ .

We are reduced to showing  $L(2, k)$  is a summand of  $e_0BP$ . Let  $Q = \langle x^p, y \rangle \cong \mathbf{Z}/p \times \mathbf{Z}/p$ . Since  $L(2, k)$  corresponds to the simple  $\mathbf{F}_p \text{Out}(Q)$ -module  $M_k = \text{St} \otimes (\det)^k$ , we can use the criterion developed in [MP1]. That is, since  $Q$  is not a retract of  $P$  and  $C_P(Q) = Q$ , we must show

$$\overline{N_G(Q)/Q} \cdot M_k \neq 0$$

where  $\overline{H} = \sum h$  summed over  $h \in H \leq \mathbf{F}_p(H)$ . Since  $C_G(Q)/Q$  is a  $p'$ -group, this is equivalent to

$$\overline{N_G(Q)/C_G(Q)} \cdot M_k \neq 0$$

where  $N_G(Q)/C_G(Q) \leq GL_2(\mathbf{F}_p)$ . An explicit description of the Steinberg module  $St$  may be given as follows:  $St = \langle u^{p-1}, u^{p-2}v, \dots, uv^{p-2}, v^{p-1} \rangle$  is the  $\mathbf{F}_p$ -module of polynomials in indeterminates  $u, v$  of homogeneous degree  $p-1$  with  $GL_2(\mathbf{F}_p)$  acting on  $\langle u, v \rangle$  in the standard way [G]. Furthermore

$$\overline{N_G(Q)/C_G(Q)} = \overline{N_G(Q)/PC_G(Q)} \cdot \overline{P/Q}.$$

According to [D, Prop. 4.6 and Proof]

$$\overline{P/Q} \cdot M_k = \langle v^{p-1} \rangle.$$

Since  $N_G(Q)/C_G(Q)$  is a  $p'$ -group normalizing  $P/Q$ , we may assume it is isomorphic to a subgroup of the Borel subgroup of upper triangular matrices, i.e., the matrices of the form

$$w = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Thus  $w(v^{p-1}) = b^{p-1}v^{p-1} = v^{p-1}$ , and so  $\overline{N_G(Q)/PC_G(Q)} \cdot v^{p-1} \neq 0$ .  $\square$

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