COHOMOLOGY OF GROUPS WITH
METACYCLIC SYLOW $p$-SUBGROUPS

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Abstract. We determine the cohomology algebras $H^*(G; \mathbb{F}_p)$ for all groups $G$ with a metacyclic Sylow $p$-subgroup. The complete $p$-local stable decomposition of the classifying space $BG$ is also determined.

1. Introduction and statement of results

Let $P$ be a non-abelian metacyclic $p$-group of odd order and $G$ a finite group with $P$ as a Sylow $p$-subgroup. In this note we classify all possible mod-$p$ cohomology algebras $H^*(G)$ and determine complete $p$-local stable splittings for the classifying spaces $BG$. Much of the topological part of this work was done by the first author in [D]; recent results on Swan groups [MP] have enabled us to show that in all cases $H^*(G)$ is given by a ring of invariants. Similar but less complete information for metacyclic 2-groups was obtained in [D1, MP2, M].

A metacyclic $p$-group is a $p$-group $P$ which is an extension of a cyclic group by a cyclic group. Following [D] we say that $P$ is split if $P$ can be expressed by some split extension. We recall that up to isomorphism any non-abelian metacyclic $p$-group can be expressed as

$$P = P(p^m, p^n, p^l+1, p^q) = \langle x, y \mid x^{p^m} = 1, y^{p^n} = x^{p^q}, yxy^{-1} = x^{p^l+1} \rangle$$

for positive integers $m, n, l, q$ satisfying $l, q \leq m$, $(p^l+1)p^q \equiv 1 \mod p^m$, $(p^l+1)p^q \equiv p^q \mod p^m$, $n + l \geq m$ and $q + l \geq m$. In these terms $P$ splits unless $m \neq q$ and $l < q < n$ [D, Thm. 3.1].

Let $W_G(P) = N_G(P)/P \cdot C_G(P)$; then $W_G(P) \leq Out(P)$. If $P$ is split, then $Out(P) \cong O_p Out(P) \times \mathbb{Z}/(p-1)$ where $O_p Out(P)$ is a Sylow $p$-subgroup [D, Prop. 3.2]. Therefore $W_G(P) = Z/d$ where $d$ is a divisor of $p-1$. If $P$ is non-split, $Out(P)$ is a $p$-group and so $W_G(P) = 1$. We denote by $F_p[\cdot]$ and $E[\cdot]$ the polynomial and exterior algebras over $F_p$.

Theorem 1.1. As an algebra, $H^*(G)$ has one of the following forms:

1. If $P$ is split and $l \neq m - n$, then

$$H^*(G) \cong H^*(P)^{W_G(P)} = F_p[a_d, v] \otimes E[a_d, b]$$
where \(|u_d| = 2d, |v| = 2, |a_d| = 2d - 1, |b| = 1.\)

(2) If \(P\) is split and \(l = m - n,\) then

\[
H^*(G) \cong H^*(P)^{W_G(P)} = \mathbb{F}_p[\alpha_2, \alpha_3, \ldots, \alpha_{2i-1}, i = 1, \ldots, p]/R
\]

where the relations \(R\) are given by

\[
\alpha_{2i-1}\alpha_{2j-1} = 0, \quad 1 \leq i, j \leq p,
\]

\[
\alpha_{2i-1}v = 0, \quad 1 \leq i \leq p - 1,
\]

and \(|b| = 1, |v| = 2, |z| = 2p, |\alpha_{2i-1}| = 2i - 1 + 2pd(i),\) where \(0 \leq d(i) < d\) is the residue of \(i - l\) mod \(d.\)

(3) If \(P\) is non-split, then \(H^*(G) = H^*(P)\) is isomorphic to the algebra of (1) with \(d = 1\) if \(m = l + q\) and to that of (2) with \(d = 1\) if \(m < l + q.\)

Generators for these cohomology groups are specified explicitly in the proof.

Remark 1.2. Groups exemplifying the cases above are easily given by \(G = P \rtimes \mathbb{Z}/d.\)

R. Lyons has suggested other, more natural examples, which occur as automorphism groups of Chevalley groups. For example, let \(\mathbb{F}_q\) be a finite field of characteristic different from \(p\) such that the Sylow \(p\)-subgroup of \(PSL_2(\mathbb{F}_q)\) has order \(p,\) i.e., \(q^2 - 1\) is divisible by \(p\) but not by \(p^2.\) Then the Sylow \(p\)-subgroup of \(H = PSL_2(\mathbb{F}_q^*)\) is cyclic of order \(p^2.\) Let \(\phi\) be the Frobenius automorphism of \(\mathbb{F}_q^*\) of order \(p.\) Then it is easy to see that \(\phi\) fixes a cyclic subgroup \(C \leq H\) of order \(p^2 + 1\) which contains one such Sylow \(p\)-subgroup. Thus \(G = Aut(H) = PSL_2(\mathbb{F}_q) \rtimes \mathbb{Z}/p(\phi)\) has \(P = M_3(p)\) as a Sylow \(p\)-subgroup. Furthermore \(N_H(C)\) is a dihedral group \([Hu, II, 8.4 Satz]\) containing the permutation matrix of order two. Since this matrix is fixed by \(\phi, W_G(P) = \mathbb{Z}/2\) and \(H^*(G)\) is of type (2) in Theorem 1.1 with \(d = 2.\)

The group cohomology of a group \(G\) is the cohomology of the classifying space \(BG\) of \(G.\) The space \(BG\) is stably homotopy equivalent to a wedge product of indecomposable spectra,

\[BG \simeq X_1 \vee X_2 \vee \cdots \vee X_n.\]

A complete stable decomposition of \(BG\) is a splitting into indecomposable spectra. The decomposition is unique up to stable homotopy type and ordering. If \(G\) is a \(p\)-group, then all of these spectra are \(p\)-local. Otherwise, if \(G\) is a Sylow \(p\)-subgroup of \(G,\) then a simple transfer argument shows the \(p\)-localization of \(BG\) is a stable summand of \(BP,\)

\[BP \simeq BG_p \vee Y,\]

where \(BG_p\) is the \(p\)-localization of \(BG.\) Hence \(BG_p\) consists of some, but possibly not all, of the summands of \(BP.\) Note \(H^*(BG_p; \mathbb{F}_p) = H^*(BG; \mathbb{F}_p).\)

Each indecomposable spectrum \(X\) of \(BP\) corresponds up to conjugacy to a primitive idempotent \(e\) in the ring of stable self-maps \((BP, BP).\) The spectrum \(X\) is the infinite mapping telescope or homotopy colimit of \(e,\)

\[X \simeq eBP = Tel(BP \xrightarrow{e} BP) = hocolim(BP \xrightarrow{e} BP \xrightarrow{e} \cdots).\]

For more information see either \([BF]\) or \([MP1].\)
For the remainder of the paper all spectra are localized at the prime $p$. If $P$ is a Swan group, then $BG \simeq BN_G(P) \approx B(P \times W_G(P))$. Thus determining the stable homotopy type of $BG$ involves determining which summands have their cohomology left invariant by the action of the Weyl group of $G$.

$Z_pOut(P) \subseteq \{BP, BP\}$ is a subring, in fact a retract. Therefore, certain indecomposable summands of $BP$ correspond to simple modules of the outer automorphism group $Out(P)$. A summand corresponding to a simple $Out(P)$-module is said to originate in $BP$. A summand originating in $BP$ does not occur as the summand of the classifying space of any proper subgroup of $P$.

In this paragraph we introduce some notation for Theorem 1.3 below. $L(2,k)$ originates in $B(Z/p \times Z/p)$ and corresponds to $St \otimes (det)^k$ where $St$ is the Steinberg module for $F_pGL_2(F_p)$ and $det$ is the determinant module. It is well known that the group ring $F_p[Z/(p-1)]$ has a complete set of orthogonal primitive idempotents $e_0, \ldots, e_{p-2}$ [D]. Lifting these idempotents to $Z_p[Z/(p-1)]$ determines a complete stable splitting of $BZ/p^n \simeq p-2 \bigvee_{l=0}^{m-n} L(1, l, i)$, where $L(1, n, i)$ originates in $BZ/p^n$. For more information on these summands see [HK] and [D].

If $P$ is a split metacyclic $p$-group, then since $W_G(P)$ is a $p'$-group, we have $W_G(P) \leq Z/(p-1)$. Thus the primitive idempotents $e_0, \ldots, e_{p-2}$ above determine a stable splitting of $BG$. If $P$ is non-split, then $BP$ is stably indecomposable [D, Thm. 1.3].

Among the split metacyclic groups there is one which plays a special role, the extra-special modular group $M_3(p) = P(p^2, p, p + 1)$. It is characterized by its order and exponent which are $p^3$ and $p^2$ respectively.

**Theorem 1.3.** (1) If $P$ is split and $P \neq M_3(p)$, then

$$e_0BP = X_0 \lor B(Z/p^n), \quad e_iBP = X_i, \quad l = m - n, \quad 1 \leq i \leq p - 2.$$  

$$e_0BP = X_0 \lor B(Z/p^n) \lor L(1, n, 0), \quad e_iBP = X_i \lor L(1, n, i), \quad l \neq m - n, \quad 1 \leq i \leq p - 2.$$  

(2) If $P = M_3(p)$, then

$$e_0BP = X_0 \lor p-2 \bigvee_{i=0}^{p-2} L(2, i) \lor L(1, 1, i), \quad e_iBP = X_i, \quad 1 \leq i \leq p - 2,$$

where $X_i$ originates in $BP$.

(3) In both cases this yields a complete stable decomposition of $BP$ and

$$BG \simeq \bigvee_{i=0}^{(p-1/d)-1} e_{id}BP.$$
Corollary 1.4. Localized at $p$, the complete stable decomposition of $BG$ is given by one of the following:

1. If $P$ is non-split, then $BG \simeq BP$.

2. If $P$ is split and $P \neq M_3(p)$, then

$$BG \simeq \bigvee_{i=0}^{(p-1/d)-1} X_{id} \vee \bigvee_{j=0}^{p-2} L(1, n, j), \quad l = m - n.$$  

$$BG \simeq \bigvee_{i=0}^{(p-1/d)-1} X_{id} \vee L(1, n, id) \vee \bigvee_{j=0}^{p-2} L(1, n, j), \quad l \neq m - n.$$  

3. If $P = M_3(p)$, then

$$BG \simeq \bigvee_{i=0}^{(p-1/d)-1} X_{id} \vee L(2, i) \vee L(1, 1, i).$$  

If $P$ is split and $P \neq M_3(p)$, then Theorem 1.3 and Corollary 1.4 were proved by the first author [D, Thm. 1.3].

Throughout we assume $P$ is a non-abelian metacyclic $p$-group and $p$ is an odd prime. All cohomology is taken with simple coefficients in $\mathbb{F}_p$ and all spaces are considered stably, localized at $p$.

2. Proofs

The classifying space $BP$ is indecomposable if and only if $P$ is non-split [D, Thm. 1.1]. Thus, since $BG$ is a summand in $BP$, we have $BG \simeq BP$ and $H^*(G) \cong H^*(P)$. In this case the cohomology algebras are given by [Hb, Thm. B if $m < l + q$ and Thm. E if $m = l + q$]. This completes the proof of both theorems for the non-split case.

Before turning to the proof of Theorem 1.1 we recall the notion of a Swan group [MP]. A $p$-group $P$ is called a Swan group if the cohomology of any group $G$ with $P$ as a Sylow $p$-subgroup is given by its invariants, i.e.,

$$res : H^*(G; \mathbb{F}_p) \cong H^*(P; \mathbb{F}_p)^{W_G(P)}.$$  

The following result of Dietz and Glauberman [MP] is fundamental to our classification.

Theorem 2.1. If $P$ is a metacyclic group of odd order, then $P$ is a Swan group.

Proof of Theorem 1.1 for $P$ split. Let $\Phi(P) = \langle x^p, y^p \rangle$ be the Frattini subgroup. Then $P/\Phi(P) \cong \mathbb{Z}/p \times \mathbb{Z}/p = \langle \pi, \gamma \rangle$. Thus quotienting by $\Phi(P)$ induces a homomorphism $Out(P) \twoheadrightarrow \text{Aut}(P/\Phi(P)) = GL_2(\mathbb{F}_p)$. By [D, Prop. 3.2] if $P$ is split, then $Out(P) \cong O_pOut(P) \rtimes \mathbb{Z}/(p - 1)$; moreover,

$$\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \in \text{Im}(\pi).$$  

Now consider the extension

$$1 \rightarrow N \rightarrow P \rightarrow K \rightarrow 1$$

where $N \cong \mathbb{Z}/p^n = \langle x \rangle$ and $K \cong \mathbb{Z}/p^n = \langle \bar{y} \rangle$. The Lyndon-Hochschild-Serre spectral sequence has $E_2 = \mathbb{F}_p[u,v] \otimes E[a,b]$ where $H^*(N) = \mathbb{F}_p[u] \otimes E[a], H^*(K) = \mathbb{F}_p[v] \otimes E[b], |a| = |b| = 1, |u| = |v| = 2$. Explicitly, $a, b$ are given as canonical homomorphisms dual to $x, y$ respectively, and $u = \beta_n(a), v = \beta_n(b)$ are their respective Bocksteins.

If $l \neq m - n$, then this spectral sequence collapses at $E_2 [\text{Dh, Thm.1}].$ Since $P$ is a Swan group, we need only compute invariants. Let $\zeta$ be a generator of $\mathbb{Z}/(p - 1) \leq \text{Out}(P)$ so that $\gamma = \zeta^{p-1/d}$ generates $W_G(P) = \mathbb{Z}/d$. By (1) $\gamma^*(a) = c \cdot a$ where $c^d = 1$ is a primitive $d$-th root of unity; $\gamma^*(u) = c \cdot u$ by application of the Bockstein. Similarly $\gamma^*$ is trivial on $v, b$. Computing we find $\gamma^*(u^k v^l a^b) = u^k v^l a^b$ iff $k + \epsilon \equiv 0 \mod d$. Theorem 1.1 (1) follows with $u_d = u^d, a_d = u^{d-1}a$.

If $l = m - n$ the spectral sequence collapses at $E_3 [\text{Dh, Thm.2}].$ We have $E_3 = \mathbb{F}_p[z, v] \otimes E[b, \xi_{2i-1}, i = 1, \ldots, p]/R$ where $z = u^p, \xi_{2i-1} = au^{i-1}$. Relations are given by

$$\xi_{2i-1} \xi_{2j-1} = 0, \quad 1 \leq i, j \leq p,$$

$$\xi_{2i-1} v = 0, \quad 1 \leq i \leq p - 1.$$

In this case, $\gamma^*(z) = c \cdot z, \gamma^*(\xi_{2i-1}) = c^i \cdot \xi_{2i-1}$. For $d > 1$ we have invariants $z^d$ and $\alpha_{2i-1} = \xi_{2i-1} z^{d(i)}$, where $0 \leq d(i) < d$ is the residue of $-i$ mod $d$. The result follows. For $d = 1, H^*(G) = H^*(P)$ and the result holds setting $\alpha_{2i-1} = \xi_{2i-1}$ since $d(i) = 0$ in this case. \hfill \square

**Proof of Theorem 1.3 for $P = M_3(p)$**. In this case we have [\text{D, Thm. 1.1 (3)}]

$$BP \simeq \bigvee_{i=0}^{p-2} X_i \vee \bigvee_{k=0}^{p-2} L(2, k) \vee \bigvee_{k=0}^{p-2} L(1, 1, k).$$

Since $P$ is a Swan group, we may assume $G = N_G(P)$ and $C_G(P) < P$, i.e., $P \leq G$ and $G = P \rtimes C$, where $C \leq \mathbb{Z}/(p - 1)$. From (1) in the proof of Theorem 1.1 it is clear that the subgroup $\langle x \rangle \rtimes C$ is normal in $G = P \rtimes C$. Therefore, $\mathbb{Z}/p(y)$ is a retract of $G$; hence, $B\mathbb{Z}/p$ is a summand of $BG$ for every $G$ with $M_3(p)$ as a Sylow $p$-subgroup. Thus

$$B\mathbb{Z}/p = \bigvee_{k=0}^{p-2} L(1, 1, k)$$

is a summand of $e_0BP = B(P \rtimes \mathbb{Z}/(p - 1)).$

We are reduced to showing $L(2, k)$ is a summand of $e_0BP$. Let $Q = \langle x^p, y \rangle \cong \mathbb{Z}/p \times \mathbb{Z}/p$. Since $L(2, k)$ corresponds to the simple $\mathbb{F}_p\text{Out}(Q)$-module $M_k = St \otimes (det)^k$, we can use the criterion developed in [MP1]. That is, since $Q$ is not a retract of $P$ and $C_P(Q) = Q$, we must show

$$\overline{N_G(Q)/Q} \cdot M_k \neq 0$$

where $\overline{H} = \sum h$ summed over $h \in H \leq \mathbb{F}_p(H)$. Since $C_G(Q)/Q$ is a $p'$-group, this is equivalent to

$$\overline{N_G(Q)/C_G(Q)} \cdot M_k \neq 0$$
where $N_G(Q)/C_G(Q) \leq GL_2(F_p)$. An explicit description of the Steinberg module $St$ may be given as follows: $St = \langle u^{p-1}, u^{p-2}v, \ldots, uv^{p-2}, v^{p-1} \rangle$ is the $F_p$-module of polynomials in indeterminates $u,v$ of homogeneous degree $p - 1$ with $GL_2(F_p)$ acting on $\langle u,v \rangle$ in the standard way [G]. Furthermore

$$\frac{N_G(Q)}{C_G(Q)} = \frac{N_G(Q)}{PC_G(Q)} \cdot \frac{P}{Q}.$$ 

According to [D, Prop. 4.6 and Proof]

$$\frac{P}{Q} \cdot M_k = \langle v^{p-1} \rangle.$$

Since $N_G(Q)/C_G(Q)$ is a $p'$-group normalizing $P/Q$, we may assume it is isomorphic to a subgroup of the Borel subgroup of upper triangular matrices, i.e., the matrices of the form

$$w = \left( \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right).$$

Thus $w(v^{p-1}) = b^{p-1}v^{p-1} = v^{p-1}$, and so $\frac{N_G(Q)}{PC_G(Q)} \cdot v^{p-1} \neq 0$. □

References


[MP1] ______, The complete stable splitting of the classifying space of a finite group, Topology 31 (1992), 143–156. MR 93d:55012