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# ON THE GLOBAL DIMENSION OF QUASI–HEREDITARY ALGEBRAS WITH TRIANGULAR DECOMPOSITION

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ABSTRACT. Let $A$ be a quasi–hereditary algebra with triangular decomposition ${}_CA_{C^{op}} \simeq C \otimes_S C^{op}$ such that all Verma modules are semisimple over $C^{op}$. Then we show: $gldim(A) = 2 \cdot gldim(C)$. Applying this formula to the more special class of twisted double incidence algebras of finite partially ordered sets, we get a proof of a conjecture of Deng and Xi. Another application is to the so-called dual extensions of algebras.

## 1. INTRODUCTION

**Quasi–hereditary algebras** were introduced by Cline, Parshall and Scott [12, 1] in order to study highest weight categories, which occur frequently in representation theory of Lie algebras and algebraic groups, by means of finite dimensional algebras. The main examples of quasi–hereditary algebras are the algebras associated with blocks of the category $\mathcal{O}$ of a semisimple complex Lie algebra and the generalized Schur algebras associated with the rational representation theory of semisimple algebraic groups in any characteristic. These quasi–hereditary algebras, and many others, have a **triangular decomposition** [7, 8, 9]. That is, such an algebra $A$ can be written as ${}_CA_{C^{op}} \simeq C \otimes_S C^{op}$ where $(C, \geq)$ is a directed subalgebra of $A$ (see the next section for precise definitions), and the isomorphism is given by multiplication in $A$. An equivalent statement [8] is: $A$ has a strong exact Borel subalgebra $C^{op}$ and a strong $\Delta$–subalgebra $C$. So, $A$ is constructed from $C$ by a kind of doubling process. The aim of this note is to show that under additional assumptions this doubling also occurs in global dimension:

**Theorem 1.1.** *Let $(C, \geq)$ be a directed algebra. Let $A = C \otimes_S C^{op}$ be an algebra with triangular decomposition such that $S \simeq C/rad(C) \simeq A/rad(A)$. Then $(A, \leq)$ is quasi–hereditary.*

*Assume in addition that all Verma modules of $A$ are semisimple over $C^{op}$. Then $gldim(A) = 2 \cdot gldim(C)$.*

The assertion on quasi–heredity is just a special case of the theorem in [8]. What we really have to prove is the formula on the global dimension.

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Unfortunately, for a quasi–hereditary algebra with a triangular decomposition which does not satisfy the additional assumption in the theorem, such a nice formula need not be true.

A natural way to construct quasi–hereditary algebras with triangular decomposition is to start with $C$ and then to try defining an algebra structure on the bimodule $C \otimes_S C^{op}$. Deng and Xi have studied two such constructions: the twisted double incidence algebras of partially ordered sets [3] (this generalizes a special case of a construction of Dyer [6]) and the dual extensions of directed algebras [13, 2]. For quasi–hereditary algebras obtained by the first construction they conjectured a formula as in the theorem. We prove this formula (thus the conjecture) for both classes of algebras as an application of the theorem.

After a first version of this paper had been written, Xi [14] generalized the formula for quasi–hereditary dual extensions of algebras to a larger class of (not necessarily quasi–hereditary) algebras thus obtaining a new construction of algebras with large global dimension (compare [15]).

## 2. Some definitions

Quasi–hereditary algebras have been defined by Cline, Parshall and Scott [12, 1]. A detailed report on quasi–hereditary algebras can be found in [11].

**Definition 2.1.** Let $A$ be a finite dimensional algebra over a field and $I$ the set of isomorphism classes of simple $A$–modules. Choose representatives $L(i)$ of the elements of $I$. Let $\leq$ be a partial order on $I$. Then $(A, \leq)$ is called **quasi–hereditary** if and only if the following assertions are true:

(a) For each $i \in I$, there exists a finite dimensional $A$–module $\Delta(i)$ with an epimorphism $\Delta(i) \to L(i)$ such that the composition factors $L(j)$ of the kernel satisfy $j < i$.

(b) For each $i \in I$, a projective cover $P(i)$ of $L(i)$ maps onto $\Delta(i)$ such that the kernel has a finite filtration with sections $\Delta(j)$ satisfying $j > i$.

The module $\Delta(i)$ is called the **Verma module** of index $i$. The objects of the full subcategory $\mathfrak{F}(\Delta)$ of $A - mod$ by definition are the $A$–modules having a finite filtration with Verma modules as sections. Injective $A$–modules are filtered by modules $\nabla(i)$ (which are the Verma modules of the quasi–hereditary algebra $(A^{op}, \leq)$).

An algebra $(A, \leq)$ is called **directed** if it is quasi–hereditary with simple Verma modules. Equivalently, the projective covers $P(i)$ of $L(i)$ and $P(j)$ of $L(j)$ satisfy $Hom_A(P(i), P(j)) = 0$ unless $i \geq j$ and $End_A(P(i)) = End_A(L(i))$ for all indices $i$ and $j$.

Strong exact Borel subalgebras and strong $\Delta$–subalgebras have been introduced in [7]. Their existence has been shown for the blocks of category $\mathcal{O}$, for generalized Schur algebras and for many other quasi–hereditary algebras (see [7], Scott's appendix to [7], and [9]). They are defined as follows:

Let $(A, \leq)$ be quasi–hereditary and $S \simeq A/rad(A)$ a maximal semisimple subalgebra of $A$ (which exists at least if the base field $k$ is algebraically closed). A subalgebra $B$ of $A$ which contains $S$ is called a **strong exact Borel subalgebra** of $(A, \leq)$ if $(B, \leq)$ is directed, and the induction functor $A \otimes_B -$ is exact and produces Verma modules from simple modules: $A \otimes_B L(i) \simeq \Delta(i)$. A **strong $\Delta$–subalgebra** $C$ of $(A, \leq)$ contains $S$ and has the property that for each primitive idempotent $e(i) \in S$ the epimorphism $A \cdot e(i) \to \Delta(i)$ restricts to an isomorphism

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$C \cdot e(i) \simeq \Delta(i)$; thus, Verma modules over $A$ are indecomposable projective over $C$. The algebra $C$ is a strong $\Delta$–subalgebra of $(A, \leq)$ if and only if $C^{op}$ is a strong exact Borel subalgebra of $(A^{op}, \leq)$. In particular, $C$ is directed with respect to the partial order $\geq$ which is opposite to $\leq$.

The (more general) main theorem in [8] implies that $(A, \leq)$ has a strong exact Borel subalgebra $B$ and a strong $\Delta$–subalgebra $C$ which intersect in $S$ if and only if the multiplication in $A$ induces an isomorphism $C \otimes_S B \simeq A$ of left $C$– and right $B$–modules. If in addition $C = B^{op}$, then such an isomorphism will be called a **triangular decomposition** of $(A, \leq)$.

The theorem contains the additional strong assumption that the Verma modules over $A$ are semisimple if restricted to $B$. In [10] several equivalent versions of this condition have been given; one equivalent form which will be used later on is the condition that all $C$–homomorphisms between $A$–Verma modules are already $A$–linear. For the blocks of category $\mathcal{O}$ this occurs precisely in the multiplicity free case (see [10]). Other examples with this property will be discussed in section 4.

## 3. Proof of the main result

We split up the proof into a sequence of claims; the last claim implies the theorem. Claims 1 to 5 are valid in more generality. To simplify notation we are however keeping all assumptions of the theorem throughout. Claim 6 is the crucial step where the assumption on the semisimplicity of the $A$–Verma modules over $B := C^{op}$ is needed.

*Claim* 1. Let $\mathfrak{P}$ be an $A$–projective resolution of an $A$–module $M$. Then as a complex of $C$–modules, $\mathfrak{P}$ is the direct sum of a minimal $C$–projective resolution of $M$ and some complexes which up to shift have the form $0 \to \Delta(i) \overset{scalar \neq 0}{\to} \Delta(i) \to 0$ (with the scalar being an element of the division ring $End_A(L(i))$).

*Proof.* The algebra $A$ is projective as a left $C$–module, and an $A$–Verma module is indecomposable projective over $C$. Hence the decomposition of $\mathfrak{P}$ over $C$ follows. Summands with trivial homology are of the stated form because of the directedness of $C$. □

*Claim* 2. For each $i \in I$: $pdim_B L(i) = pdim_A \Delta(i)$. Thus

$$gldim(B) = max\{pdim_A(M) : M \in \mathfrak{F}(\Delta)\}.$$

*Proof.* The algebra $A$ is projective as a right $B$–module, hence the induction functor $A \otimes_B -$ is exact. The adjunction formula implies $Ext^n_A(\Delta(i), -) \simeq Ext^n_B(L(i), -)$ for each $i$, hence the first part of the claim. The second assertion follows by induction on the number of Verma modules in a filtration of $M$. □

*Claim* 3. An $A$–module $M$ lies in $\mathfrak{F}(\nabla)$ if and only if for all $i \geq 1$ and for all simple $A$–modules $L$: $Ext^i_B(L, M) = 0$, that is, if $M$ is injective over $B$. An $A$–module $M$ lies in $\mathfrak{F}(\Delta)$ if and only if for all $i \geq 1$ and for all simple $A$–modules $L$: $Ext^i_C(M, L) = 0$, that is, $M$ is projective over $C$.

*Proof.* We prove the first statement; the second is dual, since $(A^{op}, \leq)$ has triangular decomposition $B \otimes_{S^{op}} C$. The isomorphism in the proof of Claim 2 shows that $Ext^i_B(L, M) = 0$ for all $L$ if and only if $Ext^i_A(\mathfrak{F}(\Delta), M)$ equals zero. Thus the statement follows from a well–known characterisation of $\mathfrak{F}(\nabla)$ (see Theorem 1 in [5]). □

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*Claim* 4. Let $M$ be an $A$–module, $\mathfrak{P}$ an $A$–projective resolution of $M$ and $P$ a projective $A$–module which occurs in $\mathfrak{P}$ at step $i \geq pdim_C(M)$. Let $N$ be the submodule of $P$ which is mapped to zero (by the map in $\mathfrak{P}$). Then $N \in \mathfrak{F}(\Delta)$.

*Proof.* The projective module $P$ has a filtration by Verma modules, thus as a $C$–module it is projective. A decomposition of the $C$–module $P$ into Verma modules can be choosen in such a way that each direct summand is mapped (in $\mathfrak{P}$) either to zero or isomorphically. Therefore, $N$ is filtered by those Verma modules in this filtration which are mapped to zero. □

*Claim* 5. $gldim(A) \leq gldim(B) + gldim(C) = 2 \cdot gldim(C)$.

*Proof.* Consider the minimal $A$–projective resolution $\mathfrak{P}$ of an $A$–module $M$. Let $i$ denote $gldim(C)$, and let $N$ be the kernel at step $i$ in $\mathfrak{P}$, so $N = \Omega_i(M)$ (if we start counting with $N = \Omega_0(N)$). By Claim 4, $N \in \mathfrak{F}(\Delta)$. Thus by Claim 2, the projective dimension of $N$ over $A$ is bounded by $gldim(B)$. □

*Claim* 6. Let $P(i)$ be an $A$–projective cover of $L(i)$ and $\Delta(l)$ a Verma module over $A$. Then each homomorphism $\varphi : P(i) \to \Delta(l)$ factors over the epimorphism $\kappa : P(i) \to \Delta(i)$.

*Proof.* Write $P$ as $A \cdot e$ for an idempotent $e \in A$. Then $x := \varphi(e)$ generates $\varphi(P(i))$ as an $A$–module. Since $\Delta(i)$ is the $C$–projective cover of $L(i)$, there is a $C$–homomorphism $\psi : \Delta(i) \to \Delta(l)$ which has $x$ in its image. By the assumption in the theorem, the map $\psi$ even is an $A$–homomorphism. Hence its image contains $\varphi(P(i))$. For $i < j$, there is no composition factor $L(j)$ in $\Delta(i)$, hence $[\varphi(P(i)) : L(j)] = 0$. This implies the desired factorization. □

*Claim* 7. Let $L$ and $L'$ be simple $A$–modules and $n$ an integer. Then $Ext^n_C(L, L')$ is a direct summand of $Ext^n_A(L, L')$ via the embedding of $C$ into $A$. In other words, if the indecomposable projective $C$–module $C \cdot e$ occurs with multiplicity $m$ at step $n$ in a minimal $C$–projective resolution of $L$, then the indecomposable projective $A$–module $A \cdot e$ also occurs with multiplicity at least $m$ at the $n$th step of the minimal $A$–projective resolution of $L$.

*Proof.* We proceed by induction on $n$. If $n$ is zero, the assertion is clear. So let us assume $n > 0$. We are given a minimal $A$–projective resolution $\mathfrak{P}$ of the simple module $L$. By Claim 1, the minimal $C$–projective resolution of $L$ is a direct summand of $\mathfrak{P}$. So, if $C \cdot e = C(i)$ occurs in the $C$–projective resolution at step $n$, then $\Delta(i)$ must occur (with the same or even larger multiplicity) in the $A$–resolution at step $n$. Pick such a copy of $\Delta(i)$, which of course is mapped non–trivially, that is, neither goes to zero nor is mapped isomorphically. Assume $\Delta(i)$ occurs in the filtration of the indecomposable projective $A$–module $P(j)$ (of course, at step $n$ of $\mathfrak{P}$). Then $j \leq i$ and we have to prove $j = i$. So we assume now $j < i$. Since $\Delta(i)$ occurs in the minimal $C$–projective resolution of $L$, there is an indecomposable projective module $P(l)$ in $\mathfrak{P}$ at step $n - 1$ such that the restriction to $\Delta(i)$ of the map $P(j) \to P(l)$ in $\mathfrak{P}$ is not zero. Because of the induction step, even the composition of this map with the projection $P(l) \to \Delta(l)$ is not zero. Altogether we get a contradiction to Claim 6. □

*Claim* 8. Let $M \in \mathfrak{F}(\Delta)$ have a filtration in which $\Delta(i)$ occurs with multiplicity one and all other Verma modules in the filtration have smaller projective dimension than $\Delta(i)$. Then $pdim_A(M) = pdim_A(\Delta(i))$.

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*Proof.* Use induction on the number of Verma modules in a filtration of $M$, and the long exact homology sequence. □

*Claim* 9. Let $L(i)$ and $L(j)$ be simple modules such that $Ext_C^{gldim(C)}(L(i), L(j)) \neq 0$, and with $j$ as large as possible. Then the projective dimension of $L(i)$ over $A$ equals $2 \cdot gldim(C)$.

*Proof.* Let $\mathfrak{P}$ be a minimal $A$–projective resolution of $L(i)$. By Claim 7, the projective module $P(j)$ occurs at step $n = gldim(C)$ in $\mathfrak{P}$. Let $K$ be the submodule of $P(j)$ which is mapped to zero (by the map in $\mathfrak{P}$). Claim 4 implies $K \in \mathfrak{F}(\Delta)$. Moreover, if we restrict $\mathfrak{P}$ to $C$ again, the Verma module $\Delta(j)$ is the end term of the minimal projective resolution of $L$, hence it is mapped injectively (as a term in the complex $\mathfrak{P}$). Let $K'$ be the kernel of the epimorphism $P(j) \to \Delta(j)$. By Claim 7, the Verma factors in the filtration of $K'$ are not part of the minimal projective resolution of $L$. Thus they are mapped either isomorphically or trivially, the latter case occurring precisely for the Verma factors of $K$. Hence, $K'/K$ has (by Claim 4 a filtration by Verma modules. Adding $\Delta(j)$ to this filtration produces a filtration by Verma modules of $P/K$. This filtration contains one copy of $\Delta(j)$, and all other Verma modules in this filtration have strictly larger indices. The choice of $j$ implies that $Ext_B^{gldim(C)}(L(l), -) = Ext_C^{gldim(C)}(-, L(l))$ equals zero for all indices $l > j$. By Claim 8 and the choice of $j$ and $i$, the $A$–projective dimension of $P/K$ equals that of $\Delta(i)$, hence, by Claim 2, the $B$–projective dimension of $L(i)$. Because of $C = B^{op}$, the claim follows. □

## 4. Applications: the algebras of Deng and Xi

We first discuss twisted double incidence algebras of partially ordered sets in more detail, and after that we briefly mention dual extensions of directed algebras.

The following definitions are taken from [3]. We use a slightly different notation, however.

Let $X$ be a finite partially ordered set (=poset). By $\mathfrak{I}(X)$ we denote its **incidence algebra**. We write $\mathfrak{I}(X)$ by quiver and relations: $\mathfrak{I}(X) = kQ/I$. The vertices of the quiver $Q$ are the elements of the poset. There is an arrow $x \to y$ precisely if $x < y$, and there is no $z$ such that $x < z < y$ (in that case we call $x$ and $y$ **neighbours**). The ideal $I$ in $kQ$ is generated by all commutativity relations, that is, by elements $\alpha - \beta$ where $\alpha$ and $\beta$ are two paths in $Q$ starting at a common vertex $a$ and ending at a common vertex $b$.

By a **mesh** in the poset $X$ we denote a sequence $(x; y_1, \dots, y_n; z)$ such that the following conditions are satisfied: for each $i$ we have $x > y_i < z$, each $y_i$ is a neighbour of both $x$ and $z$, and the $y_i$ are all the elements in $X$ satisfying these conditions. If in addition we are given an element $w$ with $x < w > z$ and $w$ is a neighbour of both $x$ and $z$, then we call the mesh together with $w$ an **extended mesh**.

A **label** of an extended mesh is a choice of scalars $m(w, y_i)$ for all $i$. If we choose such a label for each extended mesh in $X$, we say that $X$ has the labelling $M$ (where $M$ implicitly means the collection of all these labels).

Reversing the ordering in $X$ defines the opposite poset $X^{op}$ and the opposite set of relations $I^{op}$.

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Now we form a new quiver $\tilde{Q}$ which is the fibre product of $Q$ and its opposite $Q^{op}$ along the vertices. That is, $\tilde{Q}$ has the same vertices as $Q$ and for each arrow $x \to y$ in $Q$ the new quiver $\tilde{Q}$ contains two arrows $x \to y$ and $y \to x$.

**Definition 4.1.** Let $X$ be a finite poset as above. Choose a labelling $M$ for the extended meshes in $X$. Then the algebra $\mathfrak{A}(X, M)$ is defined by quiver and relations as follows: $\mathfrak{A}(X, M) = k\tilde{Q}/\tilde{I}$ where $\tilde{Q}$ is as above, and the relation ideal $\tilde{I}$ is generated by the union of $I$ and $I^{op}$, and the set of all relations $x \to w \to z = \sum_i m(w, y_i)(x \to y_i \to z)$ for all extended meshes (with notation as above). The algebra $\mathfrak{A}(X, M)$ is called the ***M*–twisted double incidence algebra** of the poset $X$ with labelling $M$.

The algebra $\mathfrak{A}(X, M)$ in general need not be quasi–hereditary. But if it is, then Deng and Xi [3] conjectured the following result:

**Theorem 4.1.** *If $\mathfrak{A}(X, M)$ is quasi–hereditary, then*

$$gldim(\mathfrak{A}(X, M)) = 2 \cdot gldim(\mathfrak{I}(X, M))$$

*Proof.* Deng and Xi have shown in [3] that $\mathfrak{I}(X, M)$ is a strong exact Borel subalgebra of $\mathfrak{A}(X, M)$, and that $\mathfrak{I}(X, M)^{op}$ is a strong $\Delta$–subalgebra. Hence there is triangular decomposition. Moreover, a composition multiplicity $[\Delta(i) : L(j)]$ is either zero or one, and it is one if and only if $j \leq i$. Hence $Hom_A(\Delta(j), \Delta(i))$ equals $Hom_C(\Delta(j), \Delta(i))$ (since both are $End_S(L(j))$ or 0, depending on $j \leq i$ or not). Thus all Verma modules are semisimple over $B$. (The last assertion also might be verified directly by induction without using the equivalence from [10].) Thus the formula in the theorem follows from Theorem 1.1. □

Another class of quasi–hereditary algebras defined by Deng and Xi [13, 2] are the dual extensions: Let $(C, \geq)$ be directed with maximal semisimple subalgebra $S$. Then putting $rad(C^{op}) \cdot rad(C) = 0$ defines an algebra structure on $A = C \otimes_S C^{op}$ which is called the **dual extension** of $C$. The algebra $A$ can be defined by quiver and relations in a similar way as above. Put $\tilde{Q}$ as above, and define as new relations $x \to w \to z = 0$ for all pairs of neighbours $x < w$ and $w > z$.

For such an algebra $A$ the formula $gldim(A) = 2 \cdot gldim(C)$ again is a special case of Theorem 1.1. Here the triangular decomposition is already given by the definition of $A$ as a dual extension. And the $C^{op}$–semisimplicity of Verma modules comes from the fact that $C$ is a quotient algebra of $A$; hence $C$–homomorphisms between Verma modules automatically are $A$–homomorphisms.

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