

MULTINOMIAL EXPANSIONS AND THE PYTHAGOREAN THEOREM

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Dedicated to the memory of Professor Morris Marden

ABSTRACT. An addition formula for homogeneous polynomials is used to obtain a generalization of the Pythagorean theorem and a new view of the multinomial expansion.

Let $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$, and let P denote the set of all polynomials

$$p(x) = \sum_{j=1}^m a_j x_1^{\alpha_j^1} x_2^{\alpha_j^2} \dots x_n^{\alpha_j^n}, \quad a_j \in \mathbf{R}.$$

Letting $\partial/\partial x = (\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_n)$, we define on P the inner product

$$\begin{aligned} (p, q) &= p \left(\frac{\partial}{\partial x} \right) q(x) \Big|_{x=0} \\ (1) \quad &= \left(\sum_{j=1}^m a_j \frac{\partial^{\alpha_j^1}}{\partial x_1^{\alpha_j^1}} \frac{\partial^{\alpha_j^2}}{\partial x_2^{\alpha_j^2}} \dots \frac{\partial^{\alpha_j^n}}{\partial x_n^{\alpha_j^n}} \right) q(x) \Big|_{x=0}. \end{aligned}$$

Let $H_k \subset P$ denote the vector space of homogeneous polynomials of degree k . Note that the dimension of H_k is

$$\dim H_k = d_k = \frac{(n+k-1)!}{(n-1)! k!}.$$

We first obtain a simple analog of the Funk-Hecke theorem [1, p. 247] for homogeneous polynomials.

Theorem 1. *If $p(x)$ is a homogeneous polynomial of degree k , then*

$$((x_1 y_1 + x_2 y_2 + \dots + x_n y_n)^k, p(y)) = k! p(x).$$

Proof. Expanding the power,

$$\begin{aligned} (x_1 y_1 + \dots + x_n y_n)^k &= \sum_{\alpha} \frac{k!}{\alpha_1! \alpha_2! \dots \alpha_n!} (x_1 y_1)^{\alpha_1} (x_2 y_2)^{\alpha_2} \dots (x_n y_n)^{\alpha_n} \\ &= k! \sum_{\alpha} \frac{x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}}{\alpha_1! \alpha_2! \dots \alpha_n!} y_1^{\alpha_1} y_2^{\alpha_2} \dots y_n^{\alpha_n}. \end{aligned}$$

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Thus if $p(y) = \sum_{\alpha} a_{\alpha} y_1^{\alpha_1} y_2^{\alpha_2} \dots y_n^{\alpha_n}$, we have

$$\begin{aligned} & ((x_1 y_1 + \dots + x_n y_n)^k, p(y)) \\ &= \left(k! \sum_{\alpha} \frac{x_1^{\alpha_1} \dots x_n^{\alpha_n}}{\alpha_1! \dots \alpha_n!} \frac{\partial^{\alpha_1}}{\partial y_1^{\alpha_1}} \dots \frac{\partial^{\alpha_n}}{\partial y_n^{\alpha_n}} \sum_{\beta} a_{\beta} y_1^{\beta_1} \dots y_n^{\beta_n} \right) \\ &= k! \sum_{\alpha} a_{\alpha} x_1^{\alpha_1} \dots x_n^{\alpha_n} = k! p(x). \end{aligned}$$

Our next result is an addition formula for orthonormal homogeneous polynomials.

Theorem 2 (Addition Formula). *Let $\{p_j(x)\}_{j=1}^{d_k}$ be an orthonormal basis for H_k . Then*

$$(2) \quad (x_1 y_1 + x_2 y_2 + \dots + x_n y_n)^k = k! \sum_{j=1}^{d_k} p_j(x) p_j(y).$$

Proof. Let $x = (x_1, x_2, \dots, x_n)$ and $\alpha_j = (\alpha_j^1, \alpha_j^2, \dots, \alpha_j^n)$. For convenience we write $x^{\alpha_j} = x_1^{\alpha_j^1} x_2^{\alpha_j^2} \dots x_n^{\alpha_j^n}$. Then expanding the power, we have

$$(x_1 y_1 + \dots + x_n y_n)^h = \sum c_j x^{\alpha_j} y^{\alpha_j},$$

where x^{α_j} and y^{α_j} are homogeneous monomials of degree k . Since $\{p_j(x)\}_{j=1}^{d_k}$ is a basis for H_k , there exist constants a_{jm} such that

$$y^{\alpha_j} = \sum_{m=1}^{d_k} a_{jm} p_m(y).$$

Thus

$$\begin{aligned} (x_1 y_1 + \dots + x_n y_n)^k &= \sum_j c_j x^{\alpha_j} \left(\sum_m a_{jm} p_m(y) \right) \\ &= \sum_j g_j(x) p_j(y), \end{aligned}$$

where the last expression is a rearrangement of the previous sum. The $g_j(x)$ are linear combinations of the monomials x^{α_j} , and thus are homogeneous polynomials of degree k in x . Further, since the $p_j(x)$ are orthonormal,

$$((x_1 y_1 + \dots + x_n y_n)^k, p_l(y)) = \left(\sum_j g_j(x) p_j(y), p_l(y) \right) = g_l(x).$$

But by the identity given in Theorem 1,

$$((x_1 y_1 + \dots + x_n y_n)^k, p_l(y)) = k! p_l(x).$$

Thus,

$$g_j(x) = k! p_j(x), \quad j = 1, 2, \dots, d_k,$$

which completes the proof.

If $\alpha_j = (\alpha_j^1, \alpha_j^2, \dots, \alpha_j^n)$, for convenience we write $\alpha_j! = \alpha_j^1! \alpha_j^2! \dots \alpha_j^n!$. The result of Theorem 2 shows that the multinomial expansion

$$(x_1y_1 + x_2y_2 + \dots + x_ny_n)^k = \sum \frac{k!}{\alpha_j!} x^{\alpha_j} y^{\alpha_j} = k! \sum \frac{x^{\alpha_j}}{\sqrt{\alpha_j!}} \frac{y^{\alpha_j}}{\sqrt{\alpha_j!}}$$

is no more than an addition formula for the orthonormal basis $\{x^{\alpha_j}/\sqrt{\alpha_j!}\}_{j=1}^{d_k}$ of H_k . The significance of the result (2) lies with the fact that it holds for any orthonormal basis $\{p_j(x)\}_1^{d_k}$, not merely the monomials $\{x^{\alpha_j}/\sqrt{\alpha_j!}\}_1^{d_k}$.

The addition formula of Theorem 2 yields an identity which contains the Pythagorean theorem $\sin^2 \theta + \cos^2 \theta = 1$ as a special case:

Theorem 3 (Pythagorean Identity). *Suppose $\{p_j(x)\}_{j=1}^{d_k}$ is an orthonormal basis for H_k . Then on the unit sphere $|s| = \sqrt{s_1^2 + s_2^2 + \dots + s_n^2} = 1$,*

$$\sum_{j=1}^{d_k} [p_j(s)]^2 = \frac{1}{k!}.$$

Proof. Letting $x = y$ in the addition formula yields

$$(x_1^2 + x_2^2 + \dots + x_n^2)^k = k! \sum_{j=1}^{d_k} [p_j(x)]^2,$$

from which the result is immediate.

Most inner products are integrals. Although the inner product we have used here is defined by a derivative, it is far from artificial. Indeed, this inner product allows us to view Taylor series as Fourier series. That is, suppose we have the Taylor series

$$f(x) = \sum_{j=0}^{\infty} a_j x^{\alpha_j}.$$

Rewriting the series as

$$f(x) = \sum_{j=1}^{\infty} b_j \frac{x^{\alpha_j}}{\sqrt{\alpha_j!}},$$

and noting that the monomials $\{x^{\alpha_j}/\sqrt{\alpha_j!}\}_{j=1}^{\infty}$ are orthonormal with respect to the inner product (1), we have

$$(x^{\alpha_j}/\sqrt{\alpha_j!}, f(x)) = b_j.$$

These simple results appear to have considerable analytic significance. To see this, one need only consider the kernel

$$\sum_{k=0}^{\infty} \sum_{j=1}^{d_k} p_k^j(x) p_k^j(y) = \sum_{k=0}^{\infty} \frac{(x \cdot y)^k}{k!} = e^{x \cdot y},$$

and the reproducing formula

$$(f(y), e^{x \cdot y}) = f(x).$$

Here $\{p_k^j(x)\}_{j=1}^{d_k}$, $k = 0, 1, 2, \dots$, are arbitrary orthonormal bases for the homogeneous polynomials H_k of degree k .

REFERENCES

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