ENDOMORPHISM RINGS OF COMPLETELY PURE-INJECTIVE MODULES

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Abstract. Let $R$ be a ring, $E = E(R_R)$ its injective envelope, $S = \text{End}(E_R)$ and $J$ the Jacobson radical of $S$. It is shown that if every finitely generated submodule of $E$ embeds in a finitely presented module of projective dimension $\leq 1$, then every finitely generated right $S/J$-module $X$ is canonically isomorphic to $\text{Hom}_R(E, X \otimes_S E)$. This fact, together with a well-known theorem of Osofsky, allows us to prove that if, moreover, $E/JE$ is completely pure-injective (a property that holds, for example, when the right pure global dimension of $R$ is $\leq 1$ and hence when $R$ is a countable ring), then $S$ is semiperfect and $R_R$ is finite-dimensional. We obtain several applications and a characterization of right hereditary right noetherian rings.

Introduction

Let $R$ be a ring, $M_R$ a right $R$-module, and $S = \text{End}(M_R)$. Then there exists an adjoint pair:

$$\text{Hom}_R(M, -) : \text{Mod } R \rightleftarrows \text{Mod } : - \otimes_S M$$

which induces a functorial morphism $\alpha : 1_{\text{Mod } S} \to \text{Hom}_R(M, - \otimes_S M)$. If $X$ is a right $S$-module such that $\alpha_X$ is an isomorphism, we will say that $X_S$ is $M$-invariant. It is well known that when every right $S$-module $X$ is $M$-invariant, useful information can be passed from $M_R$ to $S$. This is what happens, for example, when $M_R$ is a finitely generated projective module, which makes it possible to characterize properties of the endomorphism ring $S$ in terms of $M_R$. This property also holds when $M_R$ is finitely presented and $S$ is a (von Neumann) regular ring and this, coupled with Osofsky’s theorem \cite{8, 9} that asserts that a ring whose cyclic right modules are all injective is semisimple, has been exploited in \cite{3} to obtain an easy proof of the result of Damiano that shows that a right PCI ring (i.e., a ring with each proper cyclic right module injective) is right noetherian.

This technique was also (implicitly) applied in \cite{1} to a right hereditary ring $R$ whose injective envelope $E(R_R)$ is projective, showing that $R$ is, in this case, a (two-sided) hereditary artinian QF-3 ring. An extension in \cite[Corollary 6]{3} shows that if $E(R_R)$ is just finitely presented (instead of projective), then $R$ is a right

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artinian ring with Morita duality. The key point of this proof is to show that $R$ is right finite-dimensional. But, as the endomorphism ring $S$ of $E = E(R_R)$ is regular, all the cyclic right $S$-modules are $E$-invariant. This makes it possible to transfer the injectivity property and then to use Osofsky’s theorem to show that $S$ is semisimple.

In this paper we consider the rather more general situation that arises when the injective envelope $E_R = E(R_R)$ has the property that every finitely generated submodule embeds in a finitely presented module whose projective dimension is $\leq 1$ (this includes the right hereditary rings with finitely presented injective envelope, but also the rings $R$ such that every finitely generated submodule of $E_R$ embeds in a free module). If $S = \text{End}(E_R)$ and $J$ is the radical of $S$, we prove in Theorem 1.6 that each finitely generated right $S/J$-module is $E$-invariant—a result that will be our main tool in the rest of the paper. This allows us to apply the transfer techniques sketched above to the ring $S/J$ and hence substantially broaden the scope of these methods. In this setting, we usually cannot expect that the endomorphism ring $S$ is semisimple. In general, it is not even regular. However, we show that when certain quotients of $E_R$ are pure-injective, then $S$ is semiperfect and hence $R_R$ is finite-dimensional. More specifically, we assume that $E/JE$ is a completely pure-injective $R$-module, i.e., a module such that each pure quotient of itself is pure-injective. We give several applications and we extend [3, Corollary 6] by proving that if $R$ is right hereditary and every finitely generated submodule of $E_R$ is finitely presented, then $R$ is right noetherian.

In the last part of the paper we consider rings $R$ whose right pure global dimension (cf. [6, 7]) is $\leq 1$. This includes all countable rings. If every finitely generated submodule of $E_R$ embeds in a finitely presented module of projective dimension $\leq 1$, then we show that $E/JE$ is pure-injective (Theorem 2.1), so that $E/JE$ is completely pure-injective in this case and hence $R$ is, again, finite-dimensional. As an application we show that, for these rings, the property that $R$ is right nonsingular and every finitely generated right $R$-module embeds in a free module is right-left symmetric.

We refer to [5] and [11] for all undefined notions used in the text.

1. $M$-invariant modules

Let $S M_R$ be a bimodule. We have a pair of adjoint functors $\text{Hom}_R(M, -) : \text{Mod} - R \leftrightarrows \text{Mod} - S : - \otimes_S M$ and the corresponding adjunction morphisms $\alpha_X$, for every $X \in \text{Mod} - S$. The right $S$-modules $X$ such that $\alpha_X$ is an isomorphism will, again, be called $M$-invariant. The following result is well known (cf. [12], [11]).

**Proposition 1.1.** Let $S M_R$ be a bimodule. Then the following assertions hold:

(i) If $L_R$ is pure-injective, then $\text{Hom}_R(M, L)$ is a pure-injective right $S$-module.

(ii) If $S M$ is flat and $L_R$ is $M$-injective, then $\text{Hom}_R(M, L)$ is injective.

Our interest in $M$-invariant modules is motivated by the fact that certain injectivity properties are easily transferred to these modules. From Proposition 1.1 we have:

**Proposition 1.2.** Let $S M_R$ be a bimodule and $X$ an $M$-invariant right $S$-module. Then the following assertions hold:

(i) If $X \otimes_S M$ is pure-injective, then $X$ is pure-injective.

(ii) If $S M$ is flat and $X \otimes_S M$ is $M$-injective, then $X$ is injective.
In order to exploit Proposition 1.2 we need to have $M$-invariant $S$-modules. Recall that if $E_R$ is (quasi-)injective (or pure-injective), then $S/J$ (where $S = \text{End}(E_R)$ and $J = J(S)$) is a regular ring and idempotents lift modulo $J$. We want to apply Osolinsky’s theorem to $S/J$ and for this we need to prove that the cyclic right $S/J$-modules are $E$-invariant. We start by giving a useful sufficient condition for $\alpha_X$ to be a monomorphism.

**Proposition 1.3.** Let $P_R$ be a finitely generated projective module, $E = E(P_R)$ and $S = \text{End}(E_R)$. Then $\alpha_X$ is a monomorphism for each finitely generated right $S/J$-module $X$.

**Proof.** Since $X$ is an $S/J$-module and $XJ = 0$, we have a free presentation of $X$ in $\text{Mod}-S$, say $S^t \xrightarrow{h} S^n \xrightarrow{p} X \to 0$, where $J^n = J(S^n) \subseteq \text{Ker } p = \text{Im } h$. Applying $- \otimes_S E$ we obtain an exact sequence in $\text{Mod}-R$

$$E^t \xrightarrow{h} E^n \xrightarrow{p^*} X \otimes_S E \to 0.$$

Let $Z := \text{Im } h = \text{Ker } p$, with canonical projection $v : E^t \to Z$ and canonical injection $u : Z \to E^n$. Then each $f \in \text{Hom}_R(E, E^n)$ such that $p^* \circ f = 0$ factors in the form $f = u \circ f'$, where $f' \in \text{Hom}_R(E, Z)$. Since $P$ is projective, we obtain a morphism $g : P \to E^t$ that makes the diagram

$$
\begin{array}{ccc}
P & \xrightarrow{j} & E \\
\downarrow g & & \downarrow f' \\
E^t & \xrightarrow{v} & Z
\end{array}
$$

commute, where $j$ is the canonical inclusion. Since $P$ is finitely generated, $g(P) \subseteq E^t$ for some finite subset $F$ of $I$. As $E$ is injective, there exists a homomorphism $t : E \to E^t$ such that $t \circ j = g$. Hence $h_* \circ t \circ j = h_* \circ g = f \circ j$, so that $(h_* \circ t - f) \circ j = 0$. Since $j$ is an essential monomorphism by hypothesis, $\text{Ker } (h_* \circ t - f)$ is essential in $E$. Consider the following commutative diagram of right $S$-modules:

$$
\begin{array}{ccc}
S^t & \xrightarrow{h} & S^n & \xrightarrow{p} & X & \to 0 \\
\downarrow \alpha_{S^t} & & \downarrow \alpha_{S^n} & & \downarrow \alpha_X \\
\text{Hom}_R(E, E^t) & \xrightarrow{h_*} & \text{Hom}_R(E, E^n) & \xrightarrow{p^*} & \text{Hom}_R(E, X \otimes_S E)
\end{array}
$$

Then $f \in \text{Hom}_R(E, E^n)$ and $f \in \text{Ker } p^*$, so there exists $t \in \text{Hom}_R(E, E^t)$ such that $h_*(t) - f$ has essential kernel and, hence, belongs to $J(S^n)$. Thus $h_*(t) - f \in \alpha_{S^n}(\text{Ker } p)$. On the other hand, since $\text{Im } t \subseteq E^t$ for $F$ finite, there exists $q \in S^t$ such that $t = \alpha_{S^t}(q)$ and so $h_*(t) = (\alpha_{S^n} \circ h)(q) \in \alpha_{S^n}(\text{Ker } p)$. Thus we have that $f \in \alpha_{S^n}(\text{Ker } p)$ and this implies that $\alpha_X$ is a monomorphism. \hfill $\Box$

Recall that $R$ is called a right Kasch ring whenever $E(R_R)$ is a cogenerator of $\text{Mod}-R$. From the preceding result we immediately obtain:

**Corollary 1.4.** Let $R$ be a right Kasch ring. Then $\text{End}(E(R_R))$ is also a right Kasch ring.

**Proof.** Let $E = E(R_R)$, $S = \text{End}(E_R)$ and $J = J(S)$. If $C$ is a simple right $S$-module, then $CJ = 0$ and so $C$ is an $S/J$-module. Thus $\alpha_C$ is a monomorphism by
Proposition 1.3 and, as $C \otimes_S E$ is cogenerated by $E$, we obtain a monomorphism $C \xrightarrow{\alpha_x} \text{Hom}_R(E, C \otimes_S E) \to \text{Hom}_R(E, E^1) \cong S^1$, for some set $I$. Hence $C$ embeds in $S_J$. \[\square\]

Now, in order to obtain $E$-invariant modules from Proposition 1.3, we need to give conditions for $\alpha_X$ to be an epimorphism. The following lemma will be crucial for this purpose.

**Lemma 1.5.** Let $P_R$ be a finitely generated projective right $R$-module, $E = E(P_R)$ its injective hull, and $S = \text{End}(E_R)$. Assume that each finitely generated submodule of $E$ embeds in a finitely presented module of projective dimension $\leq 1$. Then, for each finitely generated right $S/J$-module $X$, $\text{Hom}_R(E/P, X \otimes_S E) = 0$.

**Proof.** Let $f \in \text{Hom}_R(E/P, X \otimes_S E)$ and $\pi : E \to E/P$ the canonical projection. We want to prove that $g = f \circ \pi = 0$. Since $P$ is finitely generated, $E$ is the direct limit of all its finitely generated submodules that contain $P$. Thus it will be enough to show that if $P \subseteq Z \subseteq E$ and $Z$ is finitely generated, then $g(Z) = 0$. By hypothesis, there exists a finitely presented right $R$-module $F$ such that $\text{pd}(F) \leq 1$, and a monomorphism $\varphi : Z \to F$. Then, regarding $P$ as a submodule of $F$, we get the following commutative diagram:

$$
\begin{array}{ccc}
Z & \xrightarrow{\varphi} & E \\
\downarrow \pi' & \swarrow \gamma & \downarrow \pi \\
Z/P & \xrightarrow{\psi} & E/P \\
\downarrow \beta & & \downarrow \alpha \\
F/P & & F/P
\end{array}
$$

where $\beta$ is the monomorphism induced by $\varphi$, $\gamma$ is obtained by the injectivity of $E$, and $\delta$ is induced by $\gamma$. We have that $F/P$ is a finitely presented module. Consider the functorial exact sequence

$$0 = \text{Ext}^1_R(P, -) \to \text{Ext}^2_R(F/P, -) \to \text{Ext}^2_R(F, -) = 0.$$

Since $\text{pd}(F) \leq 1$, the last term is zero, and so $\text{pd}(F/P) \leq 1$. Next let $S^{(I)} \to S^n \xrightarrow{p} X \to 0$ be a free presentation of $X$ in $\text{Mod-}S$ and consider the induced exact sequence in $\text{Mod-}R$, $E^{(I)} \to E^n \xrightarrow{\varphi \otimes E} X \otimes_S E \to 0$. Set $Y = \text{Ker}(p \otimes_S E)$. From the short exact sequence $0 \to K \to E^{(I)} \to Y \to 0$ we obtain the natural exact sequence

$$\text{Ext}^1_R(F/P, E^{(I)}) \to \text{Ext}^1_R(F/P, Y) \to \text{Ext}^2_R(F/P, K).$$

Since $\text{pd}(F/P) \leq 1$, we have that $\text{Ext}^2_R(F/P, K) = 0$ and, as $F/P$ is finitely presented and $E$ is injective, $\text{Ext}^1_R(F/P, E^{(I)}) \cong \text{Ext}^1_R(F/P, E^{(I)}) = 0$. Thus $\text{Ext}^1_R(F/P, Y) = 0$ and so we have an exact sequence

$$\text{Hom}_R(F/P, E^n) \xrightarrow{(\varphi \otimes E)} \text{Hom}_R(F/P, X \otimes E) \to \text{Ext}^1_R(F/P, Y) = 0$$

which shows that $(\varphi \otimes E)_\alpha = \text{Hom}_R(F/P, p \otimes E)$ is an epimorphism. Hence, there exists a morphism $\epsilon : F/P \to E^n$ such that $f \circ \delta = (p \otimes E) \circ \epsilon$. But, as $E^n$ is injective and $v$ is a monomorphism, $\epsilon \circ \beta : Z/P \to E^n$ can be extended to a map $\mu : E/P \to E^n$ such that $\mu \circ v = \epsilon \circ \beta$. This gives $(p \otimes E) \circ \mu \circ v = (p \otimes E) \circ \epsilon \circ \beta = f \circ \delta \circ \beta = f \circ \beta.$
Let $S/J \leq S/J$. The projectivity of $\phi$ and, if $X$ shows that $\alpha$ is an element of $S$ whose kernel contains $P$. Therefore $p_i \circ \mu \circ \pi \in J(S)$. Now, let $x$ be an element of $E$ and set $e_i = (\delta_{ij})_{i=1,\ldots,n} \in S$. Since $XJ = 0$ and $p_i \circ \mu \circ \pi \in J$,

\[
((p \otimes E) \circ \mu \circ \pi \circ u)(x) = (p \otimes E)((\mu \circ \pi)(x)) = \sum_{i=1}^n p(e_i) \circ (p_i \circ \mu \circ \pi)(x) = \sum_{i=1}^n p(e_i) \cdot (p_i \circ \mu \circ \pi) \otimes x = 0.
\]

This completes the proof. \hfill \Box

**Theorem 1.6.** Let $P_R$ be a finitely generated projective module, $E = E(P_R)$ and $S = \text{End}(E_R)$. Assume that each finitely generated submodule of $E$ embeds in a finitely presented module of projective dimension $\leq 1$. Then each finitely generated right $S/J$-module is $E$-invariant.

**Proof.** Let $X$ be a finitely generated right $S/J$-module. By Proposition 1.3 $\alpha_X$ is a monomorphism. It remains to prove that $\alpha_X$ is an epimorphism. Consider a free presentation $S^{(1)} \to S^n \xrightarrow{p} X \to 0$ of $X$ in $\text{Mod}-S$. Tensoring with $S E$ yields an exact sequence in $\text{Mod}-R$, $E^{(1)} \to E^n \xrightarrow{p \otimes E} X \otimes_S E \to 0$. Now, if $\varphi \in \text{Hom}_R(E, X \otimes_S E)$ and $j : P \to E$ is the canonical inclusion, there is by the projectivity of $P$ a morphism $t : P \to E^n$ such that $\varphi \circ j = (p \otimes E) \circ t$. Then, as $E$ is injective, there exists $h : E \to E^n$ such that $h \circ j = t$. Thus we have $(p \otimes E) \circ h \circ j = (p \otimes E) \circ t = \varphi \circ j$, so that $(\varphi - (p \otimes E) \circ h) \circ j = 0$. Hence $g := \varphi - (p \otimes E) \circ h$ factors through the projection $\pi : E \to E/P$, say as $g = f \circ \pi$. By Lemma 1.5 we have that $f = 0$, and so $g = 0$ and $\varphi = (p \otimes E) \circ h$. Thus we see that $(p \otimes E)_*$ is an epimorphism and the commutative diagram:

\[
\begin{array}{ccc}
S^n & \xrightarrow{P} & X \\
\downarrow^{\alpha_X} & & \downarrow^{\alpha_X} \\
\text{Hom}_R(E, E^n) & \xrightarrow{(p \otimes E)_*} & \text{Hom}_R(E, X \otimes_S E)
\end{array}
\]

shows that $\alpha_X$ is indeed an epimorphism. \hfill \Box

If $E_R$ is quasi-injective and $S = \text{End}(E_R)$, then $S/J$ is a regular right self-injective ring. If we set $\tilde{E} := (S/J) \otimes_S E = E/J E$, then we have a bimodule $S/J \tilde{E}$ and, if $X \in \text{Mod}-S/J$ we have that

\[
X \otimes_S E \cong (X \otimes_{S/J} S/J) \otimes_S E \cong X \otimes_{S/J} ((S/J) \otimes_S E) \cong X \otimes_{S/J} \tilde{E}.
\]

Thus, if we identify $X \otimes_S E$ with $X \otimes_{S/J} \tilde{E}$, and if $\tilde{\alpha}_X : X \to \text{Hom}_R(\tilde{E}, X \otimes_{S/J} \tilde{E})$ is the canonical morphism and $p : E \to \tilde{E}$ the canonical projection, we see that $\text{Hom}_R(p, X \otimes_S E) \circ \tilde{\alpha}_X = \alpha_X$. Since $\text{Hom}_R(p, X \otimes_S E)$ is a monomorphism, if $X$ is $E$-invariant, then $X$ is $E$-invariant.

Specifically, if $X = S/J$, then we have proved

**Corollary 1.7.** Let $P_R$ be a finitely generated projective module, $E = E(P_R)$, $S = \text{End}(E_R)$ and $J = J(S)$. If every finitely generated submodule of $E$ embeds in a finitely presented module of projective dimension $\leq 1$, there is a canonical isomorphism $S/J \cong \text{End}(E/J E)$.

**Proposition 1.8.** Let $E_R$ be quasi-injective (or pure-injective) and let $X$ be a right $S/J$-module which is $E$-invariant. If $X \otimes_S E$ is either $E$-injective or pure-injective, then $X$ is $E$-invariant.
Proof. Let $\bar{E} = E/J E$. Since $X$ is $E$-invariant, it is also $\bar{E}$-invariant. On the other hand, as $S/J$ is regular, $S/J \bar{E}$ is flat. By Proposition 1.2 applied to the adjunction defined by $S/J \bar{E}_R$, if we assume that $X \otimes_S E \cong X \otimes_{S/J} \bar{E}$ is $E$-injective, we get that $X_{S/J}$ is injective. Similarly, if $X \otimes_{S/J} \bar{E}$ is pure-injective, then $X_{S/J}$ is pure-injective and hence, since $S/J$ is regular, injective.

We will say that a module $M$ is completely pure-injective when every pure quotient of $M$ is pure-injective. (Note the change of terminology with respect to [3].)

Corollary 1.9. Let $P_R$ be a finitely generated projective module, $E = E(P_R)$, $S = \text{End}(E_R)$, and $J = J(S)$. Assume that every finitely generated submodule of $E_R$ embeds in a finitely presented right $R$-module of projective dimension $\leq 1$ and that $E/J E$ is completely pure-injective. Then $S$ is semiperfect and $P_R$ is finite-dimensional.

Proof. By Theorem 1.6, each finitely generated right $S/J$-module $X$ is $E$-invariant. Since the canonical projection $S/J \twoheadrightarrow X$ is a pure epimorphism (since $S/J$ is regular), we have that the induced $R$-epimorphism $E/J E \to X \otimes_S E$ is also pure. Thus $X \otimes_S E$ is a pure-injective right $R$-module by hypothesis, and by Proposition 1.8, $X_{S/J}$ is injective. Then, by Osofsky's theorem [8, 9], $S/J$ is semisimple and hence $S$ is semiperfect. This is equivalent to $E_R$ (and hence to $P_R$) being finite-dimensional.

The preceding corollary can be regarded as a generalization of [3, Corollary 6]. A more specific extension of this result is the following:

Corollary 1.10. Let $R$ be a right hereditary ring. Then $R$ is right noetherian if and only if every finitely generated submodule of $E(R_R)$ is finitely presented.

Proof. If every finitely generated submodule of $E(R_R)$ is finitely presented, then $R_R$ is right finite-dimensional by Corollary 1.9. Thus, using [5, Corollary 5.20], we see that $R$ is right noetherian. The converse is clear.

2. Rings of pure global dimension less than or equal to one

Recall that the pure-injective dimension of a right $R$-module $M$ is defined as the smallest nonnegative integer (or $\infty$) such that there exists an exact sequence $0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$, where the $E_i$, $i = 0, \ldots, n$, are pure-injective modules and the associated short exact sequences are pure exact. The supremum of the pure-injective dimensions of the right $R$-modules is called the right pure global dimension of $R$ [7, 6], and is denoted by $r.\text{pgldim}(R)$. Thus the rings $R$ such that $r.\text{pgldim}(R) \leq 1$ provide a natural source of completely pure-injective modules. The following theorem will be useful in order to apply our results to these rings.

Theorem 2.1. Let $R$ be a ring, $E = E(R_R)$, $S = \text{End}(E_R)$ and $J = J(S)$. If every finitely generated submodule of $E_R$ embeds in a finitely presented module of projective dimension $\leq 1$, then $E/J E$ is a pure-injective $R$-module.

Proof. Let $\bar{E} = E/J E$. Consider the exact sequence in $\text{Mod-}R$, $0 \to R \xrightarrow{j} E \to E/R \to 0$, and let $g \in \text{Hom}_R(R, \bar{E}) \cong \bar{E}$. Then $g$ induces a homomorphism $h : R_R \to E$ such that if $g : E \to \bar{E}$ is the canonical projection, then
By the injectivity of $E$, $h$ extends to $t : E \to E$, so $g$ extends to a morphism $q \circ t : E \to \bar{E}$. Thus, in the exact sequence

$$\text{Hom}_R(E/R, \bar{E}) \to \text{Hom}_R(E, \bar{E}) \xrightarrow{j^*} \text{Hom}_R(R, \bar{E})$$

$j^*$ is an epimorphism and hence an isomorphism since $\text{Hom}_R(E/R, \bar{E}) = 0$ by Lemma 1.5. Since $S/J$ is $E$-invariant by Theorem 1.6, we have isomorphisms of left $S/J$-modules:

$$\bar{E} \cong \text{Hom}_R(E, \bar{E}) \cong \text{Hom}_R(E, (S/J) \otimes_S E) \cong S/J.$$

Let $\bar{E}^* = \text{Hom}_{S/J}(\bar{E}, S/J)$. Since $\bar{E}$ is reflexive as a $S/J$-module,

$$\bar{E} \cong \text{Hom}_{S/J}(\bar{E}^*, S/J).$$

Since $S/J$ is right self-injective, applying Proposition 1.1 to the bimodule $R\bar{E}^*_{S/J}$ we obtain that $\bar{E}$ is a pure-injective right $R$-module.

Remark. As a consequence of Theorem 2.1 we see that, in Corollary 1.9, it is enough to assume that every proper pure quotient of $E/JE$ is pure-injective, instead of requiring that $E/JE$ be completely pure-injective.

**Corollary 2.2.** Let $R$ be a ring such that $r.pgldim(R) \leq 1$. Assume, further, that every finitely generated submodule of $E(R_R)$ embeds in a finitely presented module of projective dimension $\leq 1$. Then $R$ is right finite-dimensional.

**Proof.** If $E = E(R_R)$ we have, by Theorem 2.1, that $E/JE$ is pure-injective and hence completely pure-injective. Then $R$ is right finite-dimensional by Corollary 1.9.

An interesting class of rings of right pure global dimension $\leq 1$ is the class of countable rings [6, 7]. For instance, it follows from the preceding results that every countable ring $R$ such that every finitely generated submodule of $E(R_R)$ embeds in a finitely presented module of projective dimension $\leq 1$ is finite-dimensional.

The following result is a partial generalization of [1, Theorem 3.2], and shows that the rings such that $r.pgldim(R) \leq 1$ and $E(R_R)$ is projective are not far from being right QF-3 rings (but they need not be, as the ring $R = (\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix})$ shows).

**Corollary 2.3.** Let $R$ be a ring such that $r.pgldim(R) \leq 1$ and $E(R_R)$ is projective. Then $R$ has a faithful injective right ideal.

**Proof.** By Corollary 2.2 $R$ is right finite-dimensional and, using [10, Lemma 2], we obtain the result.

The rings $R$ such that every finitely generated right $R$-module embeds in a free module have been called right FGF by Faith [2]. It is still an open problem whether a right FGF ring must be QF.

**Corollary 2.4.** Let $R$ be a right FGF ring such that $r.pgldim(R) \leq 1$ and $R$ has essential right socle. Then $R$ is QF.

**Proof.** $R$ is right finite-dimensional by Corollary 2.2. Thus $\text{Soc}(R_R)$ is finitely generated and, as $R_R$ has essential socle, we see that $R_R$ has finite essential socle. Since each finitely generated right module embeds in a (finitely generated) free right $R$-module, we see that every finitely generated right module has finite essential socle, so that $R$ is right artinian. Then $R$ is QF by [2].
Recall that a ring homomorphism \( \varphi : R \to Q \) is a right flat epimorphism of rings (or a perfect right localization of \( R \)) precisely when \( RQ \) is flat and the canonical morphism \( Q \otimes_R Q \to Q \) is an isomorphism. Goodearl proved that if \( Q \) is the right maximal quotient ring of a right nonsingular ring \( R \), then the canonical morphism \( R \to Q \) is a left flat epimorphism if and only if every finitely generated nonsingular right \( R \)-module embeds in a free module [4, Theorem 7]. In general, this condition is not right-left symmetric, as is shown by the endomorphism ring of an infinite-dimensional vector space over a field. However, if \( r.p\text{gl.dim}(R) \leq 1 \), then we have symmetry.

**Corollary 2.5.** Let \( R \) be a ring such that \( r.p\text{gl.dim}(R) \leq 1 \). Then the following conditions are equivalent:

(i) \( R \) is right nonsingular and every finitely generated nonsingular right \( R \)-module embeds in a free module.

(ii) \( R \) is left nonsingular and every finitely generated nonsingular left \( R \)-module embeds in a free module.

(iii) \( R \) has a semisimple two-sided maximal quotient ring.

**Proof.** (i)\( \Rightarrow \) (iii) Let \( Q = Q_{\text{max}}(R) \) be the maximal right quotient ring of \( R \). By Corollary 2.2, \( R \) is right finite-dimensional and so \( Q \) is semisimple [11, Theorem XII.2.5]. Further, \( QR \) is flat by the result of Goodearl mentioned above [cf. also [5, Theorem 5.17] and [11, Theorem XII.7.1]]. But then it follows from [11, Corollary XII.7.3] that \( Q \) is also the maximal left quotient ring of \( R \).

(iii)\( \Rightarrow \) (i) Since \( Q \) is semisimple, \( R \) is right nonsingular by [11, Proposition XII.2.2]. Also, since the left maximal quotient ring \( Q \) of \( R \) is semisimple, the canonical homomorphism \( R \to Q \) is a left flat epimorphism. Then, using again [5, Theorem 5.17], we see that every finitely generated nonsingular right \( R \)-module embeds in a free module.

Finally, observe that the proof can be completed by symmetry, bearing in mind that condition (iii) is left-right symmetric.

An entirely similar argument can be applied to the characterization given by Cateforis and Goodearl of the right nonsingular rings such that every finitely generated nonsingular right \( R \)-module is projective [5, Theorem 5.18]. This class of rings is not right-left symmetric in general [5] but, from the preceding corollary and [5, Theorem 5.18], we have:

**Corollary 2.6.** Let \( R \) be a ring such that \( r.p\text{gl.dim}(R) \leq 1 \) and \( Q \) its maximal right quotient ring. Then the following conditions are equivalent:

(i) \( R \) is right nonsingular and every finitely generated nonsingular right \( R \)-module is projective.

(ii) \( R \) is left nonsingular and every finitely generated nonsingular left \( R \)-module is projective.

(iii) \( R \) is left and right semihereditary, and \( Q \) is a semisimple two-sided maximal quotient ring of \( R \).

**References**


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