ON THE ZERO SETS OF CERTAIN ENTIRE FUNCTIONS

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Abstract. We consider the class $B$ of entire functions of the form

$$f = \sum p_j \exp g_j,$$

where $p_j$ are polynomials and $g_j$ are entire functions. We prove that the zero-set of such an $f$, if infinite, cannot be contained in a ray. But for every region containing the positive ray there is an example of $f \in B$ with infinite zero-set which is contained in this region.

Let $B$ be Borel’s class of entire functions of one complex variable that are finite sums of entire functions with only finitely many zeros (possibly none). Clearly $f \in B$ if and only if

$$f = \sum_{j=0}^{n} p_j \exp g_j,$$

where the $p_j$ are polynomials and the $g_j$ are entire functions. This class is called $B_1$ in [HRS].

Theorem 1. No function in $B$ can have as its zero set an infinite set of positive real numbers.

Theorem 2. Given any open set $\Omega$ in the complex plane that contains the positive real axis, there is a function $f$ in $B$ whose zero set is an infinite subset of $\Omega$.

Proof of Theorem 1. We will use H. Cartan’s theory of holomorphic curves [C, L]. An $n + 1$-vector of entire functions $(f_0, \ldots, f_n)$ without zeros common to all $f_j$ defines a holomorphic curve $F$ which is a holomorphic map of the complex plane $\mathbb{C}$ into the complex projective space $\mathbb{P}^n$. The characteristic $T(r, F)$ is defined in the following way:

$$T(r, F) = \frac{1}{2\pi} \int_0^{2\pi} \max \{|f_0|, \ldots, |f_n|\} (re^{i\theta})d\theta.$$  

For any vector $a = (a_0, \ldots, a_n) \in \mathbb{C}^{n+1}\{0\}$ define

$$N(r, a, F) = \frac{1}{2\pi} \int_0^{2\pi} \log |a_0 f_0 + \ldots + a_n f_n| (re^{i\theta})d\theta.$$
Such a vector $\mathbf{a}$ defines a hyperplane in $\mathbb{P}^n$ by the equation $a_0x_0 + \ldots + a_nx_n = 0$. If we denote by $n(r, \mathbf{a}, F)$ the number of preimages of this hyperplane under $F$ which are contained in the disk $\{z : |z| \leq r\}$, then by the Jensen formula

$$N(r, \mathbf{a}, F) = \int_0^r \left\{ n(t, \mathbf{a}, F) - n(0, \mathbf{a}, f) \right\} \frac{dt}{t} + n(0, \mathbf{a}, F) \log r + \text{const}. $$

If $n = 1$, the Cartan characteristic $T(r, F)$ coincides (up to an additive constant) with the usual Nevanlinna characteristic of the meromorphic function $f_1/f_0$. We will use the Second Main Theorem of Cartan, which (in a simplified form) states the following: Let $\mathbf{a}_1, \ldots, \mathbf{a}_q$ be an admissible system of vectors; that is, any $n+1$ of them are linearly independent. If the components $f_0, \ldots, f_n$ of a curve $F$ are linearly independent, then

$$\sum_{j=1}^q N(r, \mathbf{a}_j, F) \geq (q - n - 1 + o(1))T(r, F), \quad r \in \mathbb{R}^+ \setminus E,$$

where $E$ is an exceptional set of finite length.

The following theorem due to E. Borel (see, for example [L, p. 186]) is a simple corollary of the Second Main Theorem of Cartan. Let $f_j = p_j \exp g_j$, where $p_j \neq 0$ are polynomials and $g_j$ are entire functions. If $\{f_0, \ldots, f_n\}$ are linearly dependent, then there are two functions $\exp g_j$ and $\exp g_k$, which are proportional (with constant coefficients).

It follows from Borel’s theorem that every function of the class $\mathcal{B}$ can be written in reduced form, namely the functions $f_j = p_j \exp g_j$ in (1) are linearly independent. Furthermore in the proof of Theorem 1 we may assume without loss of generality that $f$ is transcendental, the polynomials $p_j$ have no zeros common to all $p_j$ and that $g_0 = 0$.

With these assumptions we introduce the holomorphic curve $F$ with coordinates $f_j = p_j \exp g_j, \ 0 \leq j \leq n$, and show first that

$$r = O(T(r, F)), \quad r \to \infty. $$

Because $f$ in (1) is assumed to be transcendental, at least one of $g_j$ is not constant. Assume that $g_0 \neq \text{const}$. Then by the definition of characteristic and by our assumption that $g_0 = 0$ we have

$$2\pi T(r, F) \geq \int_0^{2\pi} \max\{\log |f_0|, \log |f_n|\} d\theta$$

$$\geq \int_0^{2\pi} \max\{0, \text{Re } g_n\} d\theta + O(\log r) \geq cr + O(\log r),$$

for some $c > 0$, which proves (3).

We need the following estimate

$$T(r, f) \leq T(r, F) + O(\log r), \quad r \to \infty. $$
To prove this we use first the inequality \( \log |a+b| \leq \max\{\log |a|, \log |b|\} + \log 2 \) and then our assumption that \( g_0 = 0 \) (so \( \log |f_0| = \log |p_0| = O(\log r) \)):

\[
2\pi T(r, f) = \int_0^{2\pi} \log^+ |f| d\theta
\leq \int_0^{2\pi} \max\{\log |f_0|, \ldots, \log |f_n|\}^+ d\theta + O(1)
= \int_0^{2\pi} \max\{0, \log |f_1|, \ldots, \log |f_n|\} d\theta + O(\log r)
\leq \int_0^{2\pi} \max\{\log |f_0|, \ldots, \log |f_n|\} d\theta + O(\log r)
= 2\pi T(r, F) + O(\log r).
\]

Now we apply the Second Main Theorem of Cartan with \( q = n+2 \), and the following vectors: \( \mathbf{a}_j \) for \( 1 \leq j \leq n+1 \) is the \( j \)-th row of the \( (n+1) \times (n+1) \) unit matrix and \( \mathbf{a}_{n+2} = (1, \ldots, 1) \) is the row of 1’s. Then we have \( N(r, \mathbf{a}_j, F) = O(\log r) \), \( r \to \infty \), and \( N(r, \mathbf{a}_{n+2}, F) = N(r, 0, f) \), the usual Nevanlinna counting function of zeros of the entire function \( f \). From (2) it follows that

\[
N(r, 0, f) \geq (1 + o(1))T(r, F), \quad r \in \mathbb{R}^+ \setminus E.
\]

Combined with (4) this implies

\[
N(r, 0, f) \sim T(r, F), \quad r \to \infty, \quad r \in \mathbb{R}^+ \setminus E.
\]

In particular, this asymptotic equality combined with (3) implies that the genus of \( f \) is at least 1 (maybe infinite).

Finally we use the following result of A. Edrei and W. Fuchs [EF] and J. Miles [M]: \textit{If \( f \) is an entire function of genus at least 1, with positive zeros, then there is a set \( E_1 \) of zero logarithmic density and a constant \( \epsilon > 0 \) such that}

\[
N(r, 0, f) \leq (1 - \epsilon)T(r, f), \quad r \in \mathbb{R} \setminus E_1.
\]

Since this estimate is incompatible with (6), Theorem 1 must hold.

\textbf{Proof of Theorem 2.} By taking a smaller region if necessary (but still including the positive real axis), we may assume that \( \Omega \) is connected and simply connected, and is bounded by a single smooth simple curve \( \gamma : [-1, 1] \to \mathbb{C} \) such that \( \gamma(t) \to \infty \) as \( t \to \pm 1 \) and \( \gamma \) intersects the real axis once (this intersection happens on the negative ray). The complement \( T \) of \( \Omega \) is an Arakelyan set, i.e. \( \Omega \) is connected and locally connected at \( \infty \) (see [GAI]). Using the Arakelyan approximation theorem [GAI] we find a non-constant entire function \( g \) with the property \( |g(z) - 1/2| < 1/4, \; z \in T \). Thus \( g^{-1}(\mathbb{Z}) \subset \Omega \) and \( f(z) = \exp[2\pi i g(z)] - 1 \) gives the required example.

\textbf{References}


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