ON THE PRIME MODEL PROPERTY

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ABSTRACT. Assume $T$ is superstable, $\Phi(x)$ is a formula over $\emptyset$, $Q = \Phi(M^*)$ is countable and $K_Q = \{M : M$ is countable and $\Phi(M) = Q\}$. We investigate models in $K_Q$ assuming $K_Q$ has the prime model property. We prove some corollaries on the number of models in $K_Q$. We show an example of an $\omega$-stable $T$ and $Q$ with $K_Q$ having exactly 3 models.

First we fix the general set-up. Throughout, $T$ is a countable complete theory in a first-order language $L$, $\Phi(x)$ is a formula of $L$ without parameters, $M^*$ is a countable model of $T$ and $Q = \Phi(M^*)$. We work within a monster model $C = C^{eq}$ of $T$. All models of $T$ we consider are elementary submodels of $C$. Let $K_Q = \{M : M$ is countable and $\Phi(M) = Q\}$ and let $I(K_Q)$ be the number of models in $K_Q$, up to isomorphism.

The goal of this note is to develop some model theory of $K_Q$. As I pointed out in [Ne1, p.651], in general $K_Q$ can not be treated with all common model-theoretic tools, since for example $K_Q$ does not necessarily have the joint embedding property. It turns out however that if we assume just that $K_Q$ has the prime model property (defined below), then it is much more manageable, at least for stable $T$. $K_Q$ really may differ from an elementary class: I have found an example of $K_Q$ with $I(K_Q) = 2$, while a theorem of Vaught [Sa] says that $I(T, \aleph_0) \neq 2$. In this example $T$ is weakly minimal, but $K_Q$ does not have the prime model property. In this paper we give an example of an $\omega$-stable $T$ with $I(T, \aleph_0) = \aleph_0$ and a strongly minimal $Q$ with $I(K_Q) = 3$.

Now we recall some basic notions from [Ne1]. We call $f \in Aut(C)$ a $Q$-mapping if $f|Q = Q$. For $a \subseteq M \in K_Q$ let $Aut_Q(C/a)$ be the group of $Q$-mappings fixing $a$ pointwise. This group acts in a Borel way on $S(Qa)$, inducing equivalence relation $E(a)$ on $S(Qa)$. The orbits of this action, that is the $E(a)$-classes, are called pseudotypes over $Qa$. When $a = \emptyset$, we use $E$ to denote $E(a)$. Pseudotypes are Borel subsets of $S(Qa)$, and $E(a)$ is analytic. We say that $p \in S(Qa)$ is $Q$-isolated over $a$ if $p/E(a)$ is not meager. $p/E(a)$ is the pseudotype of $p$ over $Qa$ (that is the equivalence class as a subset of $S(Qa)$, not a quotient of some sort). $Q$-isolation has many nice properties (see [Ne1, Ne2]), for instance if $Q$-isolated types are dense in $S(Q)$ then there is $M \in K_Q$ such that each $b \subseteq M$ satisfies a $Q$-isolated type over $Q$. Such an $M$ is called $Q$-atomic. A $Q$-atomic model is unique up to isomorphism. Also, $tp(ab/Q)$ is $Q$-isolated iff $tp(a/Q)$ is $Q$-isolated and $tp(b/Qa)$ is $Q$-isolated over $a$.
Let \( B(Q,a) = \{ p \in S(Qa) : \text{there is no } M \in K_Q \text{ containing a realization of } p \} \). \( B(Q,a) \) is a meager \( F_\sigma \)-subset of \( S(Qa) \). \( B(Q) \) denotes \( B(Q,\emptyset) \). We think of the types in \( B(Q) \) as ‘bad’ types, since they can not be used to build models in \( K_Q \). Types in \( S(Q) \setminus B(Q) \) are called good.

We say that \( K_Q \) has the prime model property if the following holds.

(PM) For every \( a \) with \( tp(a/Q) \) good there is a model \( M \in K_Q \) containing \( a \), such that for every \( M' \in K_Q \) containing \( a \), some \( Q \)-mapping fixing \( a \) embeds \( M \) into \( M' \).

The model \( M \) occurring in (PM) is called \( Q \)-prime over \( a \). ‘\( Q \)-prime’ means \( Q \)-prime over \( \emptyset \).

Remark. (1) If \( M \) is \( Q \)-prime over \( a \) then \( M \) is \( Q \)-atomic over \( a \). In particular, a \( Q \)-prime over \( a \) model is unique up to isomorphism over \( a \). (However, even a countable \( Q \)-atomic model need not be \( Q \)-prime.)

(2) If (PM) holds and \( p \in S(Qa) \) then \( p \) is \( Q \)-isolated over \( a \) iff every \( M \in K_Q \) containing \( a \) contains a realization of a type \( p' \in p/E(a) \).

(3) (PM) implies that \( Q \)-isolated over \( a \) types are dense in \( S(Qa) \) and “\( Q \)-prime over \( a \)” = “\( Q \)-atomic over \( a \)”.

Proof. (1) Wlog \( a = \emptyset \). We must show that if \( p \in S(Q) \) is not \( Q \)-isolated then no \( p' \in p/E \) is realized in \( M \). To prove this it is enough to show that some \( N \in K_Q \) omits every \( p' \in p/E \). Since \( p \) is not \( Q \)-isolated, \( p/E \) is meager, hence is covered by countably many closed nowhere dense subsets \( X_n, n < \omega \), of \( S(Q) \). A closed subset \( X_n \) of \( S(Q) \) corresponds to a type \( q_n \) over \( Q \), and \( X_n \) being nowhere dense means that \( q_n \) is non-isolated. Hence by the omitting types theorem there is a model \( N \in K_Q \) omitting every \( q_n, n < \omega \). This means that \( N \) omits every \( p' \in p/E \). (After all, the omitting types theorem is just the Baire category theorem adapted for the needs of model theorists.) Since a \( Q \)-atomic model is unique up to isomorphism, also a \( Q \)-prime model is such.

(2), (3) follow easily.

We say that \( K_Q \) has the joint embedding property if the following holds.

(JE) For every \( a \) with \( tp(a/Q) \) good, if \( M_0, M_1 \in K_Q \) contain \( a \) then for some \( M \in K_Q \) containing \( a \) there are \( Q \)-mappings \( f_0, f_1 \) fixing \( a \) and embedding \( M_0, M_1 \) respectively into \( M \).

We say that \( M \in K_Q \) is \( Q \)-saturated if for every \( a \subset M \) and good \( p \in S(Qa) \), some type in \( p/E(a) \) is realized in \( M \). \( M \in K_Q \) is \( Q \)-universal if for every \( N \in K_Q \) there is a \( Q \)-mapping embedding \( N \) into \( M \).

**Lemma 1.** (1) If \( T \) is stable, there are countably many good pseudotypes over \( Q \) and (PM) holds, then there is a \( Q \)-saturated model.

(2) A \( Q \)-saturated model is \( Q \)-universal and unique up to isomorphism.

(3) If (\( T \) is stable and (PM) holds) or (a \( Q \)-saturated model exists) then (JE) holds.

Proof. [Ne4, Corollary 2.10(3)] proves that if \( T \) is stable and (PM) holds, then whenever the types \( tp(a/Q), tp(b/Q) \) are good, then for some \( b' \) with \( tp(b'/Q)E tp(b/Q) \), the type \( tp(ab'/Q) \) is good. This amalgamation property easily implies the lemma.

The following fact ([Ne4, Corollary 2.10]) shows that the assumption of (PM) makes sense.
Fact. Assume $T$ is small, stable, $T[\Phi]$ is $\omega$-stable and one of the following conditions holds.

(i) $T[\Phi]$ is bounded, 1-based or of finite rank, and $I(T[\Phi], \aleph_0) < 2^{\aleph_0}$.
(ii) $Q$ is atomic or saturated (as a model of $T[\Phi]$).
(iii) $T$ is superstable and $T[\Phi]$ has $< 2^{\aleph_0}$ countable models.

Then $K_Q$ has the prime model property and there are countably many good pseudotypes over $Q$.

It would be very good if we could omit the assumption in (i) that $T[\Phi]$ is bounded, 1-based or of finite rank. Then we could drop (iii) in the Fact. It is open whether $T$ stable, $T[\Phi]$ $\omega$-stable and $I(T, \aleph_0) < 2^{\aleph_0}$ implies (PM). This is related to the $\tau$-stability conjecture [Ne2, Ne4]. As I mentioned above, there is an example of a weakly minimal $T$ and $K_Q$ with $I(K_Q) = 2$. In this example $K_Q$ does not have the prime model property. Now we give an example of an $\omega$-stable $T$ with $I(T, \aleph_0) = \aleph_0$ and a strongly minimal $Q$ with $I(K_Q) = 3$.

Example. The example is a modification of Shelah’s example of a theory $T_0$ with abnormal types ([Ba, XVIII,4]). Let $V$ be a countably infinite set and $P = \{c_n, n < \omega\}$ an infinite and co-infinite subset of $V$. Equip $V \setminus P$ with a structure of a model of $T_0$. This means among others that there is an equivalence relation $E'$ on $V \setminus P$ and an asymmetric relation $R$ on $(V \setminus P)/E'$ such that $(\langle V \setminus P \rangle/E'; R)$ is a directed graph without cycles such that each element has infinitely many successors and predecessors (see [Ba] for details). We can arrange that $(\langle V \setminus P \rangle/E'; R)$ is connected. So on $(V \setminus P)/E'$ there is a natural distance function $d(x, y)$. Define a ternary relation $S \subseteq P \times (V \setminus P)/E' \times (V \setminus P)/E'$ by: $S(c_n, x, y)$ iff $d(x, y) \leq n$.

Let $M^* = (V; P, S, c_n, n < \omega; \text{the structure of } V \setminus P \text{ as a model of } T_0)$, and $T = Th(M^*)$. Clearly, $T$ is $\omega$-stable, $I(T, \aleph_0) = \aleph_0$, and $\Phi(x) = P(x)$ is trivial and strongly minimal. Let $M'$ be a prime model of $T$ and $Q = P(M')$. Then for any $M \in K_Q$, $-P(M)/E$ is R-connected. By [Ba, XVIII,4.6] in $T_0$, up to isomorphism there are 3 kinds of countable connected components. Hence $I(K_Q) = 3$. $\omega$-stability implies $K_Q$ has the prime model property, hence by Theorem 1 below we can not have here $I(K_Q) = 2$. However, if $M$ is a countable model of $T$ with $P(M) \neq \{c_n, n < \omega\}$, then for $Q = \Phi(M)$, $I(K_Q)$ is infinite.

In this paper we try to generalize a result of Lachlan [Ba] saying that for superstable $T$, $I(T, \aleph_0) > 1$ implies $I(T, \aleph_0) \geq \aleph_0$. We shall use the following lemma.

Lemma 2 ([Ne1, Lemma 2.4]). If $T$ is stable, $tp(a/Qb)$ is $Q$-isolated over $b$ and $tp(a/Q)$ is not $Q$-isolated, then $a \vdash b(Q)$.

We write $a \not\vdash b(X)$ for ‘$a$ is independent from $b$ over $X$’. Regarding the theorem of Vaught we can prove the following.

Theorem 1. If $T$ is superstable and $K_Q$ has the prime model property, then $I(K_Q) \neq 2$. If moreover $T$ has finite $U$-rank and $I(K_Q) > 1$, then $I(K_Q)$ is infinite.

Proof. Suppose $I(K_Q) > 1$. If every good type in $S_n(Q), n < \omega$, is $Q$-isolated, then every model in $K_Q$ is $Q$-atomic, hence $I(K_Q) = 1$, a contradiction. So let $p \in S(Q) \setminus B(Q)$ be non-$Q$-isolated. Let $M_0$ be the $Q$-prime model.

If $(S_v(Q) \setminus B(Q))/E$ is uncountable, then clearly $I(K_Q)$ is infinite. So we can assume there are only countably many good pseudotypes over $Q$. By Lemma 1(1) we see that there is a $Q$-saturated model $M_\omega$. Since $M_\omega$ realizes a non-$Q$-isolated type, $M_0 \not\cong M_\omega$. 
Let $a$ realize $p$ and let $M$ be $Q$-prime over $a$. Wlog $M_0 \subseteq M$. Since $p$ is not $Q$-isolated, $a \in M \setminus M_0$. Since $Q = \Phi(M) = \Phi(M_0)$, $M$ and $M_0$ are a Vaughtian pair. In particular, $tp(a/M_0)$ is non-algebraic and orthogonal to $\Phi$. Choose a finite $b \subseteq M_0$ with $a \ind_{M_0} b$. So also $tp(a/Qb) \perp \Phi$. Also, $tp(a/Qb)$ is not $Q$-isolated over $b$. Otherwise, since $tp(b/Q)$ is $Q$-isolated (as $b \subseteq M_0$), by transitivity of $Q$-isolation ([Ne1]) we would get that $p = tp(a/Q)$ is $Q$-isolated.

Let $I = \{ a_n, n < \omega \}$ be a Morley sequence in $stp(a/Qb)$, with $a = a_0$. Wlog $I \subseteq M_\omega$. By superstability, for some $n'$, if $n > n'$ then $a_n \ind_{Qb} (a_{<n})$ and $tp(a_n/a_{<n})$ is stationary. Hence for $n > n'$ also $tp(a_n/Qba_{<n})$ is not $Q$-isolated over $ba_{<n}$. Indeed, since $tp(a_n/ba_{<n})$ is orthogonal to $\Phi$, by the open mapping theorem it is enough to show that $tp(a_n/ba_{<n})$ is non-isolated. Suppose otherwise. Since $a \ind_{M_0}(b)$ the construction yields $a_n \ind_{a_{<n}} (b)$. This gives that $tp(a_n/b) \models tp(a/Qb)$ is isolated and stationary, because $tp(a_n/ba_{<n})$ is stationary. Hence $tp(a/b) \models tp(a/Qb)$, and $tp(a/Qb)$ is isolated, hence $Q$-isolated over $b$, a contradiction.

The rest of the proof resembles that of [Ne1, Theorem 2.5]. Let $M_n$ be $Q$-prime over $ba_{<n}$. Clearly for $n > 0$, $M_n \not\equiv M_0$, since $p$ is realized in $M_n$. Let $m > n'$ and suppose $M_m$ and $M_n$ are isomorphic. Then within $M_m$ there is a copy $b'(a'_n, n < \omega)$ of $bI$ (via some $Q$-mapping). By the properties of $Q$-isolation, for every $c \subseteq M_m$, $M_m$ is $Q$-atomic over $ba_{<m}$. Hence $M_m$ is $Q$-atomic over $ba_{<m}b'a_{<n}$ for any $n$. In particular, for every $n > n'$ we have that $tp(a'_n/Qba_{<m}b'a_{<n})$ is $Q$-isolated over $ba_{<n}b'a_{<n}$ and $tp(a'_n/Qba_{<m}b'a_{<n})$ is not $Q$-isolated over $b'a_{<n}$. By Lemma 2 we get that for every $n > n'$, $ba_{<m} \ind_{a'_n} (Qba_{<n})$, contradicting the superstability of $T$.

The case when $U$-rank is finite is handled like in [Ne1]. We prove that if $k$ is large enough then $M_k \not\equiv M_m$.

I managed to generalize Lachlan’s theorem for $K_Q$ in case when $Q$ is weakly saturated (as a model of $T|\Phi$), that is when every type in $T|\Phi$ is realized in $Q$.

**Theorem 2.** If $T$ is superstable, $K_Q$ has the prime model property, $Q$ is weakly saturated and $I(K_Q) > 1$, then $I(K_Q)$ is infinite.

The proof relies on the following lemma.

**Lemma 3.** If $T$ is superstable and $Q$ is weakly saturated, then every type in $S(\emptyset)$ has a good extension in $S(Q)$.

**Proof.** Choose any $a$. We want to find $a' \equiv a$ with $tp(a'/Q)$ good. By superstability, for some $b \subseteq \Phi(C)$ we have $a \ind_{\Phi(C)}(b)$. This implies that if $b \subseteq Q' = \Phi(M')$ for some countable $M'$, then $tp(a/Q') \not\models B(Q')$. Indeed, otherwise for some $d \in \Phi(C)$ we would have $a \ind_{\Phi(C)(Q')}$, while $a \ind_{\Phi(C)}(Q')$, a contradiction.

Since $Q$ is weakly saturated, there is $b' \subseteq Q$ with $b \equiv b'$. Choose $a'$ with $a'b' \equiv ab$. It follows that $a' \ind_{\Phi(C)}(b')$ and $tp(a'/Q)$ is good.

Note that if in Lemma 3 we weaken the assumption of superstability to stability, then we must assume that $Q$ is universal.

**Proof of Theorem 2.** As in the proof of Theorem 1 we find $a, b$ with $tp(a/Qb)$ non-$Q$-isolated over $b$ and orthogonal to $\Phi$ (hence good). We can also assume that $a \ind Q(b)$ and $tp(a/b)$ is stationary non-isolated (see the proof of Theorem 1). Let $k = w(ab)$.
By Lemma 3, for each $n < \omega$ there is a Morley sequence $A_n$ in $stp(ab)$ of length $(k^{n+1} - 1)/(k - 1)$ such that $tp(A_n/Q)$ is good. Let $M_n$ be $Q$-prime over $A_n$. Let $|A_n|$ denote the length of $A_n$ and let $w(A_n)$ be the weight of $A_n$. Notice that

$$w(A_n) = k^{n+1} - 1 = |A_{n+1}| - 1$$

Hence for $n < m$, $w(A_n) < |A_m|$.

We shall prove that for $n < m$, $M_n$ and $M_m$ are non-isomorphic. Suppose not. Then there is an $A'_m \subseteq M_n$, a Morley sequence in $stp(ab)$ of length $|A_m|$. Since $w(A_n) < |A_m|$, for some $a'b' \in A'_m$, $a'b' \not\subseteq A_n$. In particular, $a' \not\subseteq A_n(b')$. Since $tp(a'/b')$ is orthogonal to $\Phi$, also $a' \not\subseteq A_n(Qb')$. As $M_n$ is $Q$-atomic over $A_n$, $M_n$ is also $Q$-atomic over $A_n(b')$. Hence $tp(a'/A_nQb')$ is $Q$-isolated over $A_n(b')$. As $tp(a'/b')$ is stationary, non-isolated and orthogonal to $\Phi$, $tp(a'/Qb')$ is not $Q$-isolated over $b'$. By Lemma 2, $a' \not\subseteq A_n(Qb')$, a contradiction.

**Corollary 1.** Assume $T$ is superstable, $T[\Phi$ is $\omega$-stable and $Q$ is saturated. Then $I(K_Q) > 1$ implies $I(K_Q)$ is infinite. On the other hand, $I(K_Q) = 1$ implies $T$ is $\omega$-stable and $(\omega,\omega)$-categorical relative to $\Phi$ (in the sense of [HHM]).

**Proof.** If $T$ is not small, then Lemma 3 implies that $I(K_Q)$ is infinite. So assume $T$ is small. By the Fact, $K_Q$ has the prime model property. Hence by Theorem 2, $I(K_Q) > 1$ implies $I(K_Q)$ is infinite.

Now assume $I(K_Q) = 1$. If $T$ is not $\omega$-stable then for some $a$ there is an isolated type $p \in S(a)$ without Morley rank, such that each type of smaller $\infty$-rank has Morley rank (see e.g. [Ne3]). It follows that $p$ has infinite multiplicity, hence if $b$ realizes $p$ then $p' = p|ab$ is non-isolated. By Lemma 3, we can assume that $tp(ab/Q)$ is good. Since $p' \not\subseteq \Phi$, we have $p' \vdash p''$ for some $p'' \in S(Qab)$. Clearly $p''$ is not $Q$-isolated over $ab$. Hence $I(K_Q) > 1$, a contradiction.

So $T$ is $\omega$-stable. It follows that the only model in $K_Q$ is prime over $Q$ in the usual sense.

Now suppose $Q' = \Phi(M')$ is countable. Wlog $Q' \subseteq Q$. If some $p \in S(Q')$ is non-isolated and good then also $p' = p|Q$ is non-isolated and good. $p'$ is realized in a model in $K_Q$ which is not prime over $Q$, a contradiction. We see that every model in $K_Q'$ is prime over $Q'$. Hence $T$ is $(\omega,\omega)$-categorical relative to $\Phi$ in the sense of [HHM].

I would like to stress the fact, implicit in the proof of Theorem 2, that the more ample $Q$ is, the larger $I(K_Q)$ is. To make this more explicit, we prove the following corollary, which deals with the case of strongly minimal $\Phi$, when we have a good notion of dimension of $Q$.

**Corollary 2.** Assume $T$ is small superstable, $\Phi$ is strongly minimal and $I(K_Q) > 1$. Then there are $k, l < \omega$ such that if $Q' = \Phi(M')$ is countable, $n > 2$ and

$$\text{dim}(Q') \geq \frac{k^n - k}{k - 1} + l$$

then $I(K_{Q'}) \geq n$.

**Proof.** By the Fact, $K_{Q'}$ has the prime model property. Also we can assume that there are countably many good pseudotypes over $Q'$, hence there is a $Q'$-saturated
model. Wlog the strongly minimal type \( r \in S(\emptyset) \) containing \( \Phi \) is eventually non-isolated. Let \( l \) be the largest number such that \( r^l \) is isolated. First we prove the following variant of Lemma 3.

(a) If \( p \in S(\emptyset) \) and \( \dim(Q') \geq w(p) + l \) then there is \( p' \in S(Q') \setminus B(Q') \) extending \( p \).

Indeed, let \( a \) realize \( p \) and \( C = Cb(a/\Phi(C)) \). Since \( C \subseteq acl(a) \), \( w(C) \leq w(a) \). Choose a Morley sequence \( I \) in \( r \) of size \( l \), with \( I \perp C \), and let \( Q'' \) be a model of \( T[\Phi] \) prime over \( CI \). Clearly \( w(C/I) = w(C) \). If \( d \in Q'' \) realizes \( r|I \) then \( tp(d/CI) \) is isolated and \( tp(d/I) \) is non-isolated, hence \( d \nsubseteq C(I) \). Thus \( \dim(r|I, Q'') \leq w(C) \), and \( \dim(Q') \leq w(C) + l \leq \dim(Q) \). In particular, \( Q'' \) may be embedded into \( Q' \).

Now (a) follows as in Lemma 3.

Now we find \( a, b \) as in the proof of Theorem 2. Let \( k = w(ab) \). Hence \( w(A_n) = (k^{n+2} - k)/(k - 1) \). If

\[
\dim(Q') \geq \frac{k^{n+2} - k}{k - 1} + l
\]

then by (a), for \( m \leq n \) we can find \( A'_m \equiv A_m \) with \( tp(A'_m/Q') \) good.

Let \( M_0 \) be \( Q' \)-prime, for \( 0 < m \leq n \) let \( M_m \) be \( Q' \)-prime over \( A'_m \) and let \( M_{n+1} \) be \( Q' \)-saturated. The proof of Theorem 2 shows that \( M_m \), \( m \leq n \), are non-isomorphic. The proof of Theorem 1 shows they are also non-isomorphic to \( M_{n+1} \).

Hence \( I(K_{Q'}) \geq n + 2 \).

We showed above an example of an \( \omega \)-stable \( T \) and \( Q \) with \( I(K_Q) = 3 \). It may be interesting to point that any such example should resemble the classical situation.

**Proposition.** Assume \( T \) is superstable, \( K_Q \) has the prime model property and \( I(K_Q) = 3 \). Then the 3 models in \( K_Q \) are: the \( Q' \)-prime one, the \( Q' \)-saturated one and the third model \( M \) which is neither \( Q' \)-prime nor \( Q' \)-saturated. \( M \) is characterized by the following condition.

(*) For every \( a \) with \( tp(a/Q) \) good and not \( Q' \)-isolated, \( M \) is isomorphic to a model \( Q' \)-prime over \( a \).

**Proof.** First we prove that \( M \) satisfies (*). Suppose \( tp(a/Q) \) is good and not \( Q' \)-isolated, and let \( N \) be \( Q' \)-prime over \( a \). So \( N \) is not \( Q' \)-prime. The proof of Theorem 1 shows \( N \) is not \( Q' \)-saturated. Hence \( N \) is isomorphic to \( M \). This shows (*). The other direction is equally easy.

Most of the results of this note may be adapted to the situation when \( Q \) is fixed pointwise by \( Aut(C) \), that is when the elements of \( Q \) are named by constants of the language.

**References**


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