THE INDICES, THE NULLITIES AND THE STABILITY OF TOTALLY GEODESIC SUBMANIFOLDS IN THE COMPLEX QUADRATIC HYPERSURFACES: $Q_m = SO(m+2)/SO(m) \times SO(2)$

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ABSTRACT. In the paper, the stability of totally geodesic submanifolds in the complex quadratic hypersurfaces: $Q_m = SO(m+2)/SO(m) \times SO(2)$ ($m > 1$) is discussed, and the indices, the nullities and the Killing nullities of totally geodesic submanifolds in $Q_m$ are calculated.

1. Introduction

It is always an interesting and important problem to find all stable minimal submanifolds in each symmetric space. In 1980, B.Y.Chen, P.F.Leung and T.Nagano gave the algorithm of determining the stability of totally geodesic submanifolds in compact symmetric spaces which was reformulated in 1987 by Y.Ohnita (cf. [1] and [2]). In [3], B.Y.Cheng and T.Nagano completely classified complete, connected, totally geodesic submanifolds of the complex quadratic hypersurface: $Q_m = SO(m+2)/SO(2) \times SO(m)$, $m > 1$. Using their results, we determine the indices and nullities of all totally geodesic submanifolds in $Q_m$, and discuss their stability.

Suppose $M$ is a compact minimal submanifold of a Riemannian manifold $N$ with metric $h$. The isometric immersion $\phi : M \to N$ is called stable if the second derivative of the volume $Vol(M, \phi_t^* h)$ at $t = 0$ is nonnegative for every smooth variation $\{\phi_t\}$ of $\phi$ with $\phi_0 = \phi$. Then we say that $M$ is a stable minimal submanifold of $N$. Choose a smooth variation $\{\phi_t\}$ of $\phi$ with $\phi_0 = \phi$ and $(\partial/\partial t)\phi_t(x)|_{t=0} = V_x(x \in M)$ for any vector field $V \in \Gamma(\phi^{-1}T(N))$. Then the classical second variational formula is given as follows:

$$(d^2/dt^2)Vol(M, \phi_t^* h)|_{t=0} = \int_M \langle \mathcal{J}(V^N), V^N \rangle dv$$

where $dv$ denotes the Riemannian measure of $(M, g)$ and $V^N$ the component of $V$ normal to $M$. $\mathcal{J}$ is a self-adjoint strongly elliptic linear differential operator of order 2 acting on the space $\Gamma(N(M))$ of smooth sections of the normal bundle $N(M)$, called the Jacobi operator of $\phi$. $\mathcal{J}$ has discrete eigenvalues $\mu_1 < \mu_2 < \cdots \to \infty$. Set

$$E_\mu = \{V \in \Gamma(N(M)) \mid \mathcal{J}(V) = \mu V \}.$$
We call the number $\sum_{\mu<0} \dim E_\mu$ the index of $\phi$ or the index of $M$ in $N$; denote it by $i(\phi)$ or $i(M)$. It is easy to see that $\phi$ is stable if and only if $i(\phi) = 0$. The number $\dim E_0$ is called the nullity of $\phi$ and denoted by $n(\phi)$ or $n(M)$. Define

$$P = \{ X^N \mid X \text{ is a Killing vector field on } N \} \subset \Gamma(N(M)).$$

Then $P \subset E_0$. The $\dim P$ is called the Killing nullity of $\phi$ and denoted by $n_k(\phi)$ or $n_k(M)$.

In particular, if $M$ is an $m$-dimensional compact totally geodesic submanifold immersed in a compact Riemannian symmetric space $N$ with metric $g_N$, the immersion $\phi: M \to N$ can be expressed as follows: There are compact symmetric pairs $(U, L)$ and $(G, K)$ with $N = U/L$, $M = G/K$ and

$$\phi: M = G/K \to N = U/L,$$

$$g_K \to \rho(g)L$$

where $\rho: G \to U$ is an analytic homomorphism with $\rho(K) \subset L$ and the injective differential $\rho: g \to u$ satisfying $\rho(m) \subset p$. Here $u = l + p$ and $g = k + m$ are the Cartan decompositions of $u$ and $g$, respectively. There exists an $adU$-invariant inner product $(,)$ on $u$ such that $(,)$ induces the metric $g_N$ on $N$. By $(,)$ we also denote the $adG$-invariant inner product on $g$ induced from $(,)$ through $\rho$. Let $m^+$ be the smooth orthogonal complement of $\rho(m)$ with $p$ relative to $(,)$, and $k^+$ the orthogonal complement of $\rho(k)$ in $l$. Put $g^+ = k^+ + m^+$. Then $g^+$ is the orthogonal complement of $\rho(g)$ in $u$ relative to $(,)$, and $g^+$ is $ad\rho(G)$-invariant. Let $\theta$ be the involutive automorphism of the symmetric pair $(U, L)$. Choose an orthogonal decomposition $g^+ = g_1^+ \oplus \cdots \oplus g_t^+$ such that each $g_i^+$ is an irreducible $ad\rho(G)$-invariant subspace with $\theta(g_i^+) = g_i^+$. Then the Casimir operator $C$ of the representation of $G$ on each $g_i^+$ is $a_i I$ for $a_i \in \mathbb{C}$. Put $g_i^\perp = k_i^+ + m_i^\perp$, where $k_i^+ = k^+ \cap g_i^+$ and $m_i^\perp = m^+ \cap g_i^+$.

**Theorem 1.1** ([1]). The index, nullity and Killing nullity of $\phi$ are given as follows:

1. $i(M) = \sum_{i=1}^{t} \sum_{\lambda \in D(G), a_\lambda > a_i} \dim \text{Hom}_K(V(\lambda), (m_i^\perp)^C) \dim V(\lambda);$  
2. $n(M) = \sum_{i=1}^{t} \sum_{\lambda \in D(G), a_\lambda = a_i} \dim \text{Hom}_K(V(\lambda), (m_i^\perp)^C) \dim V(\lambda);$  
3. $n_k(M) = \sum_{i=1, m_i^\perp \neq 0} \dim g_i^\perp$,

where $D(G)$ is the set of irreducible representations of $G$ and $a_\lambda$ is the eigenvalue for the Casimir operator of the irreducible $G$-module $(\lambda, V_\lambda)$ relative to $(,)$.

2. The space $Q_m$ and its totally geodesic submanifolds

The main result of [3] is the following theorem. It gave the classification of all complete, connected, totally geodesic submanifolds of the complex quadratic hypersurface: $Q_m = SO(2 + m)/SO(2) \times SO(m)$, $m > 1$.

**Theorem 2.1** ([3]). If $M$ is a maximal totally geodesic submanifold of $Q_m$, $M$ is one of the following three spaces:

1. $Q_{m-1};$
(2) a local Riemannian product of two spheres $S^p$ and $S^q$, $p + q = m$;

(3) the complex projective space $P(C^{n+1})$ of complex dimension $n$, $2n = m$.

If $M$ is a nonmaximal, totally geodesic submanifold of $Q_m$, $M$ is either contained in $Q_{m-1}$ in an appropriate position in $Q_m$, or the real projective space $P(R^{n+1})$ of real dimension $n$, $2n = m$, which is the intersection of $P(C^{n+1})$ in (3) and the local product space in (2) with $p = q = n$.

Now we have Riemannian symmetric spaces

$$N = Q_m = U/L, \ U = SO(m+2), \ L = SO(2) \times SO(m).$$

Then $u = l + p$ is the corresponding Cartan decomposition, where

$$l = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \ a \in R, \ B \in gl(m) \right\},$$

$$p = \left\{ \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} \mid A \text{ is a matrix of 2 rows and } m \text{ columns} \right\}.$$

From now on, we denote the set of matrices of $m$ rows and $n$ columns with elements in $R$ by $R(m,n)$.

If $M$ is a compact totally geodesic submanifold immersed in $N$, the immersion $\phi : M \to N$ can be expressed as follows: There exist a compact symmetric pair $(G,K)$ with $M = G/K$ and a homomorphism $\rho : G \to U$ such that $\phi$ has form as $gK \to \rho(g)L$. Then $\rho(K) \subset L$ and the injective differential $\rho : g \to u$ satisfies $\rho(m) \subset p$. We choose an ad$U$-invariant inner product on $u$ as $(X,Y) = TrXY (X,Y \in u)$; it induces the metric on $Q_m$. Let $g = k + m$ be the corresponding Cartan decomposition of $g$. Below we list the corresponding $\rho$, $k$ and $m$ in every case.

1. $M = Q_n \ (n < m), \ \rho : g \to u$ is

$$A \to \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in u, \ A \in g,$$

$$k = \left\{ \begin{pmatrix} A \\ B \\ 0 \end{pmatrix} \mid A \in gl(2), B \in gl(n), A^t = -A, B^t = -B \right\},$$

$$m = \left\{ \begin{pmatrix} 0 & A & 0 \\ -A^t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid A \in R(2,n) \right\}.$$

2. $M = S^p \times S^q \ (p + q = m), \ \rho : g \to u$ is

$$(A,B) \to \begin{pmatrix} b_{11} & 0 & b_{12} & b_{13} & \cdots & b_{1,q+1} \\ 0 & A & 0 & 0 & \cdots & 0 \\ b_{21} & 0 & b_{22} & b_{23} & \cdots & b_{2,q+1} \\ b_{31} & 0 & b_{32} & b_{33} & \cdots & b_{3,q+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{q+1,1} & 0 & b_{q+1,2} & b_{q+1,3} & \cdots & b_{q+1,q+1} \end{pmatrix}.$$
Here $A = (a_{ij}) \in gl(p+1), B = (b_{ij}) \in gl(q+1), A^t = -A, B^t = -B.$

$$k = \left\{ \begin{pmatrix} 0 & A \\ B & \end{pmatrix} \mid A \in gl(p), B \in gl(q), A^t = -A, B^t = -B \right\},$$

$$m = \left\{ \begin{pmatrix} 0 & 0 & 0 & X \\ 0 & 0 & Y & 0 \\ 0 & -Y^t & 0 & 0 \\ -X^t & 0 & 0 & 0 \end{pmatrix} \mid X \in \mathbb{R}(1,q), Y \in \mathbb{R}(1,p) \right\}.$$

3. $M = CP^n = SU(n+1)/SU(n) (m = 2n). \rho : su(n+1) \to so(2n+2)$ is $A + iB \mapsto P_{n+1}(A, B).$

Here $P_{n+1}(A, B)$ is

$$
\begin{pmatrix}
a_{11} & b_{11} & a_{12} & b_{12} & \cdots & a_{1,n+1} & b_{1,n+1} \\
-b_{11} & a_{11} & -b_{12} & a_{12} & \cdots & -b_{1,n+1} & a_{1,n+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{n+1,1} & b_{n+1,1} & a_{n+1,2} & b_{n+1,2} & \cdots & a_{n+1,n+1} & b_{n+1,n+1} \\
-b_{n+1,1} & a_{n+1,1} & -b_{n+1,2} & a_{n+1,2} & \cdots & -b_{n+1,n+1} & a_{n+1,n+1}
\end{pmatrix}
$$

where $A = (a_{ij}), B = (b_{ij}) \in gl(n+1), A^t = -A, B^t = -B, TrB = 0.$

$$k = \left\{ \begin{pmatrix} 0 & 0 \\ P_n(B, C) & \end{pmatrix} \mid B^t = -B, C^t = C, TrC = 0 \right\},$$

$$m = \left\{ \begin{pmatrix} 0 & O(A, B) \\ -O(A, B)^t & 0 \end{pmatrix} \mid O(A, B) \in \mathbb{R}(2,2n) \right\}$$

where

$$O(A, B) = \begin{pmatrix} a_{11} & b_1 & a_n & b_n \\ -b_1 & a_1 & -b_n & a_n \end{pmatrix}.$$

4. $M = RP^n = SO(n+1)/SO(n) (m = 2n). \rho : so(n+1) \to so(2n+2)$ is $A \mapsto P_{n+1}(A),$ and $P_{n+1}(A)$ equals

$$
\begin{pmatrix}
a_{11} & 0 & a_{12} & 0 & \cdots & a_{1,n+1} & 0 \\
0 & a_{11} & 0 & a_{12} & \cdots & 0 & a_{1,n+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{n+1,1} & 0 & a_{n+1,2} & 0 & \cdots & a_{n+1,n+1} & 0 \\
0 & a_{n+1,1} & 0 & a_{n+1,2} & \cdots & 0 & a_{n+1,n+1}
\end{pmatrix}
$$

where $A = (a_{ij}) \in so(n+1)$.

$$k = \left\{ \begin{pmatrix} 0 & 0 \\ P_n(A) & \end{pmatrix} \mid A \in so(n) \right\},$$

$$m = \left\{ \begin{pmatrix} 0 & T(A) \\ -T(A)^t & 0 \end{pmatrix} \mid T(A) \in \mathbb{R}(2,2n) \right\},$$

$$T(A) = \begin{pmatrix} a_1 & 0 & a_2 & 0 & \cdots & a_n & 0 \\ 0 & a_1 & 0 & a_2 & \cdots & 0 & a_n \end{pmatrix}.$$
3. The indices, nullities and stability of the geodesic submanifolds in $Q_m$

We choose the inner product $(X,Y) = TrXY$ on $u = so(m+2)$ which induces the metric on $Q_m$. In every case, we first calculate $g^\perp$, $m^\perp$ and the orthogonal decomposition of the $G$-module $(g^\perp)^C$.

1. $M = Q_n$ ($n < m$),

\[ g^\perp = \left\{ \begin{pmatrix} 0 & -B^t \\ -B & C \end{pmatrix} \mid B \in \mathbb{R}(n+2, m-n), C \in gl(m-n), C^t = -C \right\} \]

\[= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & -B^t \\ -B & 0 \end{pmatrix} \right\} = g_1^\perp \oplus g_2^\perp, \]

\[m_1^\perp = 0, \]

\[m_2^\perp = \left\{ \begin{pmatrix} 0 & 0 & D \\ 0 & 0 & 0 \\ -D^t & 0 & 0 \end{pmatrix} \mid D \in \mathbb{R}(2, m-n) \right\}. \]

Obviously the representation of $G$ on $g_1^\perp$ is trivial. Below we investigate the $G$-module $(g_2^\perp)^C$. Let $E_{ij}$ be the square matrix with entry 1 where the $i$-th row and $j$-th column meet, all other entries being 0.

1) If $n + 2 = 2s$, $m + 2 = 2(s + t)$, set $H_i = E_{2i-1,2i} - E_{2i,2i-1}$ ($1 \leq i \leq s$); then $h = span\mathbb{C}\{H_i\}$ is a Cartan subalgebra of $g^C$. Let $F_{ij} = E_{ij} - E_{ji}$ and

\[G_{jk}^+ = F_{2j-1,2k-1} \pm F_{2j,2k} + i(F_{2j-1,2k} \mp F_{2j,2k-1}) \quad (j \neq k). \]

It is obvious that

\[(g_2^\perp)^C = span\mathbb{C}\{G_{jk}^+ \mid 1 \leq j \leq s < k \leq s + t \text{ or } 1 \leq k \leq s < j \leq s + t\},\]

\[(adH)G_{jk}^+ = (e_j(H) - e_k(H))G_{jk}^+,\]

\[(adH)G_{jk}^- = sgn(j-k)(e_j(H) + e_k(H))G_{jk}^-\]

for $H \in h$, where $e_j(H_k) = -i\delta_{jk}$. So the set of weights of the $G$-module $(g_2^\perp)^C$ is

$$\Phi = \{ \pm e_1, \ldots, \pm e_s \mid 1 \leq j \leq s \}.$$ 

We can choose a set of simple roots of $(g)^C$ as

$$R = \{ \alpha_j = e_j - e_{j+1}, \alpha_s = e_{s-1} + e_s \mid 1 \leq j \leq s \}.$$ 

In this orientation, all the dominant weights which are also highest weights are $\{e_1, \ldots, e_s\}$. But the representation having the highest weight $e_1$ is exactly the first basic representation $\omega_1$ of $so(n+2)$, so

\[(g_2^\perp)^C = \omega_1 \oplus \cdots \oplus \omega_{2t}. \]

Generally, for the representation having the highest weight $\lambda = \sum_i a_i\omega_i$ of $g$, the Casimir operator $C$ has the form $C = a_\lambda I$, where

$$a_\lambda = -(a_i\omega_i + 2\rho, a_j\omega_j) = -(a_i a_j g_{ij} + 2\sum_{j,k} a_j g_{jk}).$$

Here $\rho$ is half the sum of positive roots, and the information about $g_{ij}$ may be found in [4].
Therefore, the Casimir operator $C$ of $g$ on $(g^\perp)^C$ has the action $a_iI$ ($i = 1, 2$); here $a_1 = 0$ and $a_2 = -(2s - 1)$. So

\[
\begin{align*}
\{ & \lambda \in D(G) \mid a_\lambda > a_2 \} = \{ 0 \}, \\
\{ & \lambda \in D(G) \mid a_\lambda = a_2 \} = \{ \omega_1 \}.
\end{align*}
\]

It is easy to see that $k^C = CH_1 \oplus so(n, C)$. Now $m_1^2 = 0$, and the set of weights of the $CH_1$-module $(m_2^\perp)^C$ is

\[
\{1, \ldots, 1, -1, \ldots, -1\}_{2t}.
\]

Since the action of $so(n, C)$ on $(m_2^\perp)^C$ is trivial, as a $K$-module,

\[
(m_2^\perp)^C = \bigoplus_{2t}(\lambda(1, 0, \ldots, 0) \oplus \lambda(-1, 0, \ldots, 0)).
\]

By the branching rule of representations [5], $\dim \omega_1 = 2s = n + 2$,

\[
\omega_1 = \begin{cases}
\lambda(1, 1, 0, \ldots, 0) \oplus \lambda(1, 0, \ldots, 0) \oplus \lambda(-1, 0, \ldots, 0), & s > 3 \\
\lambda(1, 0, 1, 0, \ldots, 0) \oplus \lambda(1, 0, \ldots, 0) \oplus \lambda(-1, 0, \ldots, 0), & s = 3.
\end{cases}
\]

So we have

\[
\begin{align*}
(1) \quad & i(M) = 0, \\
(2) \quad & n(M) = (n + 2)(m - n), \\
(3) \quad & n_K(M) = (n + 2)(m - n).
\end{align*}
\]

2) If $n + 2 = 2s, m + 2 = 2(s + t) + 1$, let

\[
D_j^\pm = F_{2j-1,m+2} \pm iF_{2j,m+2}, \quad 1 \leq j \leq s.
\]

Then

\[
(adH)D_j^\pm = \pm e_j(H)D_j^\pm, \quad 1 \leq j \leq s, \quad H \in h.
\]

The set of dominant weights which are also the highest weights of the $g$-module $(g^\perp)^C$ is

\[
\{\omega_1, \ldots, \omega_1\}_{2t+1}.
\]

Using methods similar to 1) we get

\[
\begin{align*}
(4) \quad & i(M) = 0, \\
(5) \quad & n(M) = 2s(2t + 1) = (n + 2)(m - n), \\
(6) \quad & n_K(M) = (n + 2)(m - n).
\end{align*}
\]

For other cases of $m + 2$ and $n + 2$, we can get the same results by similar discussion.
2. \( \phi : M = S^p \times S^q \rightarrow Q_m (p+q = m) \). By a series of similarity transformations, we have

\[
g = \left\{ \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \mid A \in so(p+1), \ B \in so(q+1) \right\},
\]

\[
k = \left\{ \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \mid A \in so(p), \ B \in so(q) \right\},
\]

\[
m = \left\{ \begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix} \mid X \in \mathbb{R}(1,p), \ Y \in \mathbb{R}(1,q) \right\},
\]

\[
g^\perp = \left\{ \begin{pmatrix} 0 & -B^t \\ -B^t & 0 \end{pmatrix} \mid B \in \mathbb{R}(p+1, q+1) \right\},
\]

\[
m^\perp = \left\{ \begin{pmatrix} 0 & Y \\ 0 & 0 \\ X & 0 \\ -X^t & 0 \end{pmatrix} \mid X \in \mathbb{R}(1,p), \ Y \in \mathbb{R}(1,q) \right\}.
\]

If \( p + 1 = 2s, \ q + 1 = 2t \), choose \( H_i = E_{2i-1,2i} - E_{2i-1,1} \) \( (1 \leq i \leq s + t) \); then \( h = \text{span}\{H_i\}_{i=0}^{s+t} \) is a Cartan subalgebra of \( g^C = (so(2s) \oplus so(2t))^C \). By a similar discussion as before, we know \( (g^\perp)^C \) is an irreducible \( g \)-module. It has the highest weight \( \lambda(1,0, \ldots, 0-1,0, \ldots, 0) \), so it is the tensor product of the first basic representations of \( so(2s) \) and \( so(2t) \). And

\[
a = - (\lambda(1,0, \ldots, 0-1,0, \ldots, 0) + 2\rho, \lambda(1,0, \ldots, 0-1,0, \ldots, 0)) \\
= - (\lambda_1(1,0, \ldots, 0) + 2\rho_1, \lambda_1(1,0, \ldots, 0)) \\
- (\lambda_2(1,0, \ldots, 0) + 2\rho_2, \lambda_2(1,0, \ldots, 0)) \\
= -2(s + t - 1),
\]

\( \{\lambda \in D(G)|a_\lambda > a\} = \{0, \lambda(1,0, \ldots, 0-0,0, \ldots, 0), \lambda(0,0, \ldots, 0-1,0, \ldots, 0)\} \),

\( \{\lambda \in D(G)|a_\lambda = a\} = \{\lambda(1,0, \ldots, 0-1,0, \ldots, 0)\} \).

Considering \( k = so(p) \oplus so(q) \), we have

\[
(m^\perp)^C = (m_1^\perp)^C \oplus (m_2^\perp)^C
\]

as a \( K \)-module, where

\[
m_1^\perp = \left\{ \begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix} \mid X \in \mathbb{R}(1,p) \right\},
\]

\[
m_2^\perp = \left\{ \begin{pmatrix} 0 & Y \\ -Y^t & 0 \end{pmatrix} \mid Y \in \mathbb{R}(1,q) \right\}.
\]
It is obvious that the actions of $so(p)$ on $m^\perp_1$ and $so(q)$ on $m^\perp_2$ are their standard representations. Thus, as a $K$-module,
\[(m^\perp)^C = \lambda'(1,0,\ldots,0-0,\ldots,0) \oplus \lambda'(0,\ldots,0-1,0,\ldots,0).\]

But
\[
\begin{align*}
\lambda(1,0,\ldots,0-0,\ldots,0) &= \lambda'(0,\ldots,0-0,\ldots,0) \oplus \lambda'(1,0,\ldots,0-0,\ldots,0), \\
\dim \lambda(1,0,\ldots,0-0,\ldots,0) &= p+1, \\
\dim \lambda(0,\ldots,0-1,0,\ldots,0) &= q+1, \\
\lambda(1,0,\ldots,0-1,0,\ldots,0) &= \lambda'(1,0,\ldots,0-1,0,\ldots,0) \\
\oplus \lambda'(1,0,\ldots,0-0,\ldots,0) &= \lambda'(0,\ldots,0-1,0,\ldots,0) \oplus \lambda'(0,\ldots,0-0,\ldots,0), \\
\dim \lambda(1,0,\ldots,0-1,0,\ldots,0) &= (p+1)(q+1).
\end{align*}
\]

So we have
\[
\begin{align*}
i(M) &= m+2, \\
n(M) &= 2(p+1)(q+1), \\
n_K(M) &= (p+1)(q+1).
\end{align*}
\]

We have the same results for the other cases of $p$ and $q$ by similar discussion.

3. $\phi : CP^n = SU(n+1)/SU(n) \rightarrow Q_{2n}$ ($m = 2n$),
\[
g^\perp = \left\{Q(A,B)|A, B \in so(n+1)\right\},
\]
\[
m^\perp = \left\{\left(\begin{array}{cc}
0 & s(a,b) \\
-s(a,b)^t & 0
\end{array}\right) \mid s(a,b) \in R(2,2n)\right\}.
\]

Here, for $A = (a_{ij}), B = (b_{ij}), Q(A,B)$ is
\[
\left(\begin{array}{ccccccc}
a_{11} & b_{11} & a_{12} & b_{12} & \cdots & a_{1,n+1} & b_{1,n+1} \\
b_{11} & -a_{11} & b_{12} & -a_{12} & \cdots & b_{1,n+1} & -a_{1,n+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{n+1,1} & b_{n+1,1} & a_{n+1,2} & b_{n+1,2} & \cdots & a_{n+1,n+1} & b_{n+1,n+1} \\
b_{n+1,1} & -a_{n+1,1} & b_{n+1,2} & -a_{n+1,2} & \cdots & b_{n+1,n+1} & -a_{n+1,n+1}
\end{array}\right),
\]
\[
s(a,b) = \left(\begin{array}{cccc}
a_1 & b_1 & \cdots & a_n \\
b_1 & -a_1 & \cdots & b_n
\end{array}\right).
\]

A Cartan subalgebra of $g$ can be imbedded in $u$ as
\[h = \{\sum_{i=1}^{n+1} c_iH_i \mid \sum_i c_i = 0\}.
\]

The set of the weights of the $g$-module $(g^\perp)^C = \sum_{j\neq k} CG_{jk}$ is
\[\{\pm(e_j + e_k) \mid j \neq k\}.
\]

Choose $\{e_1 - e_2, e_2 - e_3, \ldots, e_n - e_{n+1}\}$ as the set of simple roots of $g^C$. The dominant weights of $(g^\perp)^C$ are $e_1+e_2$ and $-e_n-e_{n+1}$. By comparing the dimensions among $(g^\perp)^C$ and the spaces of the representations having the highest weights $\{e_1 + e_2, -e_n - e_{n+1}\}$, we know
\[(g^\perp)^C = \lambda(0,1,0,\ldots,0) \oplus \lambda(0,\ldots,0,1,0).
\]

Since $(m^\perp)^C = \sum_{k=1}^{n+1}(CG_{1k}^+ \oplus CG_{k1}^-)$ and $\{\sum_{i=2}^{n+1} c_iH_i \mid \sum_i c_i = 0\}$ is a Cartan subalgebra of $k = su(n)$, the set of the weights of $(m^\perp)^C$ is $\{\pm e_k\}_{k=1}^{n+1}$. If we choose
\{e_i - e_{i-1}|2 \leq i \leq n\} as the set of simple roots of \(k\), the dominant weights of the \(k\)-module \((m^\perp)^C\) are \(e_2\) and \(-e_{n+1}\). So

\[ (m^\perp)^C = \lambda'(1,0,\ldots,0) \oplus \lambda'(0,\ldots,0,1), \]
\[ a = -\left(\lambda(0,1,0,\ldots,0) + 2\rho, \lambda(0,1,0,\ldots,0)\right) \]
\[ = -\left(\lambda(0,\ldots,0,1,0) + 2\rho, \lambda(0,\ldots,0,1,0)\right) \]
\[ = -2(n-1)(n+2)/(n+1), \]
\[ a_{\lambda_k} = -k(n-k+1)(n+2)/(n+1) \] for \(\lambda_k = \lambda(0,\ldots,0,1,0,\ldots,0)\);

\[ \{\lambda \in D(G)|a_\lambda > a\} = \{0,\lambda_1,\lambda_n\}. \]
\[ \{\lambda \in D(G)|a_\lambda = a\} = \{\lambda_2,\lambda_{n-1}\}. \]

By the branching rule of representations, we know

\[ \lambda_1 = \lambda'(1,0,\ldots,0) \oplus \lambda'(0,\ldots,0), \]
\[ \lambda_n = \lambda'(0,\ldots,0,1) \oplus \lambda'(0,\ldots,0,1,0), \]
\[ \lambda_2 = \lambda'(0,1,0,\ldots,0) \oplus \lambda'(1,0,\ldots,0), \]
\[ \lambda_{n-1} = \lambda'(0,\ldots,0,1) \oplus \lambda'(0,\ldots,0,1,0), \]
\[ \dim \lambda_1 = \dim \lambda_n = n+1, \quad \dim \lambda_2 = \dim \lambda_{n-1} = n(n+1)/2. \]

Therefore we get

(10) \quad i(M) = 2(n+1),
(11) \quad n(M) = n(n+1),
(12) \quad n_K(M) = n(n+1).

4. \(\mathbb{R}P^n = SO(n+1)/SO(n) \longrightarrow Q_{2n} (m = 2n)\). Through a series of similarity transformations, we have

\[ g = \left\{ \begin{pmatrix} A & \_

A \\
A & \
A \end{pmatrix} | A \in so(n+1) \right\}, \]
\[ k = \left\{ \begin{pmatrix} & \_

& A \\
& 0 \\
0 & & \_

& \end{pmatrix} | A \in so(n) \right\}, \]
\[ m = \left\{ \begin{pmatrix} & -X^t & 0 & 0 \\
0 & 0 & 0 & X \\
0 & 0 & 0 & -X^t \\
0 & 0 & 0 & 0 \end{pmatrix} | X \in \mathbb{R}(1,n) \right\}, \]
\[ g^\perp = g_1^\perp \oplus g_2^\perp \oplus g_3^\perp, \]
\[ m^\perp = m_1^\perp \oplus m_2^\perp \oplus m_3^\perp, \]
where
\[
g_{1}^{\perp} = \left\{ \begin{pmatrix} C \\
-C \end{pmatrix} \mid C \in so(n+1) \right\},
\]
\[
g_{2}^{\perp} = \left\{ \begin{pmatrix} 0 \\
B \\
-B^{t} \end{pmatrix} \mid B = B^{t} \in gl(n+1) \right\},
\]
\[
g_{3}^{\perp} = \left\{ \begin{pmatrix} 0 \\
B \\
-B^{t} \end{pmatrix} \mid B \in so(n+1) \right\},
\]
\[
m_{1}^{\perp} = 0, \quad m_{2}^{\perp} = \left\{ \begin{pmatrix} X \\
X^{t} \\
-X^{t} \end{pmatrix} \mid X \in R(1,n) \right\},
\]
\[
m_{3}^{\perp} = \left\{ \begin{pmatrix} X \\
X^{t} \\
-X^{t} \end{pmatrix} \mid X \in R(1,n) \right\}.
\]

It is easy to see
\[
(g_{2}^{\perp})^{C} = (g_{21}^{\perp})^{C} \oplus (g_{22}^{\perp})^{C},
\]
where
\[
(g_{21}^{\perp})^{C} = \lambda(0,\ldots,0), \quad (g_{22}^{\perp})^{C} = \lambda(2,0,\ldots,0),
\]
\[
(m_{21}^{\perp})^{C} = 0, \quad (m_{22}^{\perp})^{C} = (m_{2}^{\perp})^{C}.
\]

Without loss of generality, suppose \(n = 2s - 1\); then
\[
(g_{1}^{\perp})^{C} = \lambda(0,1,0,\ldots,0), \quad (g_{3}^{\perp})^{C} = \lambda(0,1,0,\ldots,0).\]

As a \(K\)-module, it is easy to see
\[
(m_{2}^{\perp})^{C} = \lambda'(1,0,\ldots,0), \quad (m_{3}^{\perp})^{C} = \lambda'(1,0,\ldots,0).
\]
So \(a_{21} = -4(s-2), \quad a_{3} = -4(s-1)\). For \(\lambda_{k} = \lambda(0,\ldots,0,1,0,\ldots,0)\), we have
\[
a_{\lambda_{k}} = -k(2s-k) \quad (1 \leq k \leq s-2), \quad a_{\lambda_{s}} = a_{\lambda_{-1}} = -s^{2}/2.
\]

{\lambda \in D(G) \mid a_{\lambda} > a_{21}} = \begin{cases}
\{\lambda_{1},\lambda_{2}\}, & s > 8, \\
\{\lambda_{1},\lambda_{2},\lambda_{s-1},\lambda_{s}\}, & 3 \leq s \leq 7, \\
\{\lambda_{1},\lambda_{2},\lambda_{3}\}, & s = 3,
\end{cases}

{\lambda \in D(G) \mid a_{\lambda} = a_{21}} = \lambda(2,0,\ldots,0),

{\lambda \in D(G) \mid a_{\lambda} > a_{3}} = \begin{cases}
\lambda_{1}, & s > 6, \\
\{\lambda_{1},\lambda_{s-1},\lambda_{s}\}, & s \leq 6.
\end{cases}

{\lambda \in D(G) \mid a_{\lambda} = a_{3}} = \lambda_{2}.
By the branching rule of representations, if \( s > 3 \) we get
\[
\begin{align*}
\lambda_1 &= \lambda'(1,0,\ldots,0) \oplus \lambda'(0,\ldots,0), \quad \dim \lambda_1 = n + 1, \\
\lambda(2,0,\ldots,0) &= \lambda'(2,0,\ldots,0) \oplus \lambda'(1,0,\ldots,0) \oplus \lambda'(0,\ldots,0), \\
\dim \lambda(2,0,\ldots,0) &= \frac{(n+1)(n+2)}{2} - 1, \\
\lambda_2 &= \lambda'(0,1,0,\ldots,0) \oplus \lambda'(1,0,\ldots,0), \quad \dim \lambda_2 = n(n+1)/2, \\
\lambda_s &= \lambda'(0,\ldots,0,1) = \lambda_{s-1}, \\
\dim \lambda_{s-1} &= \dim \lambda_s = 2^{s-1}.
\end{align*}
\]

If \( s = 3 \)
\[
\begin{align*}
\lambda(2,0,0) &= \lambda'(2,0), \quad \lambda_1 = \lambda'(1,0) = \lambda_3, \\
\lambda_2 &= \lambda'(0,0) \oplus \lambda'(0,1), \\
\dim \lambda(2,0,0) &= 10, \quad \dim \lambda_1 = \dim \lambda_3 = 4, \quad \dim \lambda_2 = 6.
\end{align*}
\]

Thus we have
\[
\begin{align*}
i(M) &= \begin{cases} 
(n+1)(n+4)/2, & n > 5, \\
16, & n = 5,
\end{cases} \\
n(M) &= \begin{cases} 
(n+1)^2 - 1, & n > 5, \\
0, & n = 5,
\end{cases} \\
n_K(M) &= (n+1)^2 - 1.
\end{align*}
\]

Finally, we get the following theorem:

**Theorem 3.1.** The indices, the nullities and the Killing nullities of the totally geodesic submanifolds in \( Q_m \) are listed in the following table. Among all the totally geodesic submanifolds in \( Q_m \), only \( Q_n(n < m) \) are stable; the nullity and the Killing nullity are equivalent except for \( S^p \times S^q \) (\( p + q = m \)).

<table>
<thead>
<tr>
<th>( M )</th>
<th>( i(M) )</th>
<th>( n(M) )</th>
<th>( n_K(M) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q_n )</td>
<td>0</td>
<td>((n+2)(m-n))</td>
<td>((n+2)(m-n))</td>
</tr>
<tr>
<td>( S^p \times S^q(p + q = m) )</td>
<td>( m + 2 )</td>
<td>((p+1)(q+1))</td>
<td>((p+1)(q+1))</td>
</tr>
<tr>
<td>( CP^n(m = 2n) )</td>
<td>( 2(n+1) )</td>
<td>((n+1)(n+1))</td>
<td>((n+1)(n+1))</td>
</tr>
<tr>
<td>( RP^n(m = 2n &gt; 10) )</td>
<td>((n+1)(n+4)/2)</td>
<td>((n+1)^2 - 1)</td>
<td>((n+1)^2 - 1)</td>
</tr>
</tbody>
</table>

**References**


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