THE ZEROS OF THE FIRST TWO DERIVATIVES
OF A MEROMORPHIC FUNCTION

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Abstract. We prove a theorem which implies the following: if \( f \) is meromor-
phic of finite order in the plane and \( f' \) and \( f'' \) have only finitely many zeros,
then \( f \) has only finitely many poles.

1. Introduction

We begin with the following theorem, using the term meromorphic henceforth
to mean meromorphic in the plane.

Theorem A. Suppose that \( f \) is meromorphic and that \( f \) and \( f^{(k)} \) have only finitely
many zeros, for some \( k \geq 2 \). Then \( f \) has the form \( f(z) = R(z) \exp(P(z)) \), with
\( R \) rational and \( P \) a polynomial. In particular, \( f \) has finite order and only finitely
many poles.

Theorem A was conjectured by Hayman in 1959 [9], [10], [11], and was proved
for \( k \geq 3 \) by Frank [6]. The case \( k = 2 \) was settled in [13], having been proved by
Mues [16] for functions of finite lower order. Simple examples show that Theorem
A is not true for \( k = 1 \) (see, however, [4]). Now, it is easy to give examples of
total functions \( f \) such that \( f'' \) has no zeros, but the following conjecture seems
reasonable.

Conjecture 1. If \( f \) is meromorphic of finite order and \( f'' \) has only finitely many
zeros, then \( f \) has only finitely many poles.

We remark that (see Satz 5 of [15]), for any \( k \geq 2 \) and for any transcendental
meromorphic function \( f \), setting \( L = -f^{(k+1)}/(k+1)f^{(k)} \) and applying the first
fundamental theorem to \( L' + L^2 \) gives

\[
(k - 1)N_1(r, f) \leq 2N_2(r, f) + 2N(r, 1/f^{(k)}) + O(\log rT(r, f^{(k)}))
\]

as \( r \) tends to infinity outside a set of finite measure. Here \( N_1(r, f) \) counts the simple
poles of \( f \), while \( N_2(r, f) \) counts the points at which \( f \) has multiple poles (see [10],
[12] for notation). Thus Conjecture 1 is true if \( f \) has at most finitely many multiple
poles. We remark further that it has been conjectured by Gol’dberg that \( \overline{N}(r, f) \)
may be estimated in terms of \( N(r, 1/f'') \) and an error term which is \( o(T(r, f)) \), and
we refer the reader to [7], [8], [15], [18], in particular with regard to the related
Mues conjecture that \( \sum_{a \in \mathbb{C}} \delta(a, f') \leq 1 \).
We note further that Conjecture 1 is false for $f$ of infinite order, as the following construction shows. Let $\Pi$ be any entire function having infinitely many zeros, all simple, and use the Mittag-Leffler theorem to construct an entire function $h$ such that $g(z) = \Pi^{-1}e^{h}$ has Laurent series development $g(z) = -2/(z - a) + O(|z - a|)$ near each zero $a$ of $\Pi$. It is then easy to see that $f''/f' = g$ defines a meromorphic function such that $f'$ and $f''$ have no zeros. For $f$ of finite order, no such examples are possible.

**Theorem 1.** Let $f$ be meromorphic of finite order such that $f'$ and $f''$ have only finitely many zeros. Then $f''/f'$ is rational and, in particular, $f$ has only finitely many poles.

Theorem 1 will be deduced from a result concerning the class $B$, which consists of those meromorphic functions $f$ such that the set of finite singularities of the inverse function $f^{-1}$ is bounded. This means that there is some $S > 0$ such that $f$ has no critical values or asymptotic values $w$ with $S < |w| < \infty$. The following lemma was proved by Eremenko and Lyubich [5] (see also [1], [2]).

**Lemma B.** Let $f$ be transcendental and meromorphic, in the class $B$, with $f(0) \neq \infty$. Then there exist positive constants $c, R$ such that we have, for all $z$, the estimate $|zf'(z)/f(z)| \geq c \log^{+}|f(z)/R|$.

Lemma B is proved by noting that there exists $R > 0$ such that $|f(z)| < R$ on a path from 0 to $\infty$ and then, provided $R$ is large enough, applying Bloch’s theorem to the function $\log(f^{-1}(e^{w}))$ in $\Re(w) > \log R$. We shall prove the following:

**Theorem 2.** Let $f$ be transcendental meromorphic, in the class $B$, such that $f''/f'$ has only finitely many critical values, and suppose that $f''/f'$ has only finitely many zeros. Then $f''/f'$ is rational.

The assumption in Theorem 2 that $f$ is in the class $B$ is not redundant, as the examples $f(z) = z - \tan z, g(z) = \int_{0}^{z} \int_{0}^{s} \exp(s^{2})dsdt$ show.

**Corollary.** Let $f$ be transcendental and meromorphic of finite order, with only finitely many critical values, and suppose that $f''/f'$ has only finitely many zeros. Then $f''/f'$ is rational.

This corollary obviously establishes Theorem 1, and itself follows at once from Theorem 2 because, if $f$ has finite order and only finitely many critical values, a recent result of Bergweiler and Eremenko [3] implies that $f$ has only finitely many asymptotic values.

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2. **Proof of Theorem 2**

We assume the existence of a meromorphic function $f$ in the class $B$ such that $f''/f'$ is transcendental but has only finitely many zeros. We may clearly assume that $f(0)$ is finite. We note that $f'/f''$ has only finitely many poles, and we set $u(z) = \log |f'(z)/f''(z)|$.

By a result of Lewis, Rossi and Weitsman [14], there is a path $\Gamma$, starting at $z_{0}$, say, and tending to infinity, such that $u(z)/|\log|z|| \to +\infty$ as $z \to \infty$ on $\Gamma$, while the part of $\Gamma$ joining $z_{0}$ to $z$ has length at most $\exp(o(u(z)))$. This length estimate may be found as (3.7) of [14] and is stated explicitly in [17]. We parametrize $\Gamma$ with
respect to arc length and, for $n$ a large positive integer, define $s_n$ by $s_n = \sup\{s : u(\Gamma(s)) \leq n\}$. If $n$ is large and $s_n \leq s < s_{n+1}$ we have

\[
\left| \int_{\Gamma(s)} (f''(z)/f'(z))dz \right| \leq \sum_{k=n}^{\infty} e^{-k} e^{o(k+1)} \leq c_1 e^{-n/2},
\]

using $c_j$ to denote positive constants. Thus $f'$ tends to a finite non-zero value $b$ as $z$ tends to infinity on $\Gamma$, and we can assume without loss of generality that $b = 1$. Now we have, for $n$ large and $s_n \leq s < s_{n+1}$, the estimate $|f''(\Gamma(s))-1| \leq c_2|\log |f''(\Gamma(s))|| \leq e^{-n/4}$ and so

\[
\left| \int_{\Gamma(s)} (f''(z)-1)dz \right| \leq \sum_{k=n}^{\infty} e^{-k/4} e^{o(k+1)} = o(1).
\]

Thus $f(z) = z + O(1)$ and $zf''(z)/f(z) = 1 + o(1)$ as $z$ tends to infinity on $\Gamma$, which plainly contradicts Lemma B and proves Theorem 2.

We remark finally that a modification of the above proof shows that Theorem 2 holds with $f''/f'$ replaced by $f^{(k+1)}/f^{(k)}$, for any $k \geq 2$.

REFERENCES


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