EXTREMAL PROBLEMS IN MINKOWSKI SPACE RELATED TO MINIMAL NETWORKS

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Abstract. We solve the following problem of Z. Füredi, J. C. Lagarias and F. Morgan (1991): Is there an upper-bound polynomial in \( n \) for the largest cardinality of a set \( S \) of unit vectors in an \( n \)-dimensional Minkowski space (or Banach space) such that the sum of any subset has norm less than 1? We prove that \( |S| \leq 2n \) and that equality holds if the space is linearly isometric to \( \ell_\infty^n \), the space with an \( n \)-cube as unit ball. We also remark on similar questions they raised that arose out of the study of singularities in length-minimizing networks in Minkowski spaces.

1. Introduction

In [LM] Lawlor and Morgan derived a geometrical description for the singularities (Steiner points) of a length-minimizing network connecting a finite set of points in a smooth Minkowski space (finite-dimensional Banach space). In Euclidean space the geometrical description is equivalent to the classical result that at a singularity three line segments meet at 120° angles. See also [BG], [M] and [CR] for a discussion of length-minimizing networks and their history. The geometrical description of Lawlor and Morgan leads to extremal problems of a combinatorial type in strictly convex Minkowski spaces. Such problems are considered in [FLM]. In this note we briefly remark on some of these problems and solve one of the open problems stated in [FLM] (see Theorem 3).

2. Preliminaries

We denote the real numbers by \( \mathbb{R} \) and the real vector space of \( n \)-tuples of real numbers by \( \mathbb{R}^n \). The coordinates of a vector \( \mathbf{x} \in \mathbb{R}^n \) will be denoted by \( \mathbf{x} = (x(1), x(2), \ldots, x(n)) \). The standard basis \( \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \) will be used, where \( \mathbf{e}_i \) is the vector for which \( e_i(i) = 1 \) and \( e_i(j) = 0 \) for \( i \neq j \). A Minkowski space (or finite-dimensional Banach space) \((\mathbb{R}^n, \Phi)\) is \( \mathbb{R}^n \) endowed with a norm \( \Phi \). A Minkowski space is strictly convex if \( \Phi(\mathbf{x}) = \Phi(\mathbf{y}) = 1, \mathbf{x} \neq \mathbf{y} \) implies \( \Phi(\mathbf{x} + \mathbf{y}) < 2 \).
We denote by $\ell^n_p$ the $n$-dimensional Minkowski space with norm

$$\Phi_p(x) = \left( \sum_{i=1}^n |x(i)|^p \right)^{1/p}$$

for $p \geq 1$, and by $\ell^n_\infty$ the space with norm

$$\Phi_\infty(x) = \max_{1 \leq i \leq n} |x(i)|.$$ 

We now state Auerbach’s Lemma which relates the spaces $\ell^n_p$ and $\ell^n_\infty$ to an arbitrary Minkowski space in $n$ dimensions. A proof may be found in [Pi, page 29].

**Auerbach’s Lemma.** For any Minkowski space $(\mathbb{R}^n, \Phi)$ there exists a linear isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi_\infty(x) \leq \Phi(Tx) \leq \Phi_1(x)$, i.e.

$$\max_{1 \leq i \leq n} |x(i)| \leq \Phi(Tx) \leq \sum_{i=1}^n |x(i)|.$$ 

We denote the $n$-dimensional Lebesgue measure (or volume) of measurable $V \subseteq \mathbb{R}^n$ by $\text{vol}(V)$. If $U, V \subseteq \mathbb{R}^n$, then we define $U + V = \{ u + v \mid u \in U, v \in V \}$. The Brunn-Minkowski inequality relates the volumes of compact $U$ and $V$ to that of $U + V$. A proof may be found in [BZ].

**Brunn-Minkowski inequality.** If $U, V \subseteq \mathbb{R}^n$ are compact, then

$$(\text{vol}(U + V))^{1/n} \geq (\text{vol}(U))^{1/n} + (\text{vol}(V))^{1/n}.$$ 

3. Extremal problems

From now on $S$ will denote a finite set of unit vectors in a Minkowski space. In [FLM] the following type of extremal problems is considered: Find the largest cardinality of $S$ satisfying a selection of the following conditions:

$$(A) \quad \Phi(\sum_{x \in J} x) \leq 1 \text{ for all } J \subseteq S \quad \text{(the strong collapsing condition)},$$

$$\quad \Phi(x + y) \leq 1 \text{ for all } x, y \in S, x \neq y \quad \text{(the weak collapsing condition)},$$

$$(B) \quad \sum_{x \in S} x = 0 \quad \text{(the strong balancing condition)},$$

$$(B') \quad 0 \text{ is in the relative interior of the convex hull of } S \quad \text{(the weak balancing condition)}.$$ 

See [LM] and [FLM] for the connection between these conditions and minimal networks. In [FLM] it is proved that $(A')$ and $(B')$ together give an upper bound $|S| \leq 2n$ for an arbitrary Minkowski space, and $|S| \leq n + 1$ for strictly convex Minkowski spaces. In [LM] it is proved that there exist a strictly convex norm on $\mathbb{R}^n$ and a subset $S$ of $n + 1$ unit vectors satisfying $(A)$ and $(B)$. $|S| = 2n$ is attained in, for example, $\ell^n_\infty$ with $S = \{ \pm e_i \mid 1 \leq i \leq n \}$, in which case even the strong conditions $(A)$ and $(B)$ hold. However, there are other Minkowski spaces where equality is also attained (see Theorem 1). This is to be contrasted with Theorem 3, where we show that the extreme case for $S$ satisfying $(A)$ and $(B)$ can only be attained for $\ell^n_\infty$.

**Theorem 1.** For infinitely many $n \geq 1$ there exists a set of unit vectors $S = \{ x_1, \ldots, x_{2n} \} \subseteq \ell^n_1$ satisfying $(A')$ and the strong balancing condition $(B)$. In particular, such a set exists if a Hadamard matrix of order $n$ exists.
Proof. We recall that an \( n \times n \) Hadamard matrix \( H \) consists of \((\pm 1)\)-entries such that \( HH^T = nI \), and such matrices exist for infinitely many \( n \) (see [vLW, Chapter 18]). We let \( v_1, \ldots, v_n \) be the column vectors of \( H \), and set \( x_i = \frac{1}{n} v_i \) for \( i = 1, \ldots, n \). Then \( S := \{ \pm x_i \mid 1 \leq i \leq n \} \) is a set of \( 2n \) unit vectors. Since the column vectors of \( H \) are orthogonal, \( \langle v_i, v_j \rangle = 0 \) for \( i \neq j \), implying that \( \Phi_1(x_i + x_j) = 1 \) and \( \Phi_1(x_i - x_j) = 1 \) for all \( i \neq j \). It follows that \( S \) satisfies \((A') \) and \((B) \). \( \square \)

The question now is what happens if there is no balancing condition present. In [FLM] an upper bound of \( |S| < 3^n \) is derived from the weak collapsing condition \((A') \) alone using a volume argument. Using the Brunn-Minkowski inequality we obtain a sharper bound (Theorem 2). In [FLM] a strictly convex norm and a set \( S \) of unit vectors with \( |S| \geq (1.02)^n \) satisfying \((A') \) are constructed for all sufficiently large \( n \). It would be interesting to find the greatest lower bound of the \( \alpha \)'s for which \( |S| \leq \alpha^n \) for any set \( S \) of unit vectors in an arbitrary Minkowski space satisfying \((A') \), and sufficiently large \( n \).

**Theorem 2.** If a set \( S \) of unit vectors in \( \mathbb{R}^n \) satisfies \((A') \), then \( |S| < 2^{n+1} \).

**Proof.** We denote the closed unit ball with centre \( x \) and radius \( r \) by \( B(x, r) = \{ y \in \mathbb{R}^n \mid \Phi(x - y) \leq r \} \), and the volume of a ball of unit radius by \( \beta \). For distinct \( x, y \in S \) we obtain from the triangle inequality that \( \Phi(x - y) \geq 1 \). Let \( k = |S| \). We partition \( S \) into two sets \( S_1 \) and \( S_2 \) of sizes \( k/2 \) and \( k/2 \), respectively. Let \( V_i = B(0, \frac{1}{2}) \cup \bigcup_{x \in S_i} B(x, \frac{1}{2}) \) for \( i = 1, 2 \). Clearly, each \( V_i \) consists of closed balls with disjoint interiors, and therefore, \( \text{vol}(V_1) = \beta([k/2] + 1)2^{-n} \) and \( \text{vol}(V_2) = \beta([k/2] + 1)2^{-n} \). Using \((A') \) we obtain \( V_1 + V_2 \subseteq B(0, 2) \) and \( \text{vol}(V_1 + V_2) \leq 2^n \beta \).

By the Brunn-Minkowski inequality we now have

\[
2\beta^{1/n} \geq \frac{1}{2} \beta^{1/n} (\lceil k/2 \rceil + 1)^{1/n} + \frac{1}{2} \beta (\lceil k/2 \rceil + 1)^{1/n} > \beta^{1/n} (k/2)^{1/n}
\]

and \( |S| < 2^{n+1} \). \( \square \)

From the above proof we actually find that if \( \Phi(x) = \Phi(y) = 1 \) and \( \Phi(x + y) \leq 1 \) imply \( \Phi(x - y) \geq 1 \) for \( r > 1 \), then \( |S| \leq 2(1 + 1/r)^n + 1 \) for \( S \) satisfying \((A') \). Such is the case for \( \ell_p^n \). It follows from the Clarkson inequality [C] for \( p \geq 2 \), and the Hanner inequality [H] for \( 1 < p < 2 \), that \( r \) may be taken to be \( 3^{1/p} \) for \( p \geq 2 \), and \( (2p - 1)^{1/p} \) for \( 1 < p < 2 \).

For \( \ell_2^n \) an upper bound \( |S| \leq 2^n \) holds: If the coordinates of two unit vectors \( x \) and \( y \) have the same sequence of signs, i.e. \( \text{sgn}(x(i)) = \text{sgn}(y(i)) \) for all \( i = 1, \ldots, n \), then \( \Phi_1(x + y) = 2 \), contradicting \((A') \). In the Euclidean case \( \ell_2^n \) we of course have \( |S| \leq 3 \), independent of \( n \). For \( \ell_\infty^n \), the sharp upper bound \( |S| \leq 2n \) holds: If \( |S| \geq 2n + 1 \), then by the pigeon-hole principle there are three vectors \( x, y, z \in S \) and an \( i \in \{1, \ldots, n\} \) such that \( |x(i)| = |y(i)| = |z(i)| = 1 \). Some two of these vectors will have the same sign in the \( i \)-th coordinate, and their sum will then have a norm of 2.

In [FLM, Problem 3.7] the question is asked whether the strong collapsing condition \((A) \) on its own gives an upper bound for \( |S| \) that is polynomial in \( n \). A linear upper bound may be derived by the same technique as in Theorem 2. We partition the elements of \( S \) except for at most 2 into subsets \( S_1, \ldots, S_k \) of size 3, where \( k = \lceil |S|/3 \rceil \). For \( i = 1, \ldots, k \) let

\[
V_i = \bigcup_{x \in S_i} B(x, \frac{1}{2}) \cup \bigcup_{x, y \in S_i, x \neq y} B(x + y, \frac{1}{2}).
\]
From (A) it follows that each $V_i$ consists of 6 balls with disjoint interiors and $V_1 + \cdots + V_k \subseteq B(0, \frac{1}{2}k + 1)$. By the Brunn-Minkowski inequality we obtain
\[
\frac{1}{2}k+1 \geq \frac{1}{2}6^{1/n}k \quad \text{and} \quad k \leq 2/(6^{1/n} - 1).
\]
Therefore, $|S| \leq 6/(6^{1/n} - 1) + 2 < (6/\ln 6)n$, after some calculus. This bound is not sharp, however. In the following theorem we derive the sharp upper bound $|S| \leq 2n$.

**Theorem 3.** Let $S$ be a finite set of unit vectors in a Minkowski space $(\mathbb{R}^n, \Phi)$ satisfying the collapsing condition (A). Then $|S| \leq 2n$, and equality holds iff $(\mathbb{R}^n, \Phi)$ is linearly isometric to $\ell_n^\infty$, with $S$ corresponding to the set $\{\pm e_i \mid 1 \leq i \leq n\}$ under any isometry.

**Proof.** By Auerbach’s Lemma we may assume (after applying a linear isomorphism of $\mathbb{R}^n$) that for any vector $x \in \mathbb{R}^n$ the inequalities (1) hold, with $T$ now the identity. Choose $m$ distinct vectors $x_1, \ldots, x_m$ from $S$. By (1) we have
\[
\sum_{j=1}^m |x_j(i)| \geq 1 \quad \text{for all } j = 1, \ldots, m. \tag{2}
\]
Suppose that for some coordinate $i \in \{1, \ldots, n\}$ we have
\[
\sum_{j=1}^m x_j(i) > 1.
\]
Then
\[
\Phi(\sum_{j=1}^m x_j) \geq \sum_{j=1}^m x_j(i)
\]
by (1), contradicting (A). Therefore,
\[
\sum_{j=1}^m x_j(i) \leq 1 \quad \text{for all } i = 1, \ldots, n, \tag{3}
\]
and similarly,
\[
\sum_{j=1}^m -x_j(i) \leq 1 \quad \text{for all } i = 1, \ldots, n. \tag{4}
\]
From (3) and (4) it follows that $\sum_{j=1}^m |x_j(i)| \leq 2$ for all $i = 1, \ldots, n$, and from (2) we have
\[
m \leq \sum_{j=1}^m \sum_{i=1}^n |x_j(i)| \leq 2n \tag{5}
\]
and $|S| \leq 2n$.

If $|S| = 2n$ for some set of unit vectors $S = \{x_1, \ldots, x_{2n}\}$ satisfying (A), then equality must hold in (5), (3) and (4). Therefore, $\sum_{j=1}^{2n} x_j = 0$, showing that in the extreme case the strong balancing condition (B) must be satisfied. We now show that conditions (A) and (B) together with the assumption $|S| = 2n$ imply that $(\mathbb{R}^n, \Phi)$ is linearly isometric to $\ell_n^\infty$, and $S$ corresponds to $\{\pm e_i \mid 1 \leq i \leq n\}$, as claimed in [FLM].

We recall Theorem 3.1 of [FLM]:
If \( (\mathbb{R}^n, \Phi) \) is a Minkowski space and \( S \) is a set of unit vectors satisfying \((A') \) and \((B') \), then \( |S| \leq 2n \), and if equality holds, then \( S \) corresponds to \( \{ \pm e_i \mid 1 \leq i \leq n \} \) under some linear isomorphism.

We therefore have \( S = \{ \pm x_i \mid 1 \leq i \leq n \} \), where the \( x_i \)'s are linearly independent. We first show that if \( (\mathbb{R}^n, \Phi) = \ell_\infty^n \), then \( S = \{ \pm e_i \mid 1 \leq i \leq n \} \) must hold.

For \( i = 1, \ldots, n \) choose \( j_i \in \{ 1, \ldots, n \} \) such that \( |x_i(j_i)| = 1 \). After renaming, we may assume \( x_i(j_i) = 1 \). The \( j_i \)'s must be distinct, otherwise \((A)\) is contradicted.

We may therefore rename the \( x_i \)'s to obtain \( x_i(i) = 1 \) for \( i = 1, \ldots, n \). If we have \( x_i(j) \neq 0 \) for some \( i \neq j \), then either \( \Phi_\infty(x_i + x_j) > 1 \) or \( \Phi_\infty(-x_i + x_j) > 1 \), contradicting \((A)\). Therefore, \( x_i(j) = 0 \) for all \( i \neq j \), and we have \( S = \{ \pm e_i \mid i = 1, \ldots, n \} \).

To show that in fact \( (\mathbb{R}^n, \Phi) \) is linearly isometric to \( \ell_\infty^n \), we use the following theorem of Petty [Pe] (see also [FLM, Theorem 2.1]):

If \( T \) is a subset of a Minkowski space \( (\mathbb{R}^n, \Phi) \) such that \( \Phi(x - y) = 1 \) for all \( x, y \in T, x \neq y \), then \( |T| \leq 2^n \), with equality iff \( (\mathbb{R}^n, \Phi) \) is linearly isometric to \( \ell_\infty^n \).

We will apply this theorem to the set \( T = \{ \sum_{i \in A} x_i \mid A \subseteq \{1, \ldots, n\} \} \). Obviously \( |T| = 2^n \). We now show that

\[
\Phi(\sum_{i \in A} x_i - \sum_{i \in B} x_i) = 1 \quad \text{for all } A, B \subseteq \{1, \ldots, n\}, A \neq B,
\]

thus completing the proof. Firstly, we have

\[
\Phi(\sum_{i \in A} x_i - \sum_{i \in B} x_i) = \Phi(\sum_{i \in A \setminus B} x_i + \sum_{i \in B \setminus A} -x_i) \leq 1
\]

by \((A)\). Secondly, \( A \setminus B \neq \emptyset \) or \( B \setminus A \neq \emptyset \), since \( A \neq B \). We assume without loss that \( A \setminus B \neq \emptyset \) and choose \( j \in A \setminus B \). Then

\[
2 = \Phi(2x_j) \leq \Phi(\sum_{i \in A} x_i - \sum_{i \in B} x_i) + \Phi(\sum_{i \in B \cup \{j\}} x_i - \sum_{i \in A \setminus \{j\}} x_i)
\]

\[
\leq \Phi(\sum_{i \in A} x_i - \sum_{i \in B} x_i) + 1,
\]

showing that \((6)\) holds. \( \square \)

For strictly convex norms the bound in the above theorem should perhaps be \( |S| \leq n + 1 \), but this seems to require a new idea.

References

[H] O. Hanner, On the uniform convexity of \( L^p \) and \( \mathcal{B}^p \), Ark. Mat. 3 (1956), 239–244. MR 17:987


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