

NONRESONANCE PROBLEMS
FOR DIFFERENTIAL INCLUSIONS
IN SEPARABLE BANACH SPACES

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ABSTRACT. Let X be a real separable Banach space. The boundary value problem

$$(B) \quad \begin{aligned} x' &\in A(t)x + F(t, x), \quad t \in \mathcal{R}_+, \\ Ux &= a, \end{aligned}$$

is studied on the infinite interval $R_+ = [0, \infty)$. Here, the closed and densely defined linear operator $A(t) : X \supset D(A) \rightarrow X$, $t \in \mathcal{R}_+$, generates an evolution operator $W(t, s)$. The function $F : \mathcal{R}_+ \times X \rightarrow 2^X$ is measurable in its first variable, upper semicontinuous in its second and has weakly compact and convex values. Either F is bounded and $W(t, s)$ is compact for $t > s$, or F is compact and $W(t, s)$ is equicontinuous. The mapping $U : C_b(\mathcal{R}_+, X) \rightarrow X$ is a bounded linear operator and $a \in X$ is fixed. The nonresonance problem is solved by using Ma's fixed point theorem along with a recent result of Przeradzki which characterizes the compact sets in $C_b(\mathcal{R}_+, X)$.

1. INTRODUCTION. PRELIMINARIES

Throughout the paper, X is a real, separable Banach space with dual space X^* . We also denote by $L(X)$ the space of bounded linear operators $V : X \rightarrow X$. The symbols \mathcal{R} , \mathcal{R}_+ and \mathcal{R}^+ stand for the sets $(-\infty, \infty)$, $[0, \infty)$ and $(0, \infty)$, respectively. We consider differential inclusions of the type

$$(1) \quad x' \in A(t)x + F(t, x), \quad x(0) = x_0,$$

where, for $t \in \mathcal{R}_+$, the linear operator $A(t) : X \supset D(A) \rightarrow X$ is the generator of a compact evolution operator $W(t, s) \in L(X)$, $0 \leq s \leq t < +\infty$, and $F : \mathcal{R}_+ \times X \rightarrow X$ is a set-valued mapping. We always assume that $A(t)$ is closed and densely defined on its domain $D(A) \equiv D(A(0))$, which is independent of t . In Section 2 we obtain the solvability of the associated boundary value problem

$$(B) \quad \begin{aligned} x' &\in A(t)x + F(t, x), \quad t \in \mathcal{R}_+, \\ Ux &= a, \end{aligned}$$

where $U : C_b \rightarrow X$ is a bounded linear operator and $a \in X$ is fixed. Here, $C_b \equiv C_b(\mathcal{R}_+, X)$ is the Banach space of all bounded and continuous functions

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$f : \mathcal{R}_+ \rightarrow X$ associated with the sup-norm. The “single-valued” problem (B) was studied in finite-dimensional cases by Kartsatos in [6] and [8]. The Eilenberg-Montgomery fixed point theorem on finite intervals was used by Kartsatos [7]. These results were extended by Anichini [1] to more general quasilinear cases. A boundary value problem for closed and densely defined principal parts was considered by Kartsatos [8], but the boundary conditions there could not include the periodic ones. A problem involving boundary conditions on finite intervals, like the ones used in this paper, was studied by Ward [17]. In 1979, Zecca and Zezza [18] considered the case of an infinite interval and differential inclusions in Banach spaces, but for bounded linear operators in the principal parts of the corresponding inclusions. Unbounded linear operators for differential inclusions were considered by Papageorgiou [14] and Hu and Papageorgiou [5]. It seems that the problem considered here has not been attacked, in this generality, by any other author. Further references on these problems can be found in the above cited papers and the references therein.

We let

$$\mathcal{P}_{f(c)}(X) \equiv \{A \subset X : A \text{ is nonempty, closed (and convex)}\}.$$

We also let

$$\mathcal{P}_{wkc}(X) \equiv \{A \subset X : A \text{ is nonempty, convex and weakly compact}\}.$$

We denote by \mathcal{N} the set of all positive integers. We also denote by Δ the set $\{(t, s) : 0 \leq s \leq t \leq T\}$. The symbol $B_r(0)$ stands for the open ball of X with radius $r > 0$. We denote by $\text{co}A$ (resp. $\overline{\text{co}}A$) the convex (resp. closed convex) hull of the set A . For an interval $J \subset \mathcal{R}_+$, the symbol $L^1_+(J, \mathcal{R})$ stands for the set of all nonnegative functions in $L^1(J, \mathcal{R})$.

Definition 1. Let J be an interval of \mathcal{R}_+ . Let $H : J \rightarrow \mathcal{P}_f(X)$. Then H is called “measurable” if for every open set $D \subset X$ the set $\{t \in J : H(t) \cap D \neq \emptyset\}$ is measurable. For such a measurable function H , and for $J = (0, T)$, $T \in (0, \infty]$, we set

$$S_H^1(T) \equiv \{f \in L^1(J, X) \text{ such that } f(t) \in H(t) \text{ a.e. in } J\}.$$

A function $H : J \rightarrow 2^X$ is called “integrably bounded” if there exists $m \in L^1_+(J, \mathcal{R})$ such that, for almost all $t \in J$, we have $H(t) \subset m(t)\overline{B_1(0)}$.

The following result can be found in Papageorgiou [13, Theorem 3.1].

Lemma 1. *If $H : J \rightarrow \mathcal{P}_{wkc}(X)$ is integrably bounded, where J is an interval of \mathcal{R}_+ with endpoints $0, T$ ($T \leq \infty$), then the set $S_H^1(T)$ is a nonempty, weakly compact and convex subset of $L^1((0, T), X)$.*

For various measurability results concerning multi-valued functions, the reader is referred to Himmelberg [4]. We also follow Papageorgiou [14] for facts involving the measurability/integrability of the various mappings considered in this paper.

The evolution operators $W(t, s)$ considered in this paper (cf. Friedman [3]) satisfy the following conditions.

- (j1) $W(t, s) \in L(X)$ for every $(t, s) \in \Delta$ and for each $x \in X$ the mapping $(t, s) \rightarrow W(t, s)x$ is continuous;
- (j2) $W(t, s)W(s, r) = W(t, r)$ whenever $0 \leq r \leq s \leq t \leq T$;
- (j3) $W(t, t) = I$, i.e., the identity operator on X for all $t \in [0, T]$;

(j4) the mapping $(s, T] \ni t \rightarrow W(t, s) \in L(X)$ is continuous with respect to the uniform operator topology of $L(X)$. Moreover, this continuity is uniform with respect to s lying in sets bounded away from t , i.e., as long as $t - s \geq \beta$ for any fixed $\beta > 0$.

We observe that an evolution operator $W(t, s)$ satisfying the above conditions (j1)-(j3) is uniformly bounded, i.e., there exists a constant $p(W) > 0$ such that

$$\|W(t, s)\| \leq p(W), \quad (t, s) \in \Delta.$$

This follows easily from the Uniform Boundedness Principle.

The following lemma is an important tool in this paper. Its proof is given for the sake of completeness and future reference.

Lemma 2. Let $\{f_\alpha(t)\}_{\alpha \in \mathcal{A}}$ be a set of $L^1((0, T), X)$ -functions such that

$$\|f_\alpha(t)\| \leq f(t), \quad \text{for almost all } t \in [0, T] \text{ and all } \alpha \in \mathcal{A},$$

where $f \in L^1_+(\!(0, T), \mathcal{R})$. Assume that conditions (j1)-(j4) are true for the evolution operator $W(t, s)$. Then the set

$$M_1 \equiv \{u \in C([0, T], X) : u(t) = \int_0^t W(t, s)f_\alpha(s)ds, \text{ for some } \alpha \in \mathcal{A}\}$$

is equicontinuous on the interval $[0, T]$.

Proof. From Theorem 6.5 of Himmelberg [4], we know that $s \rightarrow W(t, s)f_\alpha(s)$ is measurable. We also know that this mapping is integrably bounded over the interval $[0, t]$. Obviously, we may assume that the function $f(t)$ has a positive integral over the interval $[0, T]$. It suffices to show that M_1 is equicontinuous at each $t_0 \in [0, T]$. Fix $t_0 \in (0, T)$ and let $\epsilon > 0$ and $\lambda \in (0, t_0)$ be such that $t_0 + \lambda < T$ and

$$p(W) \int_\Omega f(s)ds < \epsilon/4,$$

where Ω is any subset of $[0, T]$ with measure $m(\Omega) \leq 2\lambda$. Now, let $\delta = \delta(t_0, \lambda, \epsilon) < \lambda$ be such that

$$(2) \quad \|W(t_0 + h, s) - W(t_0, s)\| \leq \left[\epsilon / \left(2 \int_0^T f(s)ds \right) \right], \quad s \in [0, t_0 - \lambda], \quad |h| \in [0, \delta].$$

This is possible by the uniform continuity of the mapping $[0, T] \ni t \rightarrow W(t, s) \in L(X)$, for $t - s \geq \lambda$. Then, for $u \in M_1$ and some $\alpha \in \mathcal{A}$,

$$\begin{aligned} u(t_0 + h) - u(t_0) &= \int_0^{t_0+h} W(t_0 + h, s)f_\alpha(s)ds - \int_0^{t_0} W(t_0, s)f_\alpha(s)ds \\ &= \int_{t_0-\lambda}^{t_0+h} W(t_0 + h, s)f_\alpha(s)ds - \int_{t_0-\lambda}^{t_0} W(t_0, s)f_\alpha(s)ds \\ &\quad + \int_0^{t_0-\lambda} [W(t_0 + h, s) - W(t_0, s)]f_\alpha(s)ds. \end{aligned}$$

We have

$$\left\| \int_{t_0-\lambda}^{t_0+h} W(t_0 + h, s)f_\alpha(s)ds \right\| \leq p(W) \int_{t_0-\lambda}^{t_0+h} f(s)ds < \epsilon/4,$$

$$\left\| \int_{t_0-\lambda}^{t_0} W(t_0, s) f_\alpha(s) ds \right\| \leq p(W) \int_{t_0-\lambda}^{t_0} f(s) ds < \epsilon/4.$$

From these estimates and (2), it follows that

$$\|u(t_0 + h) - u(t_0)\| < \epsilon, \text{ for all } h \in \mathcal{R} \text{ with } |h| \leq \delta.$$

Thus, M_1 is equicontinuous at every $t_0 \in (0, T)$. The equicontinuity of M_1 at $t_0 = 0$ follows from

$$\|u(h)\| \leq p(W) \int_0^h f(s) ds, \text{ for every } h \in [0, T].$$

To see the equicontinuity of M_1 at $t_0 = T$, it suffices to observe that

$$\begin{aligned} u(T) - u(T-h) &= \int_0^{T-2\lambda} W(T, s) f_\alpha(s) ds + \int_{T-2\lambda}^T W(T, s) f_\alpha(s) ds \\ &\quad - \left(\int_0^{T-2\lambda} W(T-h, s) f_\alpha(s) ds + \int_{T-2\lambda}^{T-h} W(T-h, s) f_\alpha(s) ds \right) \\ &= \int_0^{T-2\lambda} [W(T, s) - W(T-h, s)] f_\alpha(s) ds \\ &\quad + \int_{T-2\lambda}^T W(T, s) f_\alpha(s) ds - \int_{T-2\lambda}^{T-h} W(T-h, s) f_\alpha(s) ds, \end{aligned}$$

where we have assumed that: given $\epsilon > 0$ we pick $\lambda \in (0, T/2)$ so that

$$p(W) \int_\Omega f(s) ds < \epsilon/4,$$

for every measurable set $\Omega \subset [0, T]$ with $m(\Omega) < 2\lambda$. We have also assumed $\delta = \delta(\epsilon, \lambda, T) \in (0, \lambda)$ is chosen so that

$$\sup_{s \in [0, T-2\lambda]} \|W(T, s) - W(T-h, s)\| < \left[\epsilon / \left(2 \int_0^T f(s) ds \right) \right],$$

for all $h \in \mathcal{R}_+$ with $h \in [0, \delta]$. The rest of the proof follows as above. It is therefore omitted. \square

The function $F(t, x)$ is always assumed to be defined on $\mathcal{R}_+ \times X$ and such that $F(t, x(t))$ admits a measurable selection for all $x \in C(\mathcal{R}_+, X)$.

Definition 2. A function $x \in C(\mathcal{R}_+, X)$ is called a “mild solution” of (1) if there exists $f \in S_{F(\cdot, x(\cdot))}^1(\infty)$ such that

$$x(t) = W(t, 0)x_0 + \int_0^t W(t, s) f(s) dt, \quad t \in \mathcal{R}_+.$$

2. BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS

The following condition will be needed in the sequel.

(i1) The operator \tilde{U} , defined by

$$\tilde{U}x \equiv U(W(\cdot, 0)x), \quad x \in X,$$

has a bounded inverse $\tilde{U}^{-1} : X \rightarrow X$.

A mild solution $x \in C(\mathcal{R}_+, X)$ of problem (1) satisfies $Ux = a$, under condition (i1), if and only if it satisfies

$$x(t) = W(t, 0)\tilde{U}^{-1}[a - U\phi(\cdot, f)] + \int_0^t W(t, s)f(s)ds, \quad t \in \mathcal{R}_+,$$

where

$$\phi(t, f) \equiv \int_0^t W(t, s)f(s)ds.$$

The function f here is an $L^1(\mathcal{R}^+, X)$ -function such that $f(t) \in F(t, x(t))$ for almost all $t \in \mathcal{R}_+$, i.e., $f \in S_{F(\cdot, x(\cdot))}^1(\infty)$. Such a function $x(t)$, $t \in \mathcal{R}_+$, is called a “mild solution” of problem (B).

Kartsatos introduced in [6] a method for studying such problems in \mathcal{R}^n . In the present setting, this method is based on “solving” first a certain problem in each Banach space $C_m = C_m(\mathcal{R}_+, X)$ of truncated functions defined by

$$C_m = \{f \in C(\mathcal{R}_+, X) : f(t) = f(m) \text{ for all } t \in [m, \infty)\}.$$

A sequence $\{x_m\}$ of “solutions”, with $x_m \in C_m$, would then produce the desired mild solution of problem (3). Unfortunately, we have not been able to apply this method to the present setting. We leave it thus as an open problem.

In another vein, one may work directly with the space C_b . This has been possible, since 1992, because Przeradzki gave in [16] the following characterization of the compact sets in C_b .

Theorem A. *A set $B \subset C_b$ is compact if and only if the following conditions are satisfied:*

- (i) *for every $t \in R_+$, the set $\{x(t) : x \in B\}$ is relatively compact in X ;*
- (ii) *for every $\alpha > 0$, the set B is equicontinuous on the interval $[0, \alpha]$;*
- (iii) *for every $\epsilon > 0$ there exist $T = T(\epsilon) > 0$ and $\delta = \delta(\epsilon) > 0$ such that if $x, y \in B$ with $\|x(T) - y(T)\| \leq \delta$, then $\|x(t) - y(t)\| \leq \epsilon$ for all $t \in [T, \infty)$.*

Przeradzki gave his result for compact subsets B of $C(\mathcal{R}, X)$, but his proof carries over trivially to our case above. Theorem 1 below uses Theorem A in order to solve problem (B). In what follows,

$$B_M^C(0) \equiv \{f \in C_b : \|f\|_\infty \leq M\},$$

where M is a fixed positive constant. A function $F : \mathcal{R}_+ \times X \rightarrow 2^X$ is “bounded” if, for every bounded interval $J \subset \mathcal{R}_+$ and every bounded set $B \subset X$, there exists a bounded set $Q \subset X$ such that $F(t, x) \subset Q$, for almost all $t \in J$ and all $x \in B$. F is “compact” if the bounded set Q is replaced by a compact set Q .

Theorem 1. *Let $A(t) : X \supset D(A) \rightarrow X$, $t \in \mathcal{R}_+$, generate an evolution operator $W(t, s)$ satisfying conditions (j1)-(j4) on every compact interval $[0, T]$, $T > 0$. Let condition (i1) be satisfied and suppose that $F : \mathcal{R}_+ \times X \rightarrow \mathcal{P}_{fc}(X)$ is measurable in its first variable and upper semicontinuous in its second. Assume that either (a) F is bounded and $W(t, s)$ is compact for $t > s$, or (b) F is compact. Assume that for every $M > 0$ and every $\epsilon > 0$ there exists an $L^1(\mathcal{R}^+, X)$ -function $b = b(M, \epsilon)$, an integrably bounded function $H_M : \mathcal{R}_+ \rightarrow \mathcal{P}_{wkc}(X)$ and $T = T(M, \epsilon) > 0$ such that*

$$F(t, x) \subset H_M(t), \text{ for almost all } t \in \mathcal{R}_+,$$

and

$$F(t, x) - b(t) \subset \epsilon B_1(0), \text{ for almost all } t \geq T \text{ and all } x \in B_M(0).$$

Moreover, there exist constants $\alpha, N > 0$ such that

$$\|W(t, s)\| \leq N \exp\{-\alpha(t-s)\}, \text{ for all } t, s \in \mathcal{R}_+ \text{ with } t \geq s.$$

Let $a \in X$ be a fixed vector. Then the operator V defined by

$$Vx \equiv \left\{ y \in C_b : y(t) = W(t, 0)\tilde{U}^{-1}[a - U\phi(\cdot, f)] + \int_0^t W(t, s)f(s)ds, t \in \mathcal{R}_+, \text{ for some } f \in S_{F(\cdot, x(\cdot))}^1(\infty) \right\}$$

is upper semicontinuous on bounded subsets of C_b and maps bounded subsets of C_b onto relatively compact subsets of C_b . If, in addition, all possible solutions $x_\lambda \in C_b$ of the inclusion $x - \lambda Vx \ni 0$, $\lambda \in (0, 1)$, are uniformly bounded, then problem (B) has a mild solution.

Proof. We show first that V maps bounded subsets of C_b onto relatively compact subsets of the same space. We intend to prove first that the set

$$K_t = \left\{ \int_0^t W(t, s)f(s)ds : f \in S_{F(\cdot, x(\cdot))}^1(\infty), \text{ for some } x \in B_M^C(0) \right\}$$

is relatively compact in X for every $t \in \mathcal{R}_+$. This is obvious for $t = 0$. We assume that $t > 0$ and that assumption (a) holds. Then our conclusion follows as in the proof of Lemma 2.2 of Ward [17]. Let assumption (b) hold, let $x \in B_M^C(0)$ and let $f \in S_{F(\cdot, x(\cdot))}^1(\infty)$. Since $\|x(t)\| \leq M$, there exists a compact, convex set $Q \subset X$ such that

$$f(s) \in F(s, x(s)) \subset Q, \text{ for almost all } s \in [0, t].$$

We claim that the set

$$B_t \equiv W(t, [0, t])Q = \{W(t, s)x : s \in [0, t], x \in Q\}$$

is compact. To see this, we let $\{y_n\} \subset B_t$. Then there exist sequences $\{s_n\} \subset [0, t]$, $\{x_n\} \subset Q$ such that $y_n = W(t, s_n)x_n$, $n \in \mathcal{N}$. We may assume that $s_n \rightarrow s_0 \in [0, t]$ and $x_n \rightarrow x_0 \in Q$ as $n \rightarrow \infty$. Let $y_0 = W(t, s_0)x_0$. Then

$$\begin{aligned} \|y_n - y_0\| &= \|W(t, s_n)x_n - W(t, s_0)x_0\| \\ &\leq \|W(t, s_n)x_n - W(t, s_n)x_0\| + \|W(t, s_n)x_0 - W(t, s_0)x_0\| \\ &\leq \|W(t, s_n)\| \|x_n - x_0\| + \|W(t, s_n)x_0 - W(t, s_0)x_0\| \\ &\leq N \|x_n - x_0\| + \|W(t, s_n)x_0 - W(t, s_0)x_0\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Now we apply the Mean Value Theorem (cf. Aubin and Cellina [2, p. 21]) in order to get

$$\int_0^t W(t, s)f(s)ds \in t\overline{\text{co}}B_t.$$

This implies that $K_t \subset t\overline{\text{co}}B_t$ and proves the relative compactness of the set K_t , $t \in \mathcal{R}_+$.

We now show the stability of the set $D \equiv VB_M^C(0)$, i.e., we show that for every $\epsilon > 0$ there exists $T = T(\epsilon) > 0$ and $\delta = \delta(\epsilon) > 0$ such that: if $u \in Vx$ and

$v \in Vy$ are such that $\|u(T) - v(T)\| < \delta$, then $\|u(t) - v(t)\| \leq \epsilon$ for every $t \geq T$. We follow, partly, the proof of Theorem 2 of Przeradzki [16]. Fix $\epsilon > 0$ and choose $T_1 = T_1(\epsilon) > 0$ such that

$$F(t, x) - b(t) \subset [\epsilon\alpha/(8N)]B_1(0), \text{ for almost all } t > T_1 \text{ and all } x \in B_M(0).$$

Assume that the bounded set Q is such that $F(t, x) \subset Q$, for almost all $t \in [0, T_1]$ and all $x \in B_M(0)$, and let \widetilde{M} be an upper bound for Q . Choose $T = T(\epsilon) \geq T_1$ so large that for $t \geq T$ and $s \leq T_1$ we have

$$\|W(t, s)\| \leq N \exp\{-\alpha(T - T_1)\} < \frac{\epsilon}{8\widetilde{M}T_1}.$$

Letting $u \in Vx$, $v \in Vy$, for some $x, y \in B_M^C(0)$, we see that there exist $f \in S_{F(\cdot, x(\cdot))}^1(\infty)$, $g \in S_{F(\cdot, y(\cdot))}^1(\infty)$ such that

$$u(t) = W(t, 0)\widetilde{U}^{-1}[a - U\phi(\cdot, f)] + \int_0^t W(t, s)f(s)ds, \quad t \in \mathcal{R}_+,$$

and

$$v(t) = W(t, 0)\widetilde{U}^{-1}[a - U\phi(\cdot, g)] + \int_0^t W(t, s)g(s)ds, \quad t \in \mathcal{R}_+.$$

Consequently, for $t \geq T$,

$$\begin{aligned} \left\| \int_0^t W(t, s)[f(s) - g(s)]ds \right\| &\leq \left\| \int_{[0, t] \cap \{s > T_1\}} W(t, s)[f(s) - g(s)]ds \right\| \\ &\quad + \left\| \int_{[0, t] \cap \{s \leq T_1\}} W(t, s)[f(s) - g(s)]ds \right\| \\ &\leq \left\| \int_{[0, t] \cap \{s > T_1\}} W(t, s)[f(s) - b(s)]ds \right\| \\ &\quad + \left\| \int_{[0, t] \cap \{s > T_1\}} W(t, s)[g(s) - b(s)]ds \right\| \\ &\quad + \left\| \int_{[0, t] \cap \{s \leq T_1\}} W(t, s)[f(s) - g(s)]ds \right\| \\ &\leq 2 \int_{[0, t] \cap \{s > T_1\}} \|W(t, s)\| \sup_{\|x\| \leq M} |F(s, x) - b(s)| ds \\ &\quad + 2 \int_{[0, t] \cap \{s \leq T_1\}} \|W(t, s)\| \widetilde{M} ds \\ &\leq \frac{\epsilon\alpha}{4N} \int_0^t N \exp\{-\alpha(t - s)\} ds + \frac{2\widetilde{M}\epsilon T_1}{8\widetilde{M}T_1} = \frac{\epsilon}{2}. \end{aligned}$$

We can now use the fact that $H_M(t)$, $t \in \mathcal{R}_+$, is integrably bounded and the fact that

$$\|W(t, 0)\| \rightarrow 0, \text{ as } t \rightarrow \infty,$$

to show that we may choose $T \geq T_1$ so large that

$$\|u(t) - v(t)\| \leq \epsilon, \text{ for all } t \geq T.$$

This finishes the proof of the stability of the set D . In the proof of the stability of the set D , we showed the stability of the set

$$M_2 \equiv \{u \in C_b : u(t) = \int_0^t W(t,s)f(s)ds, \text{ for some } f \in S_{F(\cdot, x(\cdot))}^1(\infty) \\ \text{and some } x \in B_M^C(0)\}.$$

Since the set K_t is relatively compact in X and M_2 is equicontinuous by virtue of Lemma 2, we have the relative compactness of the set M_2 . Since the mapping $u \rightarrow W(\cdot, 0)\tilde{U}^{-1}Uu$ is a bounded linear operator on C_b and M_2 is relatively compact, it follows that the set

$$M_3 \equiv \{u \in C_b : u(t) = W(t, 0)\tilde{U}^{-1}(a - U\phi(\cdot, f)) \text{ for some } f \in S_{F(\cdot, x(\cdot))}^1(\infty) \\ \text{and some } x \in B_M^C(0)\}$$

is also relatively compact. Since $D \subset M_2 + M_3 \subset \overline{M_2} + \overline{M_3}$ and $\overline{M_1}, \overline{M_2}$ are compact, we have the relative compactness of the set D . Since $\overline{VB_M^C(0)}$ is compact, V will be upper semicontinuous, as a mapping from $B_M^C(0)$ to $\overline{VB_M^C(0)}$, if its graph is closed in $B_M^C(0) \times \overline{VB_M^C(0)}$ (cf. [2, p. 42]). Thus, it suffices to show that its graph is closed in $B_M^C(0) \times C([0, T], X)$. To show this, we let $x_n \in B_M^C(0)$, $y_n \in Vx_n$ be such that $(x_n, y_n) \rightarrow (x, y)$ uniformly on \mathcal{R}_+ . Then there exist functions $f_n \in S_{F(\cdot, x_n(\cdot))}^1(T)$ such that

$$y_n(t) = W(t, 0)\tilde{U}^{-1}[a - U\phi(\cdot, f_n)] + \int_0^t W(t,s)f_n(s)ds, \quad t \in [0, T].$$

Since $S_{H_M}^1(\infty)$ is weakly compact in $L^1(\mathcal{R}^+, X)$, by Lemma 1, we have that, passing to a subsequence if necessary, $f_n \rightarrow f \in S_{H_M}^1(\infty)$ weakly in $L^1(\mathcal{R}^+, X)$. Invoking Mazur's theorem, there exists a sequence of terms $z_n \in \text{co}(\bigcup_{k \geq n} f_k)$ such that $z_n \rightarrow f$ strongly in $L^1(\mathcal{R}^+, X)$. We may assume that $z_n(t) \rightarrow f(t)$ strongly as $n \rightarrow \infty$, for $t \in \mathcal{R}_+ \setminus \mathcal{N}$, where \mathcal{N} is a set of measure zero. We fix $t \in \mathcal{R}_+ \setminus \mathcal{N}$. Since F is upper semicontinuous in its second variable, we know that given a ball $B_r(0)$, for some $r > 0$, there exists $n \geq 1$ such that, for $k \geq n$, we have

$$F(t, x_k(t)) \subset F(t, x(t)) + B_r(0),$$

which implies

$$\overline{\text{co}} \left(\bigcup_{k \geq n} F(t, x_k(t)) \right) \subset F(t, x(t)) + \overline{B_r(0)},$$

i.e.,

$$f(t) \in F(t, x(t)) + \overline{B_r(0)}.$$

Since $r > 0$ is arbitrary, $f(t) \in F(t, x(t))$ for every $t \in \mathcal{R}_+ \setminus \mathcal{N}$. Thus, $f \in S_{F(\cdot, x(\cdot))}^1(\infty)$. It is easy to see, by using Lebesgue's Dominated Convergence Theorem, that

$$y(t) = W(t, 0)x_0 + \int_0^t W(t,s)f(s)ds, \quad t \in \mathcal{R}_+.$$

This completes the proof of the first part of the theorem.

If the assumption on the solvability of $x - \lambda Vx \ni 0$, in the second part of the theorem, is made, then for every solution x_λ of this inclusion we have $\|x_\lambda\| \leq \widetilde{M}$, where \widetilde{M} is a fixed positive constant. In order to obtain a mild solution to problem (B) by Ma's Theorem 16.1 in [12], it suffices to restrict the mapping V to the ball $B_M^C(0)$, where M is any constant in (\widetilde{M}, ∞) . \square

A method for creating concrete examples of linear and nonlinear compact evolution operators, generated by time-dependent m -accretive operators, has been developed by Kartsatos in [10].

For local existence results involving functional perturbations of nonlinear m -accretive operators, generating compact evolution operators, the reader is referred to Kartsatos and Shin [11]. It would be interesting to see extensions of our results to the existence of extremal solutions of boundary value problems as in [15].

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