

LIPSCOMB'S UNIVERSAL SPACE IS THE ATTRACTOR OF AN INFINITE ITERATED FUNCTION SYSTEM

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ABSTRACT. Lipscomb's one-dimensional space $L(A)$ on an arbitrary index set A is injected into the Tychonoff cube I^A . The image of $L(A)$ is shown to be the attractor of an iterated function system indexed by A . This system is conjugate, under an injection, with a set of right-shift operators on Baire's space $N(A)$ regarded as a code space. This view of $L(A)$ extends the fractal nature of $L(A)$ initiated in a 1992 joint paper by the author and S. Lipscomb. In addition, we give a new proof that as a subspace of Hilbert's space $l^2(A)$, the space $L(A)$ is complete and hence is closed in $l^2(A)$.

1. INTRODUCTION

In [5], Lipscomb introduced a universal one-dimensional metric space $L(A)$ by generalizing the standard construction of a non-decreasing map f of the Cantor middle-third set C onto the interval by identifying "endpoints". That map may be obtained by representing the unit interval in "ternary decimal" notation as all strings $x = .x_1x_2 \cdots$ where $x_i \in \{0, 1, 2\}$. Then $C = \{x \mid x_i \in \{0, 2\} \text{ for all } i\}$ and f may be described informally by the phrase "change all the 2's to 1's and read in binary notation". Alternately, one can describe $f(C)$ as the quotient by the equivalence relation that identifies each "right-hand endpoint $.x_1 \cdots x_{k-2}0222 \cdots$ with the next point $.x_1 \cdots x_{k-2}2000 \cdots$ in C . Lipscomb's construction $L(A)$ replaces the set $\{0, 2\}$ with an arbitrary discrete set A . Formally, let $N(A) = \prod_{i=1}^{\infty} A_i$ with the product topology. Then $N(A)$ is called Baire's space on A and $L(A)$ is the (at most) 2-to-1 quotient of $N(A)$, just as the unit interval I is the (at most) 2-to-1 quotient of C .

In [1] Lipscomb's space $L(A)$ on an arbitrary index set A was imbedded in Hilbert's space $l^2(A)$. The imbedding solved the problem: Find a metric for $L(A)$. The solution, of course, is the metric inherited from $l^2(A)$. There was a bonus to this solution. The imbedding provided a picture of a geometry for $L(A)$, namely, a geometry analogous to that of the Sierpiński triangle ω^2 in E^2 and Mandlebrot's fractal skewed web ω^3 in E^3 [2, p.142]. In particular, picturing Hilbert's space $l^2(\{1, 2\})$ as the plane E^2 , we see the imbedded image of $L(\{0, 1, 2\})$ as $\omega^2 \subset \Delta^2$, where the 2-simplex Δ^2 has its vertices at $(0, 0), (1, 0), (0, 1) \in E^2$. And picturing

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$l^2(\{1, 2, 3\})$ as 3-space E^3 , we see the imbedded image of $L(\{0, 1, 2, 3\})$ as $\omega^3 \subset \Delta^3$, where the 3-simplex Δ^3 has vertices at the origin and the terminal points of the three orthonormal basis vectors. And for $A = \{0, 1, 2, \dots, n\}$, we picture $l^2(\{1, 2, \dots, n\})$ as E^n and the n -web $\omega^n \subset \Delta^n$ as the homeomorph of $L(A)$ [1].

Recall [3] that the Sierpiński triangle $\omega^2 \subset E^2$ can be viewed as the attractor of an iterated function system (IFS) $W = \{w_0, w_1, w_3\}$ of affine transformations. Indeed, each $\omega^n \subset E^n$ is the attractor of an IFS $W = \{w_0, w_1, \dots, w_n\}$ of $n + 1$ affine transformations.

Turning to the infinite case, we shall show that the situation is entirely analogous to the finite case. At first, however, we observe that the imbedded version ω^A of $L(A)$ is a subset of a standard $|A|$ -simplex Δ^A , which is both a *subspace* of $l^2(A)$ and a *subset* of Tychonoff's cube I^A . (Recall that Tychonoff's cube I^A , having the product topology, is a compact subspace of generalized Euclidean space E^A .) To avoid confusion, let ω^A denote the imbedded version of $L(A)$, i.e., ω^A has the $l^2(A)$ -induced topology. Let ω_c^A denote the space whose underlying set is that of ω^A but whose topology is induced from the Tychonoff cube I^A . In this paper, we shall show that ω_c^A is the attractor of an IFS $W = \{w_a\}_{a \in A}$ containing affine transformations of E^A . It is an open problem to construct ω^A as the attractor of an IFS containing affine transformations of $l^2(A)$.

In addition to proving that ω_c^A is an attractor we also prove that ω^A is complete, making $L(A)$ topologically complete.¹

2. BACKGROUND AND NOTATION

We follow the notation of [1] and single out a point z of A , defining $A' = A - \{z\}$. (The notation z is a mnemonic for zero.) The points of $l^2(A)$ are collections of real numbers indexed by points of A' . Thus if E is the set of real numbers, then $x \in l^2(A)$ means $x = \{x_a\} \in E^{A'}$ such that $x_a = 0$ for all but countably many $a \in A'$ and $\sum x_a^2$ converges. The topology of $l^2(A)$ is induced from the metric $d(x, y) = \sqrt{\sum_a (x_a - y_a)^2}$. We think of x_a as the a th coordinate of x . Which coordinates happen to be non-zero will, of course, vary from point to point of $l^2(A)$. Similarly, since A will be infinite, we define

$$I^A \equiv \prod_{a \in A'} I_a,$$

where I^A is the Tychonoff cube, i.e., the product space of copies I_a of the closed unit interval $I = [0, 1]$, indexed by points of A' .

From [1], which laid the foundation for our present discussion, we also let A denote a discrete space and

$$N(A) = \prod_{n=1}^{\infty} A_n \quad (\text{each } A_n \text{ is a copy of } A),$$

the topological product of countably many copies A_n of A . The space $N(A)$ is usually known as *Baire's space* but here we shall also consider $N(A)$ as a generalization of *code space* [3] where A is viewed as the symbol set.

¹In a private communication to S. Lipscomb, Professor Ivan Ivanić of the University of Zagreb indicated that (his Ph. D. student) U. Milutinović [4] has proved the topological completeness of $L(A)$.

The points of $N(A)$ consist of all sequences $\mathbf{a} = a_1 a_2 \cdots a_n \cdots$ with $a_n \in A$. The space $N(A)$ is metric — define

$$d(\mathbf{a}, \mathbf{a}') = \begin{cases} \frac{1}{k} & \text{if } k \text{ is the first index where } a_i \neq a_i', \\ 0 & \text{if } \mathbf{a} = \mathbf{a}'. \end{cases}$$

Lipscomb's space $L(A)$ is obtained via a projection or identification map

$$p : N(A) \rightarrow L(A).$$

If $|A| = 2$, then the map p is the identification of adjacent endpoints in a Cantor space $N(A)$ and $L(A)$ is a closed interval. This case is similar to the case where A is arbitrary. Indeed, for any A , each point of $L(A)$, being an equivalence class in the partition induced by p , contains at most two members. For instance, $p(aa\cdots) = \{aa\cdots\}$ is a singleton set. If a point $a_1 a_2 \cdots \in N(A)$ has an infinite number of distinct a_i , then $p(a_1 a_2 \cdots)$ is also a singleton set. The points $a_1 a_2 \cdots a_{k-1} a_k a_k \cdots$ that are eventually constant sequences are called *endpoints* of $N(A)$. Moreover, only eventually constant sequences $a_1 a_2 \cdots$ that contain at least two distinct a_i are identified with other points. The quotient map p identifies adjacent endpoints, i.e., p identifies $a_1 \cdots a_{k-2} a_{k-1} \overline{a_k} = a_1 \cdots a_{k-2} a_{k-1} a_k a_k \cdots$ with the point $a_1 \cdots a_{k-2} a_k \overline{a_{k-1}}$. A doubleton set $\alpha \in L(A)$ is called *rational* while all others are called *irrational*. Any member $\mathbf{a} = a_1 a_2 \cdots$ of $\alpha \in L(A)$ is an *expansion* of α .

When $A = \{0, 1\}$, the space $L(A)$ is the unit interval and each expansion $a_1 a_2 \cdots$ of a point in $L(A)$ yields a binary representation “. $a_1 a_2 \cdots$,” i.e., $p(a_1 a_2 \cdots) = .a_1 a_2 \cdots$. In the general case where A is arbitrary and $a_1 a_2 \cdots$ is an expansion of $\alpha \in L(A)$, then *the A-representation* of α is “. $a_1 a_2 \cdots$.” That is,

$$p(a_1 a_2 \cdots) = \alpha = .a_1 a_2 \cdots$$

where the point preceding a_1 is called the *A-point*. (This notation is due to S. Lipscomb.)

Turning to the imbedding map $f : L(A) \rightarrow l^2(A)$ introduced in [1], let $b \in A'$, let $\mathbf{a} = a_1 a_2 \cdots \in N(A)$, and let

$$K_b(\mathbf{a}) = \{k \mid a_k = b\}.$$

Then for $.a_1 a_2 \cdots = \alpha \in L(A)$, define

$$\alpha_b = \begin{cases} \sum_{k \in K_b(\mathbf{a})} \frac{1}{2^k} & \text{if } K_b(\mathbf{a}) \neq \emptyset, \\ 0 & \text{if } K_b(\mathbf{a}) = \emptyset \end{cases}$$

and let $f : L(A) \rightarrow l^2(A)$ be given by

$$f(\alpha) = \{\alpha_b\} \in l^2(A).$$

Note that f takes the point $p(\mathbf{z}) = \{zzz\cdots\} = .zz\cdots$ whose expansion is the constant sequence $zz\cdots$ to the zero $\mathbf{0}$ of $l^2(A)$. For any other $b \in A'$, the imbedding f sends the point $.bb\cdots$ to the unit vector $\mathbf{u}_b \in l^2(A)$, where

$$\text{ath coordinate of } \mathbf{u}_b = \begin{cases} 0 & \text{if } a \neq b, \\ 1 & \text{if } a = b. \end{cases}$$

In short, for points in $L(A)$ having a constant sequence expansion, f sets up the following one-one correspondence:

$$\begin{aligned} \{zz\dots\} &\mapsto \mathbf{0} \\ \{bb\dots\} &\mapsto \mathbf{u}_b, \quad \text{for } b \in A'. \end{aligned}$$

From this correspondence it is easy to envision the action of f on $L(A)$. For example, if $\alpha = .a_1a_2\dots \in L(A)$ where $\{a_1, a_2, \dots\} = \{a, b\} \subset A'$, then there is a unique real number t , $0 \leq t \leq 1$, such that

$$f(\alpha) = t\mathbf{u}_a + (1 - t)\mathbf{u}_b.$$

Furthermore, for each triple subset $\{a, b, c\}$ of A' , the image ω^A of f meets the 3-simplex $\Delta^3 = [\mathbf{u}_a, \mathbf{u}_b, \mathbf{u}_c]$ in the (Mandelbrot's) fractal skewed web whose vertices are those of Δ^3 .

More generally, any $x \in l^2(A)$ is a countable linear combination

$$x = \sum_{A'} x_b \mathbf{u}_b.$$

That is, viewing x and each \mathbf{u}_b as function $A' \rightarrow E$,

$$x(a) = \sum_{A'} x_b \mathbf{u}_b(a) = x_a \mathbf{u}_a(a) = x_a.$$

It is therefore natural to define the *standard* $|A|$ -simplex Δ^A in $l^2(A)$ as

$$\Delta^A = \{x \in l^2(A) \mid \text{each } x_b \geq 0; \sum_{A'} x_b \leq 1\}.$$

Thus, the vertices of Δ^A consist of $\mathbf{0}$ and each \mathbf{u}_b for $b \in A'$. If we restrict the non-zero coordinates of $x \in l^2(A)$ to be finite in number, then we get every n -simplex Δ^n as a face of Δ^A . In addition, from [1, Corollary 8], f maps each $\alpha \in L(A)$ into Δ^A . We will define

$$f(L(A)) = \omega^A \subset \Delta^A.$$

As a point set we have

$$\Delta^A \subset I^A,$$

where the topology of Δ^A , being induced from $l^2(A)$, differs from that induced from I^A .

Turning to I^A , for any $x = \{x_a\}$ in I^A , let $\frac{1}{2}x = \{\frac{1}{2}x_a\} \in I^A$ and define

$$\frac{1}{2}X = \{\frac{1}{2}x \mid x \in X\}.$$

In addition, for each real number r such that $0 \leq r \leq 1$, let

$$x + r_a = x + r\mathbf{u}_a = y \in I^A, \quad \text{where } y_b = \begin{cases} x_b & \text{if } b \neq a, \\ \min\{1, x_a + r\} & \text{if } b = a. \end{cases}$$

More generally, for any $X \subset I^A$ and any real number r , let

$$X + r_a = X + r\mathbf{u}_a = \{x : x = x' + r_a, x' \in X\} \subset I^A.$$

Finally, a subbasic open set $\langle G_a \rangle$ in I^A is a product of sets, the a th factor of which is an open subset G_a of I , while for $b \neq a$, the b th factor is I_b . In terms of this subbasis, the basic open sets are of the form $G = \langle G_{a_1}, G_{a_2}, \dots, G_{a_n} \rangle = \bigcap_{i=1}^n \langle G_{a_i} \rangle$.

3. PRELIMINARY OBSERVATIONS

We begin with several lemmas.

Lemma 1. *The map $\phi(x) = \frac{1}{2}x$ on I^A is continuous.*

Proof. For each $a \in A'$, let $\phi_a : I_a \rightarrow I_a$ be the continuous map given by $x \mapsto \frac{1}{2}x$. Then observe that $\phi = \times_{A'} \phi_a$ and is a product map of continuous maps and therefore continuous. \square

Lemma 2. *For each real number r , $0 \leq r \leq 1$, and each $a \in A'$, the map $\theta(x) = x + r_a$ is continuous on I^A .*

Proof. Let $a_0 \in A'$ be fixed. Then for each $a \in A' - \{a_0\}$, let $\theta_a : I_a \rightarrow I_a$ be the continuous identity map $x \mapsto x$. And for $a = a_0$, let $\theta_a : I_a \rightarrow I_a$ be the continuous map given by $x \mapsto \min\{1, x + r\}$, which is just a shift by r , followed by a retraction to 1 if necessary to stay in $[0, 1]$. Then observe that $\theta = \times_{A'} \theta_a$ is a product map of continuous maps and therefore continuous. \square

Lemma 3. *Let G be open in I^A . Then $G \cap \omega^A$ is open in ω^A .*

Proof. Let $G = \langle G_a \rangle$ be a subbasic open set. Let $y \in G \cap \omega^A$. Then $y_a \in G_a$. There exists a neighborhood $N(y_a, \varepsilon) \subset G_a$ for some $\varepsilon > 0$. Let $w \in N(y, \varepsilon/2)$ in ω^A . Then $w \in G$. Otherwise $w_a \notin N(y_a, \varepsilon)$. This implies $(w_a - y_a) \geq \varepsilon$. Therefore $d(y, w) \geq \varepsilon$ in ω^A , contradicting the selection of w . Therefore $G \cap \omega^A$ is the union of neighborhoods about each point in $G \cap \omega^A$. Therefore $G \cap \omega^A$ is open in ω^A . \square

This lemma shows that the injection map $\omega^A \hookrightarrow I^A$ is continuous. Define ω_c^A as the image of the injection of ω^A into I^A . As a result, we shall use “ f ” to denote both the homeomorphism $f : L(A) \rightarrow \omega^A$ and the continuous injection $f : L(A) \rightarrow \omega_c^A \subset I^A$, the particular choice will be clear from the context.

These lemmas immediately yield the following lemma.

Lemma 4. *Let X be compact in I^A , let $a \in A'$, and let r be a real number, $0 \leq r \leq 1$. Then both sets $X + r_a$ and $\frac{1}{2}X$ are compact.*

Theorem 5. *Let C be a compact subset of I^A , let $B \subset A$, and let $r : B \rightarrow [0, 1]$ be any map. If for each $b \in B$, $C_b = C + r(b)\mathbf{u}_b = C + r_b$, then $K = C \cup (\bigcup_B C_b)$ is compact.*

Proof. First, observe that the previous lemma shows that each C_b is compact. So, let \mathcal{G} be a covering of K with basic open sets. Then, since C is compact, a finite subset $\mathcal{F} \subset \mathcal{G}$ covers C , i.e., $C \subset V = \bigcup_{\mathcal{F}} G$. For each $G \in \mathcal{F}$ a finite set $A_G \subset A'$ exists such that G is the intersection of the subbasic sets $\langle G_a \rangle$, $a \in A_G$. (In other words, $x \in G$ if, and only if, $x_a \in G_a$ for each $a \in A_G$.) Then, since $\bigcup_{\mathcal{F}} A_G = F$ is finite and each C_b ($b \in F$) is compact, $\bigcup_{\mathcal{F}} C_b$ is compact. It therefore follows that we can finish the proof by showing that $b \in B - F$ implies $C_b \subset V$. So, for such a b , let $x \in C_b$. Then $x = x' + r_b$, where $x' \in C$. Then $C \subset V$ implies $x' \in V$. Consequently, $G \in \mathcal{F}$ exists such that $x' \in G$. Or, in other words, $x_a' \in G_a$ for each $a \in A_G$. Then, since $b \notin A_G$, $x_a = x_a' \in G$ for each $a \in A_G$ implying $x \in G$. Since $G \subset V$, $x \in V$, which finishes the proof. \square

4. AN INFINITE HYPERBOLIC ITERATED FUNCTION SYSTEM (IFS)

For each $a \in A$, define a (clearly continuous) *shift map* $S'_a : N(A) \rightarrow N(A)$ by $a_1a_2 \cdots \mapsto aa_1a_2 \cdots$. Each such map has its counterpart $S_a : L(A) \rightarrow L(A)$ given by $.a_1a_2 \cdots \mapsto .aa_1a_2 \cdots$. It is obvious that S_a is well defined. Moreover, since

$$p \circ S'_a(a_1a_2 \cdots) = p(aa_1a_2 \cdots) = .aa_1a_2 \cdots \quad \text{and}$$

$$S_a \circ p(a_1a_2 \cdots) = S_a(.a_1a_2 \cdots) = .aa_1a_2 \cdots$$

we have $S_a \circ p = p \circ S'_a$, i.e., the top square in diagram (1) is commutative.

$$(1) \quad \begin{array}{ccc} N(A) & \xrightarrow{S'_a} & N(A) \\ p \downarrow & & \downarrow p \\ L(A) & \xrightarrow{S_a} & L(A) \\ f \downarrow & & \downarrow f \\ I^A & \xrightarrow{w_a} & I^A. \end{array}$$

It follows, then, that S_a is continuous, i.e., since p is a quotient map and $S_a \circ p (= p \circ S'_a)$ is continuous, S_a is continuous.

To make the whole diagram (1) commutative, we define $w_a : I^A \rightarrow I^A$ that is conjugate with S_a under f ($w_a f = f S_a$). More precisely, choose $a \in A$ and let

$$w_a(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2}\mathbf{u}_a & \text{if } a \neq z, \\ \frac{1}{2}x & \text{if } a = z. \end{cases}$$

We note that the collection $\{w_a\}_A$ is a natural generalization of the well-known (finite) IFS used to generate a Sierpiński triangle in the plane with vertices at $(0, 0)$, $(0, 1)$, and $(1, 0)$ (see [3]).

To see the commutativity $w_a f = f S_a$, we begin with $a \neq z$. We compare $f(.a_1a_2 \cdots)$ with $f(.aa_1a_2 \cdots)$ coordinate-wise. First, consider a coordinate $b \neq a$. Then by the definition of f , the b th coordinate of $f(.aa_1a_2 \cdots)$ is $\frac{1}{2}$ that of $f(.a_1a_2 \cdots)$, showing that w_a does indeed match these b th coordinates. Second, comparing a th coordinates, that of $f(.aa_1a_2 \cdots)$ is obtained from that of $f(.a_1a_2 \cdots)$ by multiplying by $\frac{1}{2}$ and then adding $\frac{1}{2}$. Thus, in either case, $w_a f = f S_a$ when $a \neq z$. The argument that $w_z f = f S_z$ is similar.

Lemma 6. *If $x = f(.a_1a_2 \cdots) \in \omega_c^A$, then for any $n \geq 1$, $x \in w_{a_1} \cdots w_{a_n} I^A$.*

Proof. We use $w_a f = f S_a$. First, note that

$$.a_1a_2 \cdots = S_{a_1} \cdots S_{a_n}(.a_{n+1}a_{n+2} \cdots).$$

Then calculate:

$$\begin{aligned} x &= f(S_{a_1} \cdots S_{a_n}.a_{n+1} \cdots) = w_{a_1} f(S_{a_2} \cdots S_{a_n}.a_{n+1} \cdots) \\ &= w_{a_1} w_{a_2} f(S_{a_3} \cdots S_{a_n}.a_{n+1} \cdots) = \cdots \\ &= w_{a_1} \cdots w_{a_n} f(.a_{n+1} \cdots) \in w_{a_1} \cdots w_{a_n} I^A. \quad \square \end{aligned}$$

5. THE COMPACT SPACE ω_c^A IS AN ATTRACTOR FOR $W = \{w_a\}_A$

We need some preliminary constructions and definitions.

Definition 7. For $X \subset I^A$ we say that X is invariant under W whenever

$$X = W(X) = \bigcup_{a \in A} w_a(X).$$

We inductively construct a nested sequence $\{I_n\}_{n=0}^\infty$ of compact sets. Let $I_0 = I^A$ and $I_1 = W(I_0)$. Then $I_1 \subset I_0$ and both I_0 and I_1 are compact — I_1 is compact by Theorem 5. Recursively, define $I_{n+1} = W(I_n)$. Assuming that $I_n \subset I_{n-1}$, the calculation

$$I_{n+1} = W(I_n) \subset W(I_{n-1}) = I_n$$

shows that $I_{n+1} \subset I_n$ for every $n \geq 0$. Moreover, another application of Theorem 5 shows that the compactness of I_n implies that of I_{n+1} . With $\{I_n\}_{n=0}^\infty$ defined we let $P = \bigcap_{n=0}^\infty I_n$, which is obviously also compact. (Note that $\mathbf{0} \in P \neq \emptyset$ since $z \in A$ and $w_z(\mathbf{0}) = \mathbf{0}$.)

Lemma 8. We have $\omega_c^A \subset P$.

Proof. First, observe that each

$$I_n = \bigcup_{a,b,\dots,q \in A} \underbrace{w_a w_b \cdots w_q}_n I^A.$$

Now let $x = f(.a_1 a_2 \cdots) \in \omega_c^A$. Then by Lemma 6, $x \in w_{a_1} \cdots w_{a_n} I^A \subset I_n$ for all n , showing $x \in P$. \square

To see that $P \subset \omega_c^A$, we let $x \in P$ and show that an $\mathbf{a} = a_1 a_2 \cdots \in N(A)$ exists such that

$$\begin{aligned} (2) \quad x &\in S_1 = w_{a_1} I^A \\ x &\in S_2 = w_{a_1} w_{a_2} I^A \\ &\vdots \end{aligned}$$

First, however, we need the following technical lemma.

Lemma 9. Let $a \in A$, $y \in I^A$, and $x = w_{a_1} \cdots w_{a_n} y$. Then we have the following:

- (i) If $a_1 = a \neq z$, then $x_a \geq \frac{1}{2}$.
- (ii) If j is the smallest index such that $a_j \neq a$, then

$$x_a \leq \frac{2^j - 1}{2^j} < 1.$$

- (iii) If $a_1 \neq a$, then $x_a \leq \frac{1}{2}$.
- (iv) If $a_1 \neq a$ and $x_a \geq \frac{1}{2}$, then $a_2 = \cdots = a_n = a$.

Proof. (i) If $a_1 = a \neq z$, then for $v = w_{a_2} \cdots w_{a_n} y$, it follows that $0 \leq v_a \leq 1$ and therefore $x_a = \frac{1}{2}v_a + \frac{1}{2} \geq \frac{1}{2}$. (ii) First suppose $a_1 \neq a$. Then $x_a \leq \frac{1}{2} = \frac{2^1 - 1}{2^1}$ since the a th component of $w_{a_2} \cdots w_{a_n} y \leq 1$. Second, suppose $a_1 = a$. Then $j \geq 2$ and, letting $v = w_{a_j} \cdots w_{a_n} y$, we have $x = w_{a_1} \cdots w_{a_{j-1}} v$. From the previous case the maximum possible value of v_a is $\frac{1}{2}$. In addition, the maximum possible value of x_a occurs whenever $a_1 = \cdots = a_{j-1} = a$. Thus,

$$x_a \leq \frac{2^j - 1}{2^j} < 1.$$

(iii) This is statement (ii) where $j = 1$. (iv) If a smallest index $j \geq 2$ exists such that $a_j \neq a$, then for $v = w_{a_2} \cdots w_{a_n} y$, it follows from (ii) that

$$v_a \leq \frac{2^{j-1}-1}{2^{j-1}} \text{ which implies } x_a = \frac{1}{2}v_a = \frac{2^{j-1}-1}{2^j} < \frac{1}{2},$$

which is a contradiction. □

Theorem 10. *The space $\omega_c^A = P$ and is therefore compact.*

Proof. By Lemma 6, it suffices to show $P \subset \omega_c^A$. To do this, let $x \in P$. Then, by construction of P ,

$$\begin{aligned} x &\in w_{a_{11}} I^A = T_1 \\ x &\in w_{a_{21}} w_{a_{22}} I^A = T_2 \\ &\vdots \end{aligned}$$

The trouble is we can't say anything about the limit of the initial strings of w 's, if there is one. But from $\{T_n\}$, we need to extract a sequence $\{S_n\}$ as defined in (2). The inductive construction of $\{S_n\}$ depends on the initial strings of w 's that define $\{T_n\}$. First, we select w_{a_1} . There are three cases to consider:

- (a) A w_a exists that initiates infinitely many T_n 's.
- (b) At least two w 's, say w_a and w_b , exist that initiate infinitely many T_n 's.
- (c) No initial w initiates infinitely many T_n 's.

If (a), then we define $w_{a_1} = w_a$ and $w_{a_1} I^A = S_1$. Thus, $x \in w_{a_1} I^A = S_1$. If (b), then we can assume that $a \neq z$ and that $w_a \neq w_b$. In this case, for each $n \geq 1$ we have

$$\begin{aligned} (3) \quad &x \in w_a w_{p_1} \dots w_{p_n} I^A, \\ (4) \quad &x \in w_b w_{q_1} \dots w_{q_n} I^A. \end{aligned}$$

From (i) of Lemma 9 and (3), $x_a \geq 1/2$. And from (iv) of Lemma 9 and (4), $q_1 = \dots = q_n = a$. In this case, then, we can construct all S_n because

$$x \in w_b \underbrace{w_a \cdots w_a}_n I^A \text{ for every } n > 1.$$

That is, for each $n \geq 2$, define

$$S_n = w_b \underbrace{w_a \cdots w_a}_{n-1} I^A.$$

Finally, we show (c) is impossible. For otherwise, select $a \neq z \neq b \neq a$ such that

$$\begin{aligned} (5) \quad &x \in w_a I^A, \\ (6) \quad &x \in w_b I^A. \end{aligned}$$

Furthermore, select c such that $a \neq c \neq b$, with

$$(7) \quad x \in w_c w_{p_2} \dots w_{p_n} I^A, \quad n \geq 2.$$

Again, (i) of Lemma 9 and (5) show that $x_a \geq \frac{1}{2}$ while (iv) of Lemma 9 and (7) show that $p_2 = \dots = p_n = a$. But a similar argument concerning x_b shows that $p_2 = \dots = p_n = b$, which contradicts $a \neq b$. In short, case (c) is impossible while

case (b) provides the whole sequence $\{S_n\}$. So the inductive step only need concern case (a), i.e., suppose for some fixed $k \geq 1$

$$x \in w_{a_1} \dots w_{a_k} I^A$$

where $w_{a_1} \dots w_{a_k}$ initiates infinitely many T_n . Since each w_{a_i} is one-one, a $y \in I^A$ must exist where $x = w_{a_1} \dots w_{a_k} y$. As a result y must also be contained in every T_n having $w_{a_1} \dots w_{a_k}$ as initial segment. To define a_{k+1} , then, we can repeat the argument for finding a_1 , this time however, we use y in place of x and we replace $\{T_n\}$ with $\{T'_m\}$, which we generate by "chopping off" the first k w 's from those T_n with initial segment $w_{a_1} \dots w_{a_k}$. At any rate we find $\mathbf{a} = a_1 a_2 \dots \in N(A)$ such that

$$\begin{aligned} x &\in w_{a_1} I^A \subset I_1 \\ x &\in w_{a_1} w_{a_2} I^A \subset I_2 \\ &\vdots \end{aligned}$$

Now consider $x' = f(.a_1 a_2 \dots) \in \omega_c^A$. From Lemma 6, we then have

$$x' \in w_{a_1} \dots w_{a_n} I^A \quad \text{for all } n \geq 1.$$

Thus, both x and $x' \in P$. We show $x = x'$. Otherwise, for some $a \neq z$, we have $x_a - x'_a > \varepsilon > 0$. Let N be such that $\frac{1}{2^N} < \varepsilon$. Then both

$$x \text{ and } x' \in w_{a_1} \dots w_{a_N} I^A = \prod_{a \in A'} X_a$$

where each X_a is a closed interval of length $\frac{1}{2^N}$, showing $|x_a - x'_a| < \varepsilon$. Thus, $x = x' \in \omega_c^A$. □

Theorem 11. *The compact set ω_c^A is invariant under W .*

Proof. We show $\omega_c^A = W(\omega_c^A) = \bigcup_{a \in A} w_a(\omega_c^A)$. Let $x = f(.a_1 a_2 \dots) \in \omega_c^A$. Then $f(.a_2 a_3 \dots) \in \omega_c^A$, $a_1 \in A$, and from diagram (1),

$$x = f(.a_1 a_2 \dots) = fS_{a_1}(.a_2 a_3 \dots) = w_{a_1}(f(.a_2 a_3 \dots)) \in \bigcup_{a \in A} w_a(\omega_c^A) = W(\omega_c^A).$$

Therefore $\omega_c^A \subset W(\omega_c^A)$. To see the reverse inclusion, let $x \in W(\omega_c^A)$. Then for some $.a_2 a_3 \dots \in L(A)$ and $a_1 \in A$, we have

$$x = w_{a_1} f(.a_2 a_3 \dots) = f(S_{a_1}.a_2 a_3 \dots) = f(.a_1 a_2 \dots) \in \omega_c^A. \quad \square$$

Theorem 12. *The set ω_c^A is the only W -invariant compact subset of I^A .*

Proof. We show no other closed (= compact) subset B of I^A is invariant under W . Consider first the case where $B \not\subset \omega_c^A = P = \bigcap_n I_n$. Then B fails to lie inside of some I_n . But by construction of the I_n 's, if B were invariant, then $W(B) \subset I_n$ for every n . Therefore $W(B) \neq B$, showing that B is not invariant under W . For the other case, where B is a proper subset of ω_c^A , let

$$B' = (f \circ p)^{-1}(B) \subset N(A).$$

Then, since B' is a proper closed subset of $N(A)$, a point $\mathbf{a} = a_1 a_2 \dots \in N(A) - B'$ exists and we can consider one of its neighborhoods $V = \langle a_1, \dots, a_n \rangle$ where $V \cap B' =$

\emptyset . Then for $\mathbf{b} \in B'$, since $S'_{a_1} \cdots S'_{a_n} \mathbf{b} \in V$, we have $S'_{a_1} \cdots S'_{a_n} \mathbf{b} \notin B'$. Thus, via conjugacy,

$$\begin{aligned} (f \circ p)\mathbf{b} = x \in B, \quad \text{but} \quad & (f \circ p)S'_{a_1} \cdots S'_{a_n} \mathbf{b} \notin B \\ & \Rightarrow w_{a_1} \cdots w_{a_n} (f \circ p)\mathbf{b} \notin B \\ & \Rightarrow w_{a_1} \cdots w_{a_n} (x) \notin B, \end{aligned}$$

which contradicts the invariance of B . Having exhausted all cases, the proof is complete. \square

6. TOPOLOGICAL COMPLETENESS OF $L(A)$

Convergence in $L(A)$ is ultimately convergence in $N(A)$. We therefore begin with a lemma concerning convergence in $N(A)$.

Lemma 13. *Let $\{\mathbf{a}_n\}$ be an infinite sequence in $N(A)$ with no limit point. For each index n , let $\mathbf{a}_n = a_1^n a_2^n \cdots$. Then there exists a subsequence $\{\mathbf{a}_m\}$ of $\{\mathbf{a}_n\}$ and a fixed index $i > 0$ such that $\{a_i^m\} = \{a^m\}$ is an infinite sequence in A .*

Proof. Suppose no such index exists. Then $\{a_1^n\}$ is finite and an $a_1 \in \{a_1^n\}$ exists that is the first coordinate of each member in an infinite subsequence S_1 of $\{\mathbf{a}_n\}$. Inductively, suppose $a_1 \cdots a_k$ is the initial k -segment of each member in an infinite subsequence S_k of $\{\mathbf{a}_n\}$. Then since $\{a_{k+1}^n \mid \mathbf{a}_n \in S_k\}$ is finite, one of its members, say a_{k+1} , is the $(k + 1)$ st coordinate of an infinite number of members of S_k . As a result $a_1 \cdots a_{k+1}$ is the initial $(k + 1)$ -segment of each member in an infinite subsequence S_{k+1} of $\{\mathbf{a}_n\}$. Thus, we get such a subsequence S_k for every $k > 0$. It follows that the point

$$\mathbf{a} = a_1 a_2 \cdots \in N(A)$$

is a limit point of $\{\mathbf{a}_n\}$, i.e., for any $k > 0$, the k th coordinate of \mathbf{a} agrees with infinitely many points of $\{\mathbf{a}_n\}$. \square

Theorem 14. *Lipscomb's metric space $L(A)$ is topologically complete.*

Proof. It suffices to show that the homeomorphic image $\omega_A \subset l^2(A)$ is complete in the induced metric. So let $\{x^n\}$ denote an infinite Cauchy sequence in ω^A . For each index n , choose $\mathbf{a}_n = a_1^n a_2^n \cdots \in N(A)$ such that

$$f \circ p(\mathbf{a}_n) = f(\mathbf{a}_n) = x^n.$$

Suppose $\{x^n\}$ has no limit point in ω^A . Then $\{\mathbf{a}_n\}$ is an infinite sequence in $L(A)$ with no limit point. And in turn, since p is a continuous map, $\{\mathbf{a}_n\}$ is an infinite sequence in $N(A)$ with no limit point. By Lemma 13, there exists a subsequence $\{\mathbf{a}_m\}$ of $\{\mathbf{a}_n\}$ and an index $i > 0$ such that $\{a_i^m\} = \{a^m\}$ is an infinite sequence in A . We can also suppose $z \notin \{a^m\}$. Then for every x^m , since $f(\mathbf{a}^m) = x^m$ and the i th coordinate of \mathbf{a}^m is a^m ,

$$(8) \quad x_{a^m}^m \geq 1/(2^i).$$

But $\{x^n\}$ is Cauchy, implying that for some N and all $m, k > N$,

$$(9) \quad d(x^m, x^k) < 1/(2^{i+1}).$$

Fix a $k > N$ and from $\{\mathbf{a}_m\}$ select an \mathbf{a}_m where $m > N$. Then we can show that

$$(10) \quad x_{a^m}^k > 1/(2^{i+1}).$$

To see why (10) is true, suppose otherwise. Then $x_{a^m}^k \leq 1/(2^{i+1})$ and (8) show

$$x_{a^m}^m - x_{a^m}^k \geq 1/(2^{i+1}),$$

which implies

$$d(x^m, x^k) \geq \left([x_{a^m}^m - x_{a^m}^k]^2 \right)^{1/2} \geq 1/(2^{i+1}),$$

which contradicts (9). Thus for large k , inequality (10) is true for an infinite number of m , contradicting $x^k \notin \ell^2(A)$. Thus, $\{x^n\}$ has a limit point in ω^A . \square

Since a complete subspace of a metric space must be a closed set, Theorem 14 shows that ω^A is closed in $\ell^2(A)$. Thus, we have the following corollary.

Corollary 15. *The f -image ω^A of $L(A)$ is a closed subset of $\ell^2(A)$.*

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