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A SMALL DOWKER SPACE IN ZFC

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ABSTRACT. We construct a hereditarily normal topological space whose product with the unit interval is not normal. The space is σ-relatively discrete and has cardinality of the continuum $\mathfrak{c}$.

Introduction. A *Dowker space* is a normal Hausdorff space whose product with the closed unit interval I is not normal. Whether there are such spaces at all was raised as a problem by C. H. Dowker [D] in 1951 who also gave an internal characterization of Dowker spaces. (See Lemma 0.1 below.)

Constructing Dowker spaces turned out to be a hard, many-faceted problem in general and set-theoretic topology, of basically combinatorial set-theoretic nature. It has extensive, rich literature. (See the survey paper [R_4] and also [R_6], [W_2] for some update.) There has been precisely one Dowker space built in usual (ZFC) mathematics by Mary Ellen Rudin [R_2] in 1971. This space has cardinality and weight $\aleph_\omega^\omega$, and it has few nice properties beyond normality. The search for a smaller example is referred to as the *Small Dowker Space Problem.* The effort of a number of mathematicians resulted in nice, small Dowker spaces in many models of set theory [Be], [C], [G], [JKR], [R_1], [R_5], [We], but no example different from Rudin's has been constructed in ZFC alone.

In this paper we give a solution to the Small Dowker Space Problem by constructing, without extra set-theoretic axioms, a hereditarily normal, σ-relatively discrete Dowker space of cardinality of the continuum $\mathfrak{c}$. Such a space of strongly compact cardinality (under the set-theoretic assumption that such large cardinals exist), was constructed by S. Watson [W_1]. Our proof uses a general combinatorial technique originated in M. E. Rudin's [R_3] and later used by the author [B] to construct Q-set spaces.

Our terminology and notation follow the standards of contemporary set theory and set-theoretic topology as used in [K] and [KV]. In particular, $[A]^{<\kappa}$ stands for the family of all subsets of A of cardinality $< \kappa$ and $Fn(\kappa, 2)$ denotes all finite partial functions from κ to $2 = \{0, 1\}$.

We shall make use of Dowker's internal characterization.

Lemma 0.1. *The following conditions are equivalent for a normal T_1 space X.*

(a) *X is a Dowker space.*

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(b) *X is not countably metacompact, i.e., there is an increasing open cover* $G_0 \subset G_1 \subset \ldots \subset G_n \subset \ldots, n \in \omega$, *of X such that for every sequence of closed sets* $F_n \subset G_n$, $\cup_{n\in\omega} F_n \neq X$.

The space.

Theorem 1.1. *There is a hereditarily normal, σ-relatively discrete Dowker space of cardinality $\mathfrak{c}$.*

The heart of the proof of Theorem 1.1 is the following combinatorial lemma.

Lemma 1.2. *Let $\lambda = 2^{\mathfrak{c}}$, and let $\langle c_\xi \rangle_{\xi<\lambda}$ be a one-to-one enumeration of ${}^{\mathfrak{c}}2 = \{c : c$ is a function from $\mathfrak{c}$ to $2\}$. Then there is a sequence $\langle d_\xi \rangle_{\xi<\lambda}$ of functions $d_\xi : \mathfrak{c} \to 2$ in such a way that for every $g : \mathfrak{c} \to [\lambda]^{<\omega}$, $f : \mathfrak{c} \to \omega$ and $h : \mathfrak{c} \to [\mathfrak{c}]^{<\omega}$, there are $\alpha < \beta$ in $\mathfrak{c}$ such that $f(\alpha) = f(\beta), \beta \notin h(\alpha)$ and for every $\xi \in g(\alpha), c_\xi(\alpha) = d_\xi(\beta)$.*

Proof of Lemma 1.2. Let us call a triple $\langle A, B, u \rangle$ a *control triple* if
(1) $A \in [\mathfrak{c}]^\omega, B \in [{}^A2]^{\leq\omega}$;
(2) u is a function with $\mathrm{dom}(u) \in [A]^\omega$;
(3) for every $\alpha \in \mathrm{dom}(u), u(\alpha) \in [{}^A2 \setminus B]^{<\omega}$;
(4) if $\alpha \neq \alpha'$ in $\mathrm{dom}(u)$, then $u(\alpha) \cap u(\alpha') = \emptyset$.

Let $\langle A_\beta, B_\beta, u_\beta \rangle_{\beta<\mathfrak{c}}$ be a list of all control triples mentioning each triple $\mathfrak{c}$ many times.

Suppose now that $\xi < \lambda$ and we want to define $d_\xi(\beta)$ for some $\beta < \mathfrak{c}$. We are going to distinguish among three cases.

Case 1. If $c_\xi \restriction A_\beta \in B_\beta$, then let $d_\xi(\beta) = c_\xi(\beta)$.

Case 2. If $c_\xi \restriction A_\beta \in u_\beta(\alpha)$ for some $\alpha \in \mathrm{dom}(u_\beta)$, then note first that $c_\xi \restriction A_\beta \notin B_\beta$ by (3), and that there is only one such α by (4). Then define $d_\xi(\beta) = c_\xi(\alpha)$.

Case 3. If neither Case 1 nor Case 2 holds, then set $d_\xi(\beta) = 0$.

The rest of the proof of Lemma 1.2 will be devoted to showing that the sequence $\langle d_\xi \rangle_{\xi<\lambda}$ of functions constructed above satisfies the requirements in the conclusion of Lemma 1.2. To see this, take a $g : \mathfrak{c} \to [\lambda]^{<\omega}$, an $f : \mathfrak{c} \to \omega$ and an $h : \mathfrak{c} \to [\mathfrak{c}]^{<\omega}$. For every $\alpha \in \mathfrak{c}$, let $e_\alpha \in Fn(\lambda, 2)$ be such that dom $(e_\alpha) = g(\alpha)$ and that $e_\alpha(\xi) = c_\xi(\alpha)$ for every $\xi \in g(\alpha)$.

We will produce a pair $\alpha < \beta$ in $\mathfrak{c}$ such that
(5) $f(\alpha) = f(\beta), \beta \notin h(\alpha)$ and for every $\xi \in g(\alpha), c_\xi(\alpha) = d_\xi(\beta)$.

To do this, fix two countable elementary submodels M and N of $H((2^{2^{\mathfrak{c}}})^+)$ such that $\langle c_\xi \rangle_{\xi<\lambda}, \langle d_\xi \rangle_{\xi<\lambda}, \langle e_\alpha \rangle_{\alpha<\mathfrak{c}}, g, f, h \in M$ and $M \in N$.

Let $A = \mathfrak{c} \cap N, B = \{c_\xi \restriction A : \xi \in \lambda \cap M\}$ and let us pick a partial function $u : A \to [{}^A2 \setminus B]^{<\omega}$ with the following property:

(6) whenever $v \in N$ is an infinite partial function $v : \mathfrak{c} \to [\lambda \setminus M]^{<\omega}$ and $\alpha \neq \alpha\prime$ in $\mathrm{dom}(v)$ implies $v(\alpha) \cap v(\alpha\prime) = \emptyset$, then there is an $\alpha \in \mathrm{dom}(u) \cap \mathrm{dom}(v)$ such that

$$u(\alpha) = \{c_\xi \restriction A : \xi \in v(\alpha)\}.$$

To see that there is such a u, let $\langle v_j \rangle_{j\in\omega}$ list each v as in (6). Take $\alpha_0, \alpha_1, \ldots, \alpha_j, \ldots \in A$ in such a way that $\alpha_j \in \mathrm{dom}(v_j) \cap N$ and $i < j$ implies $v_i(\alpha_i) \cap v_j(\alpha_j) = \emptyset$. Then let $\mathrm{dom}(u) = \{\alpha_j : j \in \omega\}$, and set $u(\alpha_j) = \{c_\xi \restriction A : \xi \in v(\alpha_j)\}$ for every $j \in \omega$. We have to show that u satisfies (2), (3) and (4). It obviously satisfies (2).

To see that (3) is satisfied, suppose indirectly that $u(\alpha_j) \cap B \neq \emptyset$ for some $j \in \omega$, i.e. there are $\xi \in v_j(\alpha_j) \subset N$ and $\eta \in \lambda \cap M \subset N$ such that $c_\xi \restriction A = c_\eta \restriction A$. Then $N \models c_\xi = c_\eta$, so $c_\xi = c_\eta$. Since $\langle c_\xi \rangle_{\xi < \lambda}$ is a one-to-one enumeration, it follows that $\xi = \eta$, contradicting $v_j(\alpha_j) \subset \lambda \setminus M$. The proof that u satisfies (4) is similar.

Returning to finding α and β as required in (5), let us pick $\beta < \mathfrak{c}$ to be such that $\beta > \sup A$ and $\langle A_\beta, B_\beta, u_\beta \rangle = \langle A, B, u \rangle$. To find α, let $E = g(\beta) \cap M$ and $e = e_\beta \restriction E = e_\beta \restriction (\lambda \cap M)$. Set $n = f(\beta)$. Let us say that $\langle e_\gamma \rangle_{\gamma \in D}$ $(D \subset \mathfrak{c})$ forms a $\triangle$-system with root e iff e_γ extends e for every $\gamma \in D$, and $\mathrm{dom}(e_\gamma) \cap \mathrm{dom}(e_\delta) = E$ for every $\gamma \neq \delta$ in D. Let us take a maximal $D \subset f^{\leftarrow}(\{n\})$ such that $\langle e_\gamma \rangle_{\gamma \in D}$ forms a $\triangle$-system with root e. Since $\langle e_\gamma \rangle_{\gamma \in \mathfrak{c}}, f, e \in M$, we can assume that we have taken such a $D \in M$. Then D is uncountable, or else adding β to D would contradict maximality. Thus the set

$$H = \{\gamma \in D : (\mathrm{dom}(e_\gamma) \setminus E) \cap (\lambda \cap M) = \emptyset\} \in N$$

is uncountable. Let $v : H \to [\lambda \setminus M]^{<\omega}$ be defined by

$$v(\gamma) = \mathrm{dom}(e_\gamma) \setminus E.$$

Then $v \in N$, and thus there is an $\alpha \in \mathrm{dom}(u) \cap \mathrm{dom}(v)$ such that

$$u(\alpha) = \{c_\xi \restriction A : \xi \in v(\alpha)\}.$$

This α will satisfy (5) with our β. Indeed, $\alpha \in D \subset f^{\leftarrow}(\{n\})$, so $f(\alpha) = n = f(\beta)$. Since α, $h \in N$, it follows that $h(\alpha) \subset N$. By $\beta > \sup A = \sup(\mathfrak{c} \cap N)$, we conclude $\beta \notin h(\alpha)$. Finally, let $\xi \in g(\alpha) = \mathrm{dom}(e_\alpha)$. We will distinguish between two cases.

Case A. If $\xi \in E$, then $c_\xi(\alpha) = e_\alpha(\xi) = e(\xi) = e_\beta(\xi) = c_\xi(\beta)$ and by $\xi \in E \subset \lambda \cap M$ we have $c_\xi \restriction A_\beta = c_\xi \restriction A \in B = B_\beta$. Thus by Case 1 of the definition of $d_\xi(\beta)$, $d_\xi(\beta) = c_\xi(\beta) = c_\xi(\alpha)$.

Case B. If $\xi \in \mathrm{dom}\ (e_\alpha) \setminus E = \mathrm{dom}\ v(\alpha)$, then $c_\xi \restriction A_\beta = c_\xi \restriction A \in u(\alpha) = u_\beta(\alpha)$. Thus by Case 2 of the definition of $d_\xi(\beta)$, it follows that $d_\xi(\beta) = c_\xi(\alpha)$.

The rest of the proof of Theorem 1.1 is similar to the proof of Lemma 1 in [W_1] of M. E. Rudin and S. Watson. □

The rest of the proof of Theorem 1.1. Let $X = \mathfrak{c} \times \omega$. Set, for every $n \in \omega, X_n = \mathfrak{c} \times \{n\}$ and $G_n = \mathfrak{c} \times (n+1)$. To define the topology on X, let us first define, for every $\alpha \in \mathfrak{c}, s \in [\lambda]^{<\omega}$ and $a \in [\mathfrak{c}]^{<\omega}$, the set

$$F(\alpha, s, a) = \{\beta \in \mathfrak{c} : \text{for every } \xi \in s, d_\xi\ (\beta) = c_\xi(\alpha)\} \setminus a.$$

Note that for every $\alpha \in \mathfrak{c}$,

$$\mathfrak{F}_\alpha = \{F(\alpha, s, a) : s \in [\lambda]^{<\omega}, a \in [\mathfrak{c}]^{<\omega}\}$$

is closed under finite intersections. Now, for $x = \langle \alpha, n \rangle \in X$, let

$$\mathfrak{N}_x = \begin{cases} \{\{x\}\}, \text{if } n = 0; \\ \{\{x\} \cup F \times \{n-1\}\ :\ F \in \mathfrak{F}_\alpha\}, \text{if } n \geq 1. \end{cases}$$

Let us now declare a subset U of X to be open if and only if for every $x \in U$, there is an $N \in \mathfrak{N}_x$ such that $N \subset U$. Since each $\mathfrak{N}_x$ is closed under finite intersections and $\cap \mathfrak{N}_x = \{x\}$ for every $x \in X$, this defines a T_1 topology τ. Clearly, each X_n is a discrete subspace of X (equipped with the topology τ), so X is a σ-relatively discrete T_1 space.

To see that X is hereditarily normal, note first that any two disjoint subsets of the same level $X_n = \mathfrak{c} \times \{n\}$ can be separated by disjoint open sets. To see this, we proceed by induction on $n \in \omega$. Since X_0 consists of isolated points, we are done for $n = 0$. Suppose now that $n \geq 1$, and we are done for $n - 1$. Let A_0, A_1 be disjoint subsets of X_n. Take a $\xi < \lambda$ such that $c_\xi(\alpha) = i$, whenever $\langle \alpha, n\rangle \in A_i$ ($i \in 2$). Set $B_i = \{\langle \beta, n-1\rangle : d_\xi(\beta) = i\}$. Then for every $i \in 2$ and $x \in A_i, N_x = \{x\} \cup F(\alpha, \{\xi\}, \emptyset) \times \{n-1\} \in \mathfrak{N}_x$ and $N_x \subset A_i \cup B_i$. By our inductive hypothesis, B_0 and B_1 can be put into disjoint open subsets V_0 and V_1 of $G_{n-1} = \cup_{k \leq n-1} X_k$. Then $A_0 \cup V_0$ and $A_1 \cup V_1$ are disjoint open sets separating A_0 and A_1.

To prove now that X is hereditarily normal, let H, K be subsets of X such that $\bar{H} \cap K = H \cap \bar{K} = \emptyset$. We have to show that H and K can be separated by disjoint open sets. By the standard shoestring argument, we can assume that $H \subset X_n$ for some $n \in \omega$. By passing from K to $K \cup (X_n \backslash H)$, we can also assume that $K \cap X_n \cup H = X_n$.

Claim 1. *If $m \leq n$, then H and $K \cap X_m$ can be put into disjoint open subsets.*

We have already proved Claim 1 for $m = n$, so suppose $m < n$. Take disjoint open subsets U and V of G_m such that $U \supset X_m \setminus \bar{K}$ and $V \supset X_m \cap \bar{K}$. Then $U^* = (X \setminus (G_m \cup \bar{K})) \cup U$ and $V^* = V$ are disjoint open sets separating H and $K \cap X_m$.

By Claim 1, there are disjoint open subsets W_H, W_K of G_n such that $W_H \supset H$ and $W_K \supset K \cap G_n$. Then $W_H^* = W_H$ and $W_K^* = (X \setminus (G_n \cup \bar{H})) \cup W_K$ are disjoint open sets separating H and K.

Finally, let us start the proof that X *is not countably metacompact* by defining a subset Y of $\mathfrak{c}$ to be σ-*decomposable* if we can find an $f : Y \to \omega$ and for every $\alpha \in Y$, an $F_\alpha \in \mathfrak{F}_\alpha$ in such a way that $\alpha \neq \beta$ and $f(\alpha) = f(\beta)$ implies that $\alpha \notin F_\beta$ and $\beta \notin F_\alpha$. Clearly, every countable union of σ-decomposable sets is σ-decomposable and by Lemma 1.2, $\mathfrak{c}$ is *not* σ-decomposable. Hence, whenever $\mathfrak{c}$ is the union of countably many of its subsets, at least one of those subsets is *not* σ-decomposable.

Claim 2. *If $n \in \omega$ and $Y \subset \mathfrak{c}$ is not σ-decomposable, then $Y_1 = \{\alpha \in Y : \langle \alpha, n+1\rangle \in \overline{Y \times \{n\}}\}$ is not σ-decomposable.*

To see that Claim 2 is true, let $Y_0 = Y \setminus Y_1$. Since $\alpha \in Y_0$ implies that there is an $F_\alpha \in \mathfrak{F}_\alpha$ with $F_\alpha \cap Y = \emptyset, Y_0$ is σ-decomposable (in fact, 1-decomposable). If Y_1 were now σ-decomposable, then $Y = Y_0 \cup Y_1$ would be σ-decomposable, contradicting our assumption.

Passing now to the proof that X is not countably metacompact, consider the increasing open cover $\{G_m : m \in \omega\}$ of X. We are going to show that there is no sequence of closed sets $F_m \subset G_m$ in such a way that $\cup_{m\in\omega} F_m = X$. Indeed, if there were such a sequence $\langle F_m \rangle_{m\in\omega}$, then one *of the sets* $Y_m = \{\alpha \in \mathfrak{c} : \langle \alpha, 0\rangle \in F_m\}$ would not be σ-decomposable. By Claim 2, $F_m \supset \overline{Y_m \times \{0\}}$ would not be a subset of G_m, contradiction.

Final Remarks. 1. It remains open whether there is a first countable Dowker space in ZFC. The character of spaces similar to our space has to be large if we want to carry out the construction in ZFC. Indeed, let us say that X is a *nonskipping scattered space of height* ω, if $X = \cup_{n \epsilon \omega} X_n$, the X_n are pairwise disjoint, each X_n consists of isolated points of $\cup_{j \geq n} X_j$, and whenever $n \;\epsilon\; \omega, Y \subset \cup_{k \leq n} X_n$ and $\overline{Y} \cap (\cup_{j \geq n+1} X_j) \neq \emptyset$, then $\bar{Y} \cap X_{n+1} \neq \emptyset$. Note that our space is a nonskipping scattered space of height ω. It is easy to see that regular, hereditarily collectionwise Hausdorff, nonskipping scattered spaces of height ω are paracompact. Since by Fleissner's result [F], every normal space of character $\leq 2^\omega$ is collectionwise Hausdorff in the constructible universe, we conclude that under $V = L$, there are no hereditarily normal Dowker spaces of character $\leq 2^\omega$ which are nonskipping scattered spaces of height ω.

2. On the other hand, there are consistent examples of spaces with small character similar to the space in this paper. P. deCaux's Dowker space from ♣ is of scattered height ω, and it has character $\leq \mathfrak{c}$, because it is locally countable. It is not necessarily hereditarily normal. A deCaux type space of M. E. Rudin [R_4] from $\diamondsuit$ is hereditarily normal, first countable and locally compact, but is neither of scattered height ω nor σ-relatively discrete. Dennis Burke and the author observed that an example of S. Shelah [S] can be modified to obtain a consistent example of a locally countable Dowker space (thus of character $\leq \mathfrak{c}$) which is both hereditarily normal and is of scattered height ω.

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