A SUBMARTINGALE INEQUALITY

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ABSTRACT. This paper extends Burkholder’s inequality between a nonnegative submartingale and a process strongly differentially subordinate to it.

1. STATEMENT OF THE INEQUALITY

Let $0 \leq \alpha \leq 1$ and $1 < p < \infty$. Suppose that $f = (f_n)_{n \geq 0}$ and $g = (g_n)_{n \geq 0}$ are adapted to a filtration $(\mathcal{F}_n)_{n \geq 0}$ of a probability space $(\Omega, \mathcal{F}, P)$. Here $f$ is a nonnegative submartingale and $g$ is $\mathbb{R}^\nu$-valued, where $\nu$ is a positive integer. With $f_n = d_0 + \cdots + d_n$ and $g_n = e_0 + \cdots + e_n$ ($n \geq 0$) we assume that

$$|e_n| \leq |d_n| \quad (n \geq 0),$$

$$|\mathbb{E}(e_n|\mathcal{F}_{n-1})| \leq \alpha \mathbb{E}(d_n|\mathcal{F}_{n-1}) \quad (n \geq 1).$$

Then, with $\|f\|_p = \sup_{n \geq 0} \|f_n\|_p$, we have the inequality

$$\|g\|_p \leq (r-1)\|f\|_p$$

where $r = \max\{\alpha + 1, p/(p-1)\}$ and $r$ is best possible.

Remarks. 1. The case $\alpha = 1$ is shown in Burkholder’s paper [3].

2. For martingales $f$ and $g$ that satisfy (1.1) and have values in a Hilbert space, the inequality (1.3) is shown in Burkholder’s paper [2]. There $r = \max\{p, p/(p-1)\}$.

3. Best possible means that if $r-1$ in (1.3) is replaced by a strictly smaller number $\beta$, then the opposite inequality $\|g\|_p > \beta \|f\|_p$ holds for some probability space and some $f$ and $g$ as above.

2. OUTLINE OF THE PROOF OF THE INEQUALITY

We may assume $\|f\|_p < \infty$. Also, we may assume

$$f_{n-1} > 0 \quad \text{and} \quad g_{n-1} + te_n > 0 \quad \text{for all } t \in \mathbb{R} \text{ and } n \geq 1.$$ 

Indeed, for each $0 < \varepsilon < 1$, the processes $f^\varepsilon$ and $g^\varepsilon$, where $f_n^\varepsilon = f_n + \varepsilon$ and $g_n^\varepsilon = (g_n, \varepsilon)$, satisfy (2.1) and all the assumptions in Section 1. Here $g^\varepsilon$ is a process in $\mathbb{R}^{\nu+1}$. Since $\|g\|_p \leq \|g^\varepsilon\|_p$ and $\|f^\varepsilon\|_p \leq \|f\|_p + \varepsilon$, the inequality (1.3) follows from $\|g^\varepsilon\|_p \leq (r-1)\|f^\varepsilon\|_p$ as $\varepsilon$ tends to 0.
Put $S = (0, \infty) \times \mathbb{R}^r$ and define functions $U$ and $V$ on $S$ by
\begin{align}
U(x, y) &= (|y| - (r - 1)x)(x + |y|)^{p-1}, \\
V(x, y) &= |y|^p - ((r - 1)x)^p.
\end{align}
Then the inequality (1.3) follows from
\begin{equation}
\mathbb{E}V(f_n, g_n) \leq 0 \quad \text{for all } n \geq 0
\end{equation}
which is a consequence of the inequalities:
\begin{align}
\mathbb{E}V(f_n, g_n) &\leq p(1 - 1/r)^{p-1} \mathbb{E}U(f_n, g_n) \quad \text{for all } n \geq 0,
\mathbb{E}U(f_n, g_n) &\leq \mathbb{E}U(f_{n-1}, g_{n-1}) \quad \text{for all } n \geq 1,
\mathbb{E}U(f_0, g_0) &\leq 0.
\end{align}

3. Proof of the inequality

It remains to prove the inequalities (2.5), (2.6) and (2.7).

Proof of (2.5). Notice that (2.5) is proved if $V(x, y) \leq p(1 - 1/r)^{p-1}U(x, y)$ for all $(x, y) \in S$, and if the latter inequality holds if and only if it holds with $x$ replaced by $x/(x + |y|)$ and $y$ replaced by $y/(x + |y|)$. Thus, it is enough to prove the inequality for $x \in (0, 1]$ and $|y| = 1 - x$, or, equivalently, to prove $F \leq 0$ on $[0, 1]$ where
\begin{equation}
F(x) = (1 - x)^p - [(r - 1)x] - p(1 - rx)(1 - 1/r)^{p-1}.
\end{equation}
If $0 < x < 1$, then
\begin{align}
F'(x) &= -(p - 1)[(1 - x)^{p-1} + (r - 1)p x^{p-1}] + r[p(1 - 1/r) - 1] \leq 0, \\
F''(x) &= p(p - 1)[(1 - x)^{p-2} - (r - 1)p x^{p-2}] - r(p - 1).
\end{align}
Notice that $0 < 1/r < 1$ and $F'(1/r) = F''(1/r) = 0$.

If $p = 2$, then $F''(x) = 2[1 - (r - 1)^2] \leq 0$ on $(0, 1)$ because $r \geq (\alpha + 1)/2 \geq 2$. Thus $F$ has the maximum at $x = 1/r$ which proves $F \leq 0$ on $[0, 1]$ in this case.

Now assume $1 < p < 2$. The strict concavity of $\log x$ implies that
\begin{equation}
\log 1 > a \log a + (1 - a) \log(a + 1) \quad \text{or} \quad (a + 1)^{a-1} > a^a \quad \text{for } 0 < a < 1,
\end{equation}
so $a = p - 1$ gives $(p - 1)^{p-1} < p^{p-2}$. Hence $r^{p-1} \geq (p/(p - 1))^{p-1} > p$. Thus $F'(1) = -(r - 1)p - p(1 - r)(1 - 1/r)^{p-1} - ((p - r^{p-1})(r - 1))^{p-2} < 0$. Let $x^*$ be the zero of $F''$. Computation gives $x^*[1 + (r - 1)p/(p - 2)] = 1 > 1/r < x^* < 1$, and $F''(x) < 0$ if $0 < x < x^*$. Thus the maximum of $F$ on $[0, x^*]$ is $F(1/r) = 0$. Also, $F''(x) > 0$ if $x^* < x < 1$. Thus $F$ is convex on $[x^*, 1]$, hence $F \leq 0$ on $[x^*, 1]$; because $F(x^*) \leq 0$ and $F(1) < 0$.

The case $p > 2$ is handled similarly. Here $F(0) < 0$ follows from the inequality
\begin{equation}
(a + 1)^{a-1} < a^a \quad \text{for } a > 1
\end{equation}
which follows from (3.4) upon replacing $a$ by $1/a$.

Proof of (2.6). The proof depends on the inequality: If $(x, y)$ and $(x + h, y + k)$ belong to $(0, \infty) \times \mathbb{R}^r$ with $|h| \geq |k|$, and $y + tk \neq 0$ for all $t \in \mathbb{R}$, then
\begin{equation}
U(x + h, y + k) \leq U(x, y) + \varphi(x, y)h + \psi(x, y) \cdot k
\end{equation}
where \( \varphi(x, y) = U_x(x, y) \) and \( \psi(x, y) = U_y(x, y) \), that is,

\[
\varphi(x, y) = \left| (p - r)(x + |y|) - (p - 1)rx \right| (x + |y|)^{p - 2}, \\
\psi(x, y) = \left| p(x + |y|) - (p - 1)rx \right| (x + |y|)^{p - 2}/|y|.
\]

For a proof of (3.6) in the case \( \alpha = 1 \), see [3]; the case \( 0 \leq \alpha < 1 \) requires only the further observation that \( r \geq p/(p - 1) \) and \( r \geq p \).

Now (2.1) and (3.6) give

\[
(3.7) \quad U(f_n, g_n) \leq U(f_{n-1}, g_{n-1}) + \varphi(f_{n-1}, g_{n-1})d_n + \psi(f_{n-1}, g_{n-1}) \cdot e_n
\]

where all the random variables are integrable: for example the last one is integrable because of the assumption (1.1) and the estimate \( |\psi(x, y)| \leq pr(x + |y|)^{p - 1} \).

Conditioning on \( \mathcal{F}_{n-1} \) and using the assumption (1.2), one sees from (3.7) that

\[
\mathbb{E}U(f_n, g_n) - \mathbb{E}U(f_{n-1}, g_{n-1}) \\
\leq \mathbb{E}(|\varphi(f_{n-1}, g_{n-1}) + \alpha|\psi(f_{n-1}, g_{n-1})| \mathbb{E}(d_n|\mathcal{F}_{n-1})).
\]

Here \( f \) is a submartingale, hence \( \mathbb{E}(d_n|\mathcal{F}_{n-1}) \geq 0 \). Thus the inequality (2.6) follows from the inequality \( \varphi(x, y) + \alpha|\psi(x, y)| \leq 0 \) which, by homogeneity, is equivalent to the inequality \( G(x) \leq 0 \) if \( 0 \leq x \leq 1 \), where

\[
G(x) = (p - r) - (p - 1)rx + \alpha|p - (p - 1)rx|.
\]

Clearly \( G \) is convex, hence it suffices to show \( G(0) \leq 0 \) and \( G(1) \leq 0 \). Here \( G(0) = (\alpha + 1)p - r \leq 0 \), and noting that \( p - (p - 1)r \leq 0 \), one has

\[
G(1) = (p - r) - (p - 1)r - \alpha|p - (p - 1)r| \\
= -r + (1 - \alpha)|p - (p - 1)r| \leq 0.
\]

This proves (2.6).

**Proof of (2.7).** Since \( r \geq 2 \) we have \( |y| - (r - 1)x \leq |y| - x \leq 0 \) if \( |y| \leq x \). Hence (2.7) follows from (2.2), the assumption (1.1) which gives \( |g_0| \leq |f_0| \) when \( n = 0 \), and the nonnegativity of \( f_0 \).

### 4. Sharpness of the Inequality

**Case 1.** Suppose that \( 1 < p \leq (\alpha + 2)/(\alpha + 1) \). Then \( r = p/(p - 1) \) and so \( r - 1 \) is the best constant in (1.3) since it is already the best possible constant if \( f \) is a nonnegative martingale as can be seen in (5.90) and (5.91) of [1].

**Case 2.** Here \( p > (\alpha + 2)/(\alpha + 1) \) so \( r = (\alpha + 1)p \). Choose a small \( \eta > 0 \) so that \( \eta(p - 1)/(\alpha + 1) < 2 \). Define \( (x_n)_{n \geq 1} \) and \( (\pi_n)_{n \geq 1} \) by

\[
rx_n = 2 + n(\alpha + 1)\eta \quad \text{and} \quad \pi_1 = \frac{1}{2}, \pi_{n+1} = \frac{1}{2} \prod_{k=1}^{n} \left( 1 - \frac{\eta}{x_k} \right).
\]

Now we define a filtration \( (\mathcal{F}_n)_{n \geq 0} \) on the Lebesgue probability space \([0, 1]\) as follows: \( \mathcal{F}_0 = \{\varnothing, \{0, 1\}\} \) and for \( n \geq 1 \), \( \mathcal{F}_{2n-1} = \mathcal{F}_{2n} \) is generated by the partition of \([0, 1]\) determined by \( 0 < \pi_n < \pi_{n-1} < \cdots < \pi_1 < 1 \).
Using the same notation for the interval \([a, b]\) and its characteristic function on \([0, 1]\) we put
\[
d_0 = e_0 = [0, 1), \quad d_1 = -e_1 = \left[0, \frac{1}{2}\right) + \left[\frac{1}{2}, 1\right),
\]
\[
d_{2n} = \eta(0, \pi_n), \quad e_{2n} = \alpha d_{2n},
\]
\[
d_{2n+1} = -e_{2n+1} = -\eta(0, \pi_{n+1}) + (x_n - \eta)[\pi_{n+1}, \pi_n).
\]
Let \(f_n = d_0 + \cdots + d_n\) and \(g_n = e_0 + \cdots + e_n\). Then one checks that \(f\) and \(g\) are adapted to \((\mathcal{F}_n)\), that \(f\) is a nonnegative submartingale, and that (1.1) and (1.2) are satisfied. Also,
\[
f_{2n+1} = \sum_{1 \leq k \leq n} x_k[\pi_{k+1}, \pi_k] + 2 \left[\frac{1}{2}, 1\right),
\]
\[
g_{2n+1} = rx_n[0, \pi_{n+1}] + (r-1) \sum_{k=1}^n x_k[\pi_{k+1}, \pi_k].
\]
Thus we get
\[
\|g_{2n+1}\|_p \geq (r-1)\|f_{2n+1}\|_p \left(\frac{A_n}{A_n + 2^p}\right)^{1/p}
\]
with \(A_n = a_1 + \cdots + a_n\) where \(a_n = |x_n|^p(\pi_n - \pi_{n+1})\).

Notice that each pair of stopped processes, \(f_{2n+1}^2\) and \(g_{2n+1}^2 = (g_{k+1}(2n+1))_{k \geq 0}\), satisfies all the assumptions of Section 1. Thus, in order to establish sharpness in Case 2, it is enough to show that \(\lim_{n \to \infty} A_n = \infty\). Put \(b = (\alpha + 1)\eta\). By Taylor’s formula, as \(n \to \infty\),
\[
\frac{a_{n+1}}{a_n} = \left(1 + \frac{b}{2 + nb}\right)^{p-1} \left(1 - \frac{r\eta}{2 + nb}\right)
\]
\[
= \left(1 + \frac{(p-1)b}{2 + b n} + O\left(\frac{b}{2 + bn}\right)^2\right) \left(1 - \frac{r\eta}{2 + nb}\right)
\]
\[
= \left(1 + \frac{p-1}{n} + O\left(\frac{1}{n^2}\right)\right) \left(1 - \frac{p}{n} + O\left(\frac{1}{n^2}\right)\right)
\]
\[
= 1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right).
\]
Now recall the Gauss test for the convergence of a series of positive numbers:

if \(a_n > 0\) and \(\frac{a_{n+1}}{a_n} = 1 - \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)\), then \(\sum a_n < \infty\) if and only if \(\lambda > 1\).

In our case, \(\lambda = 1\) so \(\sum a_n = \lim A_n = \infty\).

This completes the proof that \(r-1\) is the best constant in the inequality (1.3).

\textbf{Remarks.} 1. The inequality (1.3) holds even for \(\alpha > 1\). The proof in Section 3 works provided that \(\alpha \leq 1 + \sqrt{1 + (p-1)^2}/(p-1)\). For \(\alpha > 1\) sharpness of \(r\) could not be established.

2. If \(0 < \|f\|_p < \infty\), then the inequality (1.3) is strict unless \(\alpha = 0\) and \(p = 2\): this follows as in [3] because \(r > 2\) unless \(\alpha = 0\) and \(p = 2\).

3. The inequality (1.3), or, more precisely, the function \(U\) in (2.2), leads to analogous inequalities for Itô processes and smooth functions on Euclidean domains. These follow as in [3]. For example, the condition \(|\Delta v| \leq |\Delta u|\) in [3] is replaced

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by $|\Delta v| \leq \alpha|\Delta u|$. The inequality $\|v\|_p \leq (p^{**} - 1)\|u\|_p$ in [3] becomes $\|v\|_p \leq (r - 1)\|u\|_p$.

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REFERENCES


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