CRUMPLED LAMINATIONS AND MANIFOLDS
OF NONFINITE TYPE

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Abstract. Using a group-theoretic construction due to Bestvina and Brady, we build \((n + 1)\)-manifolds \(W\) which admit partitions into closed, connected \(n\)-manifolds but which do not have finite homotopy type.

At the heart of this note is an example due to Bestvina and Brady \cite{1} of an almost finitely presented group which is not finitely presented. Specifically, they describe a finitely presented group \(G\) with a perfect normal subgroup \(P\) such that \(G/P\) is not finitely presented (i.e., \(P\) fails to be the normal closure in \(G\) of a finite set); furthermore, \(P\) itself is expressed as an infinite free product \(\ast \) of finitely presented groups \(P_i\), which happen to be pairwise isomorphic.

For each positive integer \(m\) let \(\Gamma_m\) denote \(P_1 \ast P_2 \ast \cdots \ast P_m \subset P\), and let \(N_m\) denote the normal closure of \(\Gamma_m\) in \(G\). Then \(N_1 \subset N_2 \subset \cdots \subset N_m \subset N_{m+1} \subset \cdots\) and \(P = \bigcup N_m\). Set \(G'_m = G/N_m\). Note the existence of natural projections \(\psi_m : G'_m \rightarrow G'_{m+1}\); the direct limit of \(\{\psi_m\}\) is \(G/P\). Since \(P = \bigcup N_m\) fails to be finitely generated as a normal subgroup, infinitely many of \(\{\psi_m\}\) must have nontrivial kernel. This answers Questions 3.4 and 3.5 of \cite{2}. We use it here to describe crumpled laminations \(p : W \rightarrow \mathbb{R}\) on manifolds \(W\) which do not have finite homotopy type, answering Question 3.1 of \cite{2} negatively, and illustrating the sharpness of the main result (Theorem 1.1) there.

Recall that a crumpled lamination on an \((n + 1)\)-manifold \(W\) is a closed map \(p\) of \(W\) to an interval \(J\) (possibly noncompact) such that each \(p^{-1}(t)\) \((t \in J)\) is a closed, connected \(n\)-manifold.

Given a compact \(n\)-manifold \(M\), \(n \geq 5\), and a finitely generated, perfect subgroup \(H\) of \(\pi_1(M)\), the mapping cylinder construction of \cite{3} provides a map \(f : M \rightarrow M'\) from \(M\) onto another \(n\)-manifold \(M'\) and a compact \((n + 1)\)-dimensional cobordism \((W, M, M')\), where the \((n + 1)\)-manifold \(W\) is obtained from the mapping cylinder of \(f\) by attaching a collar \(M' \times [1, 2]\) to the obvious copy of \(M'\); in addition, here the inclusion \(M' \rightarrow W\) is a homotopy equivalence, \(\pi_1(M')\) is isomorphic to the quotient of \(G\) by \(N(H)\), the normal closure of \(H\), and inclusion \(M \rightarrow W\) induces the obvious projection \(G \rightarrow G/N(H)\). Since \(W\) is determined as the disjoint union

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of $M \times [0, 1)$ and $M' \times [1, 2)$, it possesses a crumpled lamination derived from an obvious map $p : W \to [0, 2]$ having $n$-manifolds as point preimages.

For $n \geq 5$ name a closed $n$-manifold $M_0$ for which $\pi_1(M_0) \cong G$. The mapping cylinder construction provides a laminated cobordism $(W_1, M_0, M_1)$ such that $M_1 \to W_1$ is a homotopy equivalence, $\pi_1(M_1) \cong \pi_1(W_1) \cong G_1'$, and the inclusion $M_0 \to W_1$ induces the natural projection $\psi_0 : G \to G_1' = G/N_1$. Applying the construction recursively, we obtain successive laminated cobordisms $(W_m, M_{m-1}, M_m)$ such that $M_m \to W_m$ is a homotopy equivalence, $\pi_1(M_m) \cong \pi_1(W_m) \cong G'_m$, and the inclusion $M_{m-1} \to W_m$ induces $\psi_{m-1} : G'_{m-1} \to G'_m$. We regard distinct $W_i, W_j$ as intersecting only if $i = j \pm 1$ and then $W_i \cap W_{i+1} = M_i$. Consequently,

$$W = (M_0 \times (-\infty, 0)) \cup \left( \bigcup_{i=1}^{\infty} W_i \right)$$

is an $(n+1)$-manifold equipped with a lamination. It follows routinely that $\pi_1(W)$ is the direct limit of the inclusion-induced sequence

$$\left\{ \pi_1 \left( (M_0 \times (-\infty, 0)) \cup \left( \bigcup_{i=1}^{m} W_i \right) \right) \to \pi_1 \left( (M_0 \times (-\infty, 0)) \cup \left( \bigcup_{i=1}^{m+1} W_i \right) \right) \right\},$$

namely, $G/P$. Hence, $W$ cannot be homotopy equivalent to a finite complex.

References


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