EACH LOCALLY ONE-TO-ONE MAP FROM A CONTINUUM
ONTO A TREE-LIKE CONTINUUM IS A HOMEOMORPHISM

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Abstract. In 1977 T. Maćkowiak proved that each local homeomorphism from a continuum onto a tree-like continuum is a homeomorphism. Recently, J. Rogers proved that each locally one-to-one (not necessarily open) map from a hereditarily decomposable continuum onto a tree-like continuum is a homeomorphism, and this paper removes “hereditarily decomposable” from the hypothesis of Rogers’ theorem.

It is not easy for a nice function to map onto a tree-like continuum without being a homeomorphism. T. Maćkowiak’s classic result [3], proved in 1977, is that each local homeomorphism from a continuum onto a tree-like continuum is a homeomorphism. Local homeomorphisms are both open and locally one-to-one, and recently J. Rogers asked if “open” could be removed from the hypothesis of the Maćkowiak theorem. In [1] Rogers proved a special case, namely that if a locally one-to-one map that is not a homeomorphism is defined on a hereditarily decomposable continuum, then the image contains a continuum that is not unicoherent. Since all tree-like continua are hereditarily unicoherent, the image cannot be tree-like. These results come from the “complicated proof” found in [2] of Corollary 5.7 in [1]. We use the noun “map” to mean continuous function, and the term “continuum” to mean a connected, compact metric space.

The theorem to follow completes the task of removing “open” from the Maćkowiak theorem. The lemma that is proved first helps to organize the covers.

Definition. A finite collection of sets has a tree-indexing if its members can be labeled \( \{L_1, L_2, \ldots, L_m\} \) so that the \( L_i \) are distinct and for each \( j \) from 2 to \( m \), \( L_j \) intersects exactly one member of the set \( \{L_1, L_2, \ldots, L_{j-1}\} \).

Tree-cover lemma. A finite collection of open sets has a tree-indexing iff its nerve is a tree.

Proof. Suppose \( \{L_1, L_2, \ldots, L_m\} \) is a tree-indexing of a finite collection \( \mathcal{V} \) of open sets. Then \( \mathcal{V} \) is coherent since if \( \mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \), two disjoint subcollections whose unions do not intersect, and if \( L_1 \in \mathcal{V}_1 \), then the element of \( \mathcal{V}_2 \) with the smallest label fails the definition of a tree-indexing. Since \( \mathcal{V} \) is coherent, its nerve is connected. Since no three elements in \( \mathcal{V} \) can intersect, its nerve is a one-dimensional graph. If the nerve of \( \mathcal{V} \) contains a simple closed curve, then the element of \( \mathcal{V} \) with the
largest label, that corresponds to some vertex in the simple closed curve, violates the tree-indexing definition. Therefore the nerve of \( \mathcal{V} \) is a tree. Now, by way of contradiction, let \( \mathcal{V} \) denote the smallest finite collection of open sets whose nerve is a tree but does not have a tree-indexing. Since the nerve of \( \mathcal{V} \) is connected, the collection \( \mathcal{V} \) must be coherent. Clearly, \( \mathcal{V} \) must have more than two elements. Since the nerve of \( \mathcal{V} \) has at least three vertices, the removal of an endpoint \( e \), and the arc that connects \( e \) to the rest of the tree, leaves a tree. By our assumption, this means that the collection \( \mathcal{V} \setminus \{ V(e) \} \), where \( V(e) \) denotes the open set in \( \mathcal{V} \) that corresponds to \( e \), has a tree-indexing \( \{ L_1, L_2, ..., L_m \} \). This generates a tree-indexing for \( \mathcal{V} \) by labeling \( V(e) \) as \( L_{m+1} \).

**Theorem.** Every locally one-to-one map from a continuum onto a tree-like continuum is a homeomorphism.

**Proof.** Suppose that \( h \) is a locally one-to-one map from a continuum \( X \) onto a tree-like continuum \( Y \). It is clear that \( h \) must be finite-to-one, and there is a positive number \( \epsilon \) such that if \( x \) and \( x' \) are points of \( X \) such that \( h(x) = h(x') \), then \( d(x, x') > 3\epsilon \). For each point \( y \) in \( Y \), there is an open set \( U(y) \) in \( Y \) containing \( y \) such that if \( h^{-1}(y) = \{ x_1, x_2, ..., x_n \} \), then \( h^{-1}(U(y)) \subseteq \bigcup_{i=1}^{n} N_{\epsilon}(x_i) \) in \( X \), where \( N_{\epsilon}(x_i) \) denotes the \( \epsilon \) neighborhood of \( x_i \) in \( X \). Now, let \( \mathcal{V} \) denote an open refinement of \( \{ U(y) \mid y \in Y \} \) that covers \( Y \) and whose nerve is a tree. By the tree-cover lemma, \( \mathcal{V} \) can be written \( \{ L_1, L_2, ..., L_m \} \), satisfying the definition of a tree-indexing. For each \( L_i \in \mathcal{V} \), let \( y_i \) denote a member of \( Y \) such that \( L_i \subseteq U(y_i) \). Index the elements of \( h^{-1}(y_i) = \{ x_1, x_2, ..., x_k \} \), and for \( j = 1, ..., k \), define \( W(i, j) = h^{-1}(L_i) \cap N_{\epsilon}(x_j) \), if this set is non-empty. Note that for each relevant \( i \) and \( j \), \( W(i, j) \) is an open set in \( X \) of diameter less than \( \epsilon \), and \( h \) is one-to-one on \( W(i, j) \). Now define \( W \) to be the set of these \( W(i, j) \)'s. Then \( W \) is an open covering of \( X \).

**3-link fact.** If \( W(i_1, j_1) \), \( W(i_2, j_2) \), and \( W(i_3, j_3) \) are distinct elements of \( W \) such that \( W(i_2, j_2) \) intersects each of the other two, then the integers \( \{ i_1, i_2, i_3 \} \) are distinct.

Let \( z_1 \) denote a point of \( W(i_1, j_1) \cap W(i_2, j_2) \), let \( z_3 \) denote a point of \( W(i_3, j_3) \cap W(i_2, j_2) \), and note that \( d(z_1, z_3) < \epsilon \). First, suppose that \( i_1 = i_2 \). By construction, \( W(i_1, j_1) \subseteq N_{\epsilon}(x_{j_1}) \) and \( W(i_2, j_2) \subseteq N_{\epsilon}(x_{j_2}) \), where \( h(x_{j_1}) = h(x_{j_2}) = y_i \). But \( z_1 \in N_{\epsilon}(x_{j_1}) \cap N_{\epsilon}(x_{j_2}) \) implies that \( d(x_{j_1}, x_{j_2}) < 2\epsilon \), that is, \( j_1 = j_2 \). This is contrary to the fact that the \( W \)'s are distinct. A similar contradiction occurs if \( i_3 = i_2 \). Secondly, suppose that \( i_1 = i_3 \). Again, by construction, \( z_1 \in W(i_1, j_1) \subseteq N_{\epsilon}(x_{j_1}) \) and \( z_3 \in W(i_1, j_3) \subseteq N_{\epsilon}(x_{j_3}) \), where \( h(x_{j_1}) = h(x_{j_3}) = y_i \). Thus \( d(x_{j_1}, x_{j_3}) > 3\epsilon \) if \( j_1 \neq j_3 \). But this is contrary to the fact that each of the following numbers is less than \( \epsilon \) : \( d(x_{j_1}, z_1) \), \( d(z_1, z_3) \), and \( d(x_{j_3}, z_3) \). This contradiction completes the proof of the 3-link fact.

Now, back to the proof of the theorem. If \( h \) is not a homeomorphism, then \( h \) is not one-to-one, so there exist two points \( x_1 \) and \( x_2 \) such that \( h(x_1) = h(x_2) \). So \( x_1 \in W(i, j) \) and \( x_2 \in W(i, k) \) for some \( i \) and \( j \neq k \), and there is a chain of elements from \( W \) with first link \( W(i, j) \) and last link \( W(i, k) \). Let \( C = \{ W(k_1, n_1), W(k_2, n_2), ..., W(k_m, n_m) \} \) denote a chain in \( W \) of shortest length such that \( k_1 = k_m \). By the 3-link fact, \( m > 3 \). The indexing on \( C \) is understood to be the usual chain indexing, where the links are distinct and \( W(k_i, n_i) \) intersects \( W(k_j, n_j) \) if \( |i - j| < 1 \). Let \( k_j \) be the smallest integer in \( \{ k_1, k_2, ..., k_m \} \), where we use \( j = 1 \) if the smallest is \( k_1 = k_m \). Then \( k_{j+1} > k_j \) and \( k_{j+2} > k_{j+1} \). To see this second inequality, note
that in $\mathcal{V}$, $L_{k_{j+1}}$ intersects $L_{k_j}$ since $W(k_{j+1}, n_{j+1})$ intersects $W(k_j, n_j)$, so $L_{k_j}$ is the only element of $\mathcal{V}$ with lower subscript that $L_{k_{j+1}}$ intersects. This means that, since $L_{k_{j+1}}$ also intersects $L_{k_{j+2}}$, the subscript $k_{j+2}$ must be greater than $k_{j+1}$. If we continue in this way we can establish the fact that $k_j < k_{j+1} < k_{j+2} < \ldots < k_m = k_1 < k_2 < \ldots < k_{j-1}$. Note that when we “turn the corner” we use the fact that $k_2 \neq k_{m-1}$, which follows since $m > 3$. The final contradiction is that the last link $L_{k_{j-1}}$ intersects both of the lower indexed links $L_{k_{j-2}}$ and $L_{k_j}$.

\[ \square \]

REFERENCES


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