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A BOUNDARY VALUE PROBLEM FOR HERMITIAN HARMONIC MAPS AND APPLICATIONS

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Abstract. We study the existence and uniqueness problems for Hermitian harmonic maps from Hermitian manifolds with boundary to Riemannian manifolds of nonpositive sectional curvature and with convex boundary. The complex analyticity of such maps and the related rigidity problems are also investigated.

1. Introduction

In [JY1], Jost and Yau introduced and studied a nonlinear elliptic system of second-order partial differential equations imposed on a map from a Hermitian manifold into a Riemannian manifold. In many cases, this system is more appropriate to Hermitian geometry than the harmonic map system. To illustrate the difference between this new system and the harmonic map system, we first fix some notation. Let M be a compact complex manifold with nonempty smooth boundary. h is a Hermitian metric on M which is compatible with the complex structure on M. Let N be a Riemannian manifold with metric g and Christoffel symbols Γ. A harmonic map f from M to N is given by, in local coordinates,

$$\frac{1}{2}\frac{\partial}{\partial z^{\overline{\beta}}}\Big(h^{\alpha\overline{\beta}}\frac{\partial f^i}{\partial z^\alpha}\Big)+\frac{1}{2}\frac{\partial}{\partial z^\alpha}\Big(h^{\alpha\overline{\beta}}\frac{\partial f^i}{\partial z^{\overline{\beta}}}\Big)+h^{\alpha\overline{\beta}}\Gamma^i_{jk}(f(z))\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\overline{\beta}}}=0 \tag{1.1}$$

for $i=1,...,\dim N$. A Hermitian harmonic map f from M to N is given by

$$h^{\alpha\overline{\beta}}\Big(\frac{\partial^2 f^i}{\partial z^\alpha\partial z^{\overline{\beta}}}+\Gamma^i_{jk}\frac{\partial f^j}{\partial z^a}\frac{\partial f^k}{\partial z^{\overline{\beta}}}\Big)=0 \tag{1.2}$$

for $i=1,\dots,\dim N$. It is well known that unless M is Kähler a holomorphic map does not have to be harmonic, i.e. satisfying (1.1). As pointed out in [JY1], (1.1) and (1.2) are equivalent if M is Kähler. Since (1.2) does not have a divergence structure nor a variational structure, (1.2) is analytically more difficult than (1.1). If N is Kähler, the Kähler structure forces all i,j,k in Γ^i_{jk} with bar, the complex conjugate, or all without bar, hence a holomorphic map satisfies (1.2) in this case.

The following existence and uniqueness result was obtained in [JY1] via the heat flow method.

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Theorem 1.1 (Jost-Yau)**.** *Let M be a compact complex manifold with nonempty smooth boundary ∂M and Hermitian metric h. Let N be a Riemannian manifold of nonpositive sectional curvature. Let $\phi : M \to N$ be continuous. Then there is a unique Hermitian harmonic map f with $f|_{\partial M} = \phi|_{\partial M}$ and f is homotopic to ϕ with respect to the fixed boundary value.*

Throughout this paper, we assume that both domain and image manifolds have smooth nonempty boundaries and the image space has nonpositive sectional curvature. In section 2, we derive a maximum principle and use it to study the parabolic system associated to the elliptic system, i.e. Hermitian harmonic maps. We show that if a solution of the parabolic system stays initially in a domain with convex boundary in the image space it remains so.

In section 3, we apply the result in section 2 and the result in [JY1] on the long time existence of solutions for the Dirichlet problem when the image has no boundary in order to solve the Dirichlet problem when the image space has nonempty smooth convex boundary. The arguments we use in sections 2 and 3 are similar to those for ordinary harmonic maps in [Ham].

In section 4, we consider the holomorpic extension problem for CR maps which map the boundary of an astheno-Kähler manifold into the boundary of a Kähler manifold with strongly seminegative sectional curvature. Note that in [JY1] a complex manifold is called astheno-Kähler if it carries a (1,1)-form with the following two properties: (1) ω^m is a positive multiple of the volume form and (2) $\partial\overline{\partial}\omega^{m-2} = 0$. Note that if $\dim_C M = 2$, then M is automatically astheno-Kähler. Since we utilize the existence of Hermitian harmonic maps, we assume the boundary of the image space is convex. As in [JY1] and [S1] a Bochner type formula plays an important role in proving the complex analyticity of Hermitian harmonic maps. We also need some convexity assumptions on the boundary of the domain space, namely the trace of the Levi form is nonnegative, to control the boundary integrals which arise from applying Stokes's theorem.

We would like to mention that there are many works on rigidity problems by using harmonic mappings, in Riemannian and Kähler settings (for example [Co], [GS], [JY1], [JY2], [JY3], [M1], [M2], [NS], [S1] and [S2]). In section 5, we apply our existence result on holomorphic extension to study the biholomorphic equivalence of an astheno-Kähler manifold M and a Kähler manifold N both with bonudaries. We show that under the assumptions on $(M, \partial M)$ and $(N, \partial N)$ as in section 4, M and N are biholomorphically equivalent provided there is a CR-diffeomorphism from ∂M to ∂N which extends to a homotopy equivalence of M and N. We would like to point out that a diffeomorphism between two CR manifolds is a CR-diffeomorphism if it preserves the Cartan-Moser chains, due to a result of J.-H. Cheng [C].

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2. A maximum principle

Let M be a compact complex manifold with nonempty smooth boundary. Let N be a compact Riemannian manifold with nonempty smooth convex boundary and S be a compact submanifold in N such that $\dim S = \dim N$ and ∂S is convex. We consider the parabolic system for Hermitian harmonic maps

$$f : M \times [0, \infty) \to N,$$
$$f(z, 0) = g(z) \text{ for } z \in M,$$

$$f(z,t) = g(z) \text{ for } z \in \partial M, 0 \le t \le \infty,$$
$$\frac{\partial f^i}{\partial t} = h^{\alpha\overline{\beta}}\Big(\frac{\partial^2 f^i}{\partial z^\alpha \partial z^{\overline{\beta}}} + \Gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\overline{\beta}}}\Big),\ i = 1, ..., \dim N.$$

Theorem 2.1. *If $f(z,a) \in S$ and $f(\partial M \times [a,b]) \in S$, then $f(M \times [a,b]) \in S$.*

Proof. Let σ be a smooth function on N such that σ is negative inside S and positive outside S; moreover, in a neighborhood of U of ∂S choose σ equal to the distance from ∂S, negative inside and positive outside. Set $\rho = \sigma \circ f$. Then ρ is smooth in the interior and continuous on $M \times [a,b]$. Now we derive a parabolic equation for ρ to satisfy. From the chain rule

$$\frac{\partial \rho}{\partial t} = \frac{\partial \sigma}{\partial y^i}\frac{\partial f^i}{\partial t}.$$

Also, we have

$$\begin{aligned}
\triangle\rho &= \sqrt{h}^{-1}\Big(\partial_\alpha(h^{\alpha\overline{\beta}}\sqrt{h}\partial_{\overline{\beta}}) + \partial_{\overline{\beta}}(h^{\alpha\overline{\beta}}\sqrt{h}\partial_\alpha)\Big)\rho \\
&= \frac{\partial\sigma}{\partial y^i}\Big(2h^{\alpha\overline{\beta}}\frac{\partial^2 f^i}{\partial z^\alpha\partial z^{\overline{\beta}}} + \sqrt{h}^{-1}(\frac{\partial(\sqrt{h}h^{\alpha\overline{\beta}})}{\partial z^{\overline{\beta}}}\frac{\partial f^i}{\partial z^\alpha} + \frac{\partial(\sqrt{h}h^{\alpha\overline{\beta}})}{\partial z^\alpha}\frac{\partial f^i}{\partial z^{\overline{\beta}}})\Big) \\
&\quad +2h^{\alpha\overline{\beta}}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\overline{\beta}}}\frac{\partial^2\sigma}{\partial y^j\partial y^k} \\
&= 2\frac{\partial\sigma}{\partial y^i}\Big(h^{\alpha\overline{\beta}}\frac{\partial^2 f^i}{\partial z^\alpha\partial z^{\overline{\beta}}} + h^{\alpha\overline{\beta}}\Gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\overline{\beta}}}\Big) \\
&\quad +2h^{\alpha\overline{\beta}}\Big(\frac{\partial^2\sigma}{\partial y^j\partial y^k} - \Gamma^i_{jk}\frac{\partial\sigma}{\partial y^i}\Big)\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\overline{\beta}}} \\
&\quad +\sqrt{h}^{-1}\frac{\partial\sigma}{\partial y^i}\Big(\frac{\partial(\sqrt{h}^{-1}h^{\alpha\overline{\beta}})}{\partial z^\alpha}\frac{\partial f^i}{\partial z^{\overline{\beta}}} + \frac{\partial(\sqrt{h}^{-1}h^{\alpha\overline{\beta}})}{\partial z^{\overline{\beta}}}\frac{\partial f^i}{\partial z^\alpha}\Big) \\
&= 2\frac{\partial\sigma}{\partial y^i}\Big(h^{\alpha\overline{\beta}}\frac{\partial^2 f^i}{\partial z^\alpha\partial z^{\overline{\beta}}} + h^{\alpha\overline{\beta}}\Gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\overline{\beta}}}\Big) \\
&\quad +2h^{\alpha\overline{\beta}}\Big(\frac{\partial^2\sigma}{\partial y^j\partial y^k} - \Gamma^i_{jk}\frac{\partial\sigma}{\partial y^i}\Big)\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\overline{\beta}}} \\
&\quad +\frac{1}{2}h^{-1}\Big(\frac{\partial h}{\partial z^\alpha}h^{\alpha\overline{\beta}}\frac{\partial\rho}{\partial z^{\overline{\beta}}} + \frac{\partial h}{\partial z^{\overline{\beta}}}h^{\alpha\overline{\beta}}\frac{\partial\rho}{\partial z^\alpha}\Big) + (\frac{\partial h^{\alpha\overline{\beta}}}{\partial z^\alpha}\frac{\partial\rho}{\partial z^{\overline{\beta}}} + \frac{\partial h^{\alpha\overline{\beta}}}{\partial z^{\overline{\beta}}}\frac{\partial\rho}{\partial z^\alpha}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\frac{\partial\rho}{\partial t} &= \frac{1}{2}\triangle\rho - h^{\alpha\overline{\beta}}\Big(\frac{\partial^2\sigma}{\partial y^j\partial y^k} - \Gamma^i_{jk}\frac{\partial\sigma}{\partial y^i}\Big)\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\overline{\beta}}} \\
&\quad -\frac{1}{4}h^{-1}\Big(\frac{\partial h}{\partial z^\alpha}h^{\alpha\overline{\beta}}\frac{\partial\rho}{\partial z^{\overline{\beta}}} + \frac{\partial h}{\partial z^{\overline{\beta}}}h^{\alpha\overline{\beta}}\frac{\partial\rho}{\partial z^\alpha}\Big) \\
&\quad -\frac{1}{2}(\frac{\partial h^{\alpha\overline{\beta}}}{\partial z^\alpha}\frac{\partial\rho}{\partial z^{\overline{\beta}}} + \frac{\partial h^{\alpha\overline{\beta}}}{\partial z^{\overline{\beta}}}\frac{\partial\rho}{\partial z^\alpha}).
\end{aligned}$$

Note that for ordinary harmonic maps, we do not have the last term in the above equation (see [Ham]). Since ∂S is convex by assumption, the second term on the right-hand side can be handled as in [Ham, p. 103]. In fact, the symmetric matrix

$$A_{jk} = (\frac{\partial^2\sigma}{\partial y^j\partial y^k} - \Gamma^i_{jk}\frac{\partial\sigma}{\partial y^i})$$

is seminegative definite. Using the argument in [Ham, p. 104] for the largest and smallest eigenvalues of A, we have

$$\begin{aligned}\frac{\partial\rho}{\partial t} \le \frac{1}{2}\triangle\rho + C\rho - \frac{1}{4}h^{-1}\Big(\frac{\partial h}{\partial z^\alpha}h^{\alpha\overline{\beta}}\frac{\partial\rho}{\partial z^{\overline{\beta}}} + \frac{\partial h}{\partial z^{\overline{\beta}}}h^{\alpha\overline{\beta}}\frac{\partial\rho}{\partial z^\alpha}\Big)\\ - \frac{1}{2}\Big(\frac{\partial h^{\alpha\overline{\beta}}}{\partial z^\alpha}\frac{\partial\rho}{\partial z^{\overline{\beta}}} + \frac{\partial h^{\alpha\overline{\beta}}}{\partial z^{\overline{\beta}}}\frac{\partial\rho}{\partial z^\alpha}\Big)\end{aligned} \tag{2.1}$$

for all z such that $f(z) \in U$, where U is some neighborhood of ∂S and the largest and smallest eigenvalues of $A(z)$ are Lipschitz (see p.104 in [Ham]). Suppose that f does not stay entirely in S. We can choose c with $a < c < b$ so that $f|_{M\times[a,c]}$ does not stay entirely in S but stays in $S \cup U$. Then we have $\rho \le 0$ on $M \times a$ and $\partial M \times [a, c]$ since f maps into S there, and (2.1) holds if $\rho > 0$. Thus, Theorem 2.1 will follow from the maximum principle below.

Lemma 2.2. *Let f be a continuous function on $M \times [a, b]$ with $f \le 0$ on $M \times a$ and on $\partial M \times [a, b]$. Suppose that wherever $f > 0$ we have that f is smooth and*

$$\begin{aligned}\frac{\partial f}{\partial t} \le \frac{1}{2}\triangle f + Cf - \frac{1}{4}h^{-1}\Big(\frac{\partial h}{\partial z^\alpha}h^{\alpha\overline{\beta}}\frac{\partial f}{\partial z^{\overline{\beta}}} + \frac{\partial h}{\partial z^{\overline{\beta}}}h^{\alpha\overline{\beta}}\frac{\partial f}{\partial z^\alpha}\Big)\\ - \frac{1}{2}\Big(\frac{\partial h^{\alpha\overline{\beta}}}{\partial z^\alpha}\frac{\partial f}{\partial z^{\overline{\beta}}} + \frac{\partial h^{\alpha\overline{\beta}}}{\partial z^{\overline{\beta}}}\frac{\partial f}{\partial z^\alpha}\Big).\end{aligned}$$

Then $f \le 0$.

We use the argument in [Ham]. Set $u = e^{-(1+C)t}f$. Then

$$\begin{aligned}&u \le 0 \text{ on } M \times a \text{ and } \partial M \times [a, b],\\ &u > 0 \text{ and } u \text{ is smooth, if } f > 0,\\ &\triangle u = e^{-(1+C)t}\triangle f,\\ &\frac{\partial u}{\partial t} = e^{-(1+C)t}\frac{\partial f}{\partial t} - (1 + C)e^{-(1+C)t}f.\end{aligned}$$

Hence, we have

$$\begin{aligned}\frac{\partial u}{\partial t} \le \frac{1}{2}\triangle u - u - \frac{1}{4}h^{-1}\Big(\frac{\partial h}{\partial z^\alpha}h^{\alpha\overline{\beta}}\frac{\partial u}{\partial z^{\overline{\beta}}} + \frac{\partial h}{\partial z^{\overline{\beta}}}h^{\alpha\overline{\beta}}\frac{\partial u}{\partial z^\alpha}\Big)\\ - \frac{1}{2}\Big(\frac{\partial h^{\alpha\overline{\beta}}}{\partial z^\alpha}\frac{\partial u}{\partial z^{\overline{\beta}}} + \frac{\partial h^{\alpha\overline{\beta}}}{\partial z^{\overline{\beta}}}\frac{\partial u}{\partial z^\alpha}\Big).\end{aligned}$$

Assume that u achieves its maximum at some point (z, t). Then if f is positive somewhere, $u(z, t)$ must be positive and $z \notin \partial M$ and $t > a$. But

$$\begin{aligned}&\frac{\partial u}{\partial t}(z, t) \ge 0,\\ &\frac{\partial u}{\partial z^i}(z, t) = 0,\\ &\triangle u(z, t) \le 0.\end{aligned}$$

Hence $u(z, t) \le 0$, which is a contradiciton. This completes the proof of Lemma 2.2. □

3. The Dirichlet problem

Now we solve a Dirichlet problem for Hermitian harmonic maps for image spaces with smooth convex nonempty boundary.

Theorem 3.1. *Let M be a compact complex manifold with smooth nonempty boundary ∂M and N be a compact Riemannian manifold with smooth nonempty convex boundary ∂N and of nonpositive sectional curvature. Let $u : M \to N$ be continuous and u map ∂M into ∂N. Then there exists a unique Hermitian harmonic map $f : M \to N$ with $f|_{\partial M} = u|_{\partial N}$ and f is homotopic to u with respect to the fixed boundary value.*

Proof. We will use an approximation argument. Set $N_\epsilon = \{y \in N : d(y, \partial N) \geq \epsilon\}$ for small positive ϵ. Consider the following Dirichlet initial-boundary problem

$$\begin{aligned}
&\frac{\partial f_\epsilon}{\partial t} = h^{\alpha\overline{\beta}}\Big(\frac{\partial^2 f_\epsilon^i(z,t)}{\partial z^a \partial z^{\overline{\beta}}} + \Gamma^i_{jk}\frac{\partial f_\epsilon^j}{\partial z^\alpha}\frac{\partial f_\epsilon^k}{\partial z^{\overline{\beta}}}\Big),\ i = 1, ..., \dim N,\\
&f_\epsilon : M \times [0,\infty) \to N,\\
&f_\epsilon(z, o) = u_\epsilon(z) \text{ for } z \in M,\\
&f_\epsilon(z,t) = u_\epsilon(z) \text{ for } z \in \partial M, 0 \leq t \leq \infty,
\end{aligned}$$

where u_ϵ is a continuous map from M into N_ϵ. For any fixed $\epsilon > 0$, N_ϵ stays away from ∂N. So we can apply the results in [JY1]. In particular, we have the long time existence for f_ϵ. Since f_ϵ initially stays in N_ϵ, Theorem 2.1 implies that f_ϵ remains inside N_ϵ because for sufficient small positive ϵ the boundary of N_ϵ is smooth and convex. Then as shown in [JY1], there exists a subsequence $f_\epsilon(z, t_n)$ that converges as t_n goes to infinity to a smooth Hermitian harmonic map, still denoted by f_ϵ, homotopic to u_ϵ and with $f_\epsilon|_{\partial M} = u_\epsilon|_{\partial M}$.

Next, we choose the maps u_ϵ such that $u_\epsilon \to u$ pointwise as $\epsilon \to 0$. Now it is easy to see that $N_\epsilon \to N$ and $f_\epsilon(\cdot, t_n)$ converge, by taking a diagonal subsequence if necessary, to a Hermitian harmonic map f with $f|_{\partial M} = u|_{\partial M}$ and homotopic to u w.r.t. the fixed boundary data. □

4. Application 1: Holomorphic extension

Let $(M, \partial M)$ and $(N, \partial N)$ be two complex manifolds with smooth boundary. A smooth map $f : \partial M \to \partial N$ is a CR-mapping if the differential df_p of f restricted to $H_p(\partial M) = T_p(\partial M) \cap J_M T_p(\partial M)$ is complex linear at every point $p \in \partial M$. f is a CR-mapping if and only if it satisfies the tangential Cauchy-Riemann equation $\overline{\partial}_b f = \overline{\partial} f \circ \pi = 0$, where π is the orthogonal projection from $T_p(\partial M)$ to $T_p(\partial M) \cap J_M T_p(\partial M)$ for any $p \in \partial M$. Recall that the Levi form can be defined in terms of the complex Hessian of a defining function for $(M, \partial M)$. In fact, if $\{z^2, \ldots, z^m\}$ is the compplex tangential holomorphic coordinates at $p \in \partial M$ and ρ is the defining function with $|\partial\rho| = 1$, then the trace of the Levi form with respect to the Hermitian metric h at the point p is

$$\sum_{i,j=2}^{m} h^{i\overline{j}}(p)\frac{\partial^2 \rho}{\partial z^i \partial \overline{z}^j}(p).$$

The boundary ∂M of M is (strictly) pseudoconvex if the Levi form is positive (definite) semidefinite at each point on ∂M. We recall a definition in [S1]. Let

R denote the Riemannian curvature tensor on N. We say that N has strongly seminegative curvature if

$$\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\overline{\beta}\gamma\overline{\delta}}(A^\alpha\overline{B^\beta} - C^\alpha\overline{D^\beta})(\overline{A^\delta}B^\gamma - \overline{C^\delta}D^\gamma)$$

is nonnegative for arbitrary complex numbers A^α, B^α, C^α and D^α when $A^\alpha\overline{B^\beta} - C^\alpha\overline{D^\beta} \neq 0$ for at least one pair of indices (α, β). Clearly strongly seminegativity of curvature implies seminegativity of sectional curvature.

Theorem 4.1. *Let M be a compact astheno-Kähler manifold with smooth non-empty boundary ∂M such that the trace of the Levi form is nonnegative at each point on ∂M. Let N be a compact Kähler manifold of strongly seminegative curvature whose boundary ∂N is convex. If $\phi \in C^2(\overline{M}, N)$ satisfies the tangential Cauchy-Riemann equation and maps ∂M to ∂N, then there exists a pluriharmonic map f from M to N such that $f|_{\partial M} = \phi|_{\partial M}$. Moreover, if the real rank of f is not less than 3 at some point on ∂M, then f is holomorphic or conjugate holomorphic.*

Proof. According to the existence theorem in [JY1], there is a Hermitian harmonic map solving the Dirichlet problem. Since

$$\partial\overline{\partial}\omega^{m-2} = 0,$$

where $m = \dim M$ and

$$\begin{aligned}&\overline{\partial}(\partial(g_{i\overline{j}}\overline{\partial}f^i \wedge \partial f^{\overline{j}}) \wedge \omega^{m-2})\\ &\qquad = \overline{\partial}\partial(g_{i\overline{j}}\overline{\partial}f^i \wedge \partial f^{\overline{j}}) \wedge \omega^{m-2} - \partial(g_{i\overline{j}}\overline{\partial}f^i \wedge \partial f^{\overline{j}}) \wedge \overline{\partial}\omega^{m-2},\end{aligned}$$

we have

$$\begin{aligned}0 &= \int_M g_{i\overline{j}}\overline{\partial}f^i \wedge \partial f^{\overline{j}} \wedge \partial\overline{\partial}\omega^{m-2}\\ &= \int_M \Big(\partial(g_{i\overline{j}}\overline{\partial}f^i \wedge \partial f^{\overline{j}} \wedge \overline{\partial}\omega^{m-2}) - \partial(g_{i\overline{j}}\overline{\partial}f^i \wedge \partial f^{\overline{j}}) \wedge \overline{\partial}\omega^{m-2})\Big)\\ &= \int_{\partial M} g_{i\overline{j}}\overline{\partial}f^i \wedge \partial f^{\overline{j}} \wedge \overline{\partial}\omega^{m-2} + \int_M \overline{\partial}(\partial(g_{i\overline{j}}\overline{\partial}f^i \wedge \partial f^{\overline{j}}) \wedge \omega^{m-2})\\ &\quad - \int_M \overline{\partial}\partial(g_{i\overline{j}}\overline{\partial}f^i \wedge \partial f^{\overline{j}}) \wedge \omega^{m-2}\\ &= \int_{\partial M} g_{i\overline{j}}\overline{\partial}f^i \wedge \partial f^{\overline{j}} \wedge \overline{\partial}\omega^{m-2} + \int_{\partial M} \partial(g_{i\overline{j}}\overline{\partial}f^i \wedge \partial f^{\overline{j}}) \wedge \omega^{m-2}\\ &\quad - \int_M \overline{\partial}\partial(g_{i\overline{j}}\overline{\partial}f^i \wedge \partial f^{\overline{j}}) \wedge \omega^{m-2}\end{aligned}$$

by Stokes' theorem. As shown in [JY1] and [S1],

$$\begin{aligned}&\int_M \overline{\partial}\partial(g_{i\overline{j}}\overline{\partial}f^i \wedge \partial f^{\overline{j}}) \wedge \omega^{m-2}\\ &\qquad = -\int_M (R_{i\overline{j}k\overline{l}}\overline{\partial}f^i \wedge \partial f^{\overline{j}} \wedge \partial f^k \wedge \overline{\partial}f^{\overline{l}} \wedge \omega^{m-2} - g_{i\overline{j}}D'\overline{\partial}f^i \wedge D''\partial f^{\overline{j}} \wedge \omega^{m-2}).\end{aligned}$$

Since ω^m is a positive multiple of the volume form of M, the integrand on the right-hand side is pointwise nonnegative as pointed out in [JY1]. Hence

$$\int_M \overline{\partial}\partial(g_{i\overline{j}}\overline{\partial}f^i \wedge \partial f^{\overline{j}}) \wedge \omega^{m-2} \leq 0. \tag{4.1}$$

Next we will prove the boundary integrals are also nonnegative by using the assumption $\overline{\partial}_b f = 0$ on ∂M and the Levi form. Since N is Kähler, we can take a holomorphic normal coordinate neighborhood of $f(p) \in N$. We have

$$\begin{aligned} B &= \partial(g_{i\overline{j}}\overline{\partial} f^i \wedge \partial f^{\overline{j}}) \wedge \omega^{m-2} + g_{i\overline{j}}\overline{\partial} f^i \wedge \partial f^{\overline{j}} \wedge \overline{\partial}\omega^{m-2} \\ &= \sum_i (\partial\overline{\partial} f^i \wedge \partial f^{\overline{i}} - \overline{\partial} f^i \wedge \partial^2 f^{\overline{i}}) \wedge \omega^{m-2} + \sum_i \overline{\partial} f^i \wedge \partial f^{\overline{i}} \wedge \overline{\partial}\omega^{m-2} \\ &= \sum_{i,\alpha,\beta} f^i_{\alpha\overline{\beta}} f^{\overline{i}}_{\gamma} dz^\alpha \wedge dz^{\overline{\beta}} \wedge dz^\gamma \wedge \omega^{m-2} + \sum_i \overline{\partial} f^i \wedge \partial f^{\overline{i}} \wedge \overline{\partial}\omega^{m-2}. \end{aligned}$$

Since f satisfies the tangential Cauchy-Riemann equation,

$$\overline{\partial}_b f = 0,$$

i.e., for any $i = 1, ..., n = \dim N$ and $\alpha = 2, ..., m$ where z^1 is the complex normal direction at $p \in \partial M$, we have

$$\overline{\partial}_\alpha f^i = 0.$$

Therefore

$$B = \sum_{i,\alpha,\beta} f^i_{\alpha\overline{\beta}} f^{\overline{i}}_1 dz^\alpha \wedge dz^{\overline{\beta}} \wedge dz^1 \wedge \omega^{m-2} + \sum_{i,\alpha,\beta} f^{\overline{i}}_1 f^i_{\overline{1}} dz^{\overline{1}} \wedge dz^1 \wedge \overline{\partial}\omega^{m-2}.$$

Note that $dz^{\overline{1}} \wedge dz^\alpha \wedge dz^\beta \wedge \omega^{m-2}$ has to contain dz^1, otherwise it equals 0. But at p, $dz^1 \wedge dz^{\overline{1}}$ is in the real normal direction of ∂M at p. Hence

$$dz^1 \wedge dz^{\overline{1}} = 0 \text{ at } p.$$

Thus, at p

$$B = \sum_{i,\alpha,\beta} f^i_{\alpha\overline{\beta}} f^{\overline{i}}_1 dz^\alpha \wedge dz^{\overline{\beta}} \wedge dz^1 \wedge \omega^{m-2}.$$

As we mentioned, given any $p \in \partial M$, we can choose local holomorphic coordinates, such that $\partial z^2, \cdots, \partial z^m$ are the complex tangential vectors at p, and ∂z^1 is the complex normal vector to ∂M at p. Moreover, we let $\rho = \rho(z^1, \cdots, z^m, \overline{z}^1, \cdots, \overline{z}^m)$ be a real-valued smooth function defined in a neighborhood U of p such that $M \cap U = \{x \in U : \rho(x) < 0\}$ and $d\rho(x) \neq 0$ on $\partial M \cap U$; and

$$\frac{\partial \rho}{\partial z^\beta}(p) = \frac{\partial \rho}{\partial z^{\overline{\beta}}}(p) = 0$$

for all $\beta = 2, \cdots, m$; and

$$\frac{\partial \rho}{\partial z^1}(p) = 1.$$

As in [NS], we can rewrite the tangential Cauchy-Riemann equation in terms of the defining function ρ as

$$\overline{\partial}_b f^i = \sum_{\alpha=2}^m (\partial_{\overline{\alpha}} f^i - \partial_{\overline{1}} f^i (\partial_{\overline{\alpha}}\rho / \partial_{\overline{1}}\rho)) d\overline{z}^i, \quad 1 \le i \le n.$$

Thus

$$\partial_{\overline{\alpha}} f^i = \partial_{\overline{1}} f^i (\partial_{\overline{\alpha}}\rho / \partial_{\overline{1}}\rho), \quad 1 \le i \le n,\ 2 \le \alpha \le m.$$

Without loss of generality, we may take the holomorphic normal coordinates at $f(p)$, i.e. $g^{i\overline{j}}(f(p)) = \delta_{ij}$ and $\partial g^{i\overline{j}}(f(p)) = \overline{\partial} g^{i\overline{j}}(f(p)) = 0$, since N is Kähler. The Kähler form ω is Hermitian symmetric, so we can assume that ω is diagonalized at p in the complex tangential directions. In this local coordinates system, $\omega(p) = \sum_{i=2} dz^\alpha \wedge dz^{\overline{\alpha}} + A$, where A is a $(1,1)$-form which contains either dz^1 or $dz^{\overline{i}}$. Also

notice that at the point p, $dz^1 + dz^{\overline{1}} = 0$, i.e., $dz^1(p) = \sqrt{-1}dy^1$. Using these facts and direct computation, at p, we have

$$\begin{aligned}
B &= \sum_{i,\alpha,\beta} f^i_{\alpha\overline{\beta}} f^{\overline{i}}_1 dz^i \wedge dz^{\overline{\beta}} \wedge dz^1 \wedge \omega^{m-2} \\
&= \sum_{i,\alpha,\beta} f^i_{\alpha\overline{\beta}} f^{\overline{i}}_1 dz^\alpha \wedge dz^{\overline{\beta}} \wedge dz^1 \wedge (\sum_{\alpha=2} (\frac{\sqrt{-1}}{2} dz^\alpha \wedge dz^{\overline{\alpha}} + A))^{m-2} \\
&= \sum_{i,\alpha,\beta} f^i_{\alpha\overline{\beta}} f^{\overline{i}}_1 dz^\alpha \wedge dz^{\overline{\beta}} \wedge dz^1 \wedge (\sum_{i=2} \frac{\sqrt{-1}}{2} dz^\alpha \wedge dz^{\overline{\alpha}})^{m-2} \\
&= \sum_{1\le i\le n,\, 2\le\alpha\le m} f^{\overline{i}}_1 f^i_{\alpha\overline{\alpha}} dz^1 \wedge dz^\alpha \wedge dz^{\overline{\alpha}} \wedge (\sum_{\alpha=2} \frac{\sqrt{-1}}{2} dz^\alpha \wedge dz^{\overline{\alpha}})^{m-2} \\
&= \sum_{1\le i\le n,\, 2\le\alpha\le m} f^{\overline{i}}_1 f^i_{\alpha\overline{\alpha}} dy^1 \wedge dx^2 \wedge dy^2 \wedge ... \wedge dx^m \wedge dy^m \\
&= (\sum_{\alpha=2}^m \frac{\partial^2\rho}{\partial z^\alpha \partial z^{\overline{\alpha}}}) \sum_i \left|\frac{\partial f^i}{\partial \overline{z}^1}\right|^2 W,
\end{aligned}$$

where W is a positive $(2m-1)$-form on ∂M. Since the trace of the Levi form on the boundary ∂M are nonnegative, we have

$$\sum_{\alpha=2}^m \frac{\partial^2\rho}{\partial z^{\overline{\alpha}} \partial z^\alpha} = \text{ trace of the Levi form of } \rho \ge 0$$

at $p \in \partial M$. Hence we conclude that at the point p

$$B \ge 0. \tag{4.2}$$

Since p is an arbitrary point of ∂M and the sign of the globally defined smooth differential form B is invariant under biholomorphic change of variables, we have by (4.1) and (4.2)

$$\begin{aligned}
0 &\ge \int_M \partial\overline{\partial}\langle g, \overline{\partial} f \wedge \partial \overline{f}\rangle_N \wedge \omega^{m-2} \\
&= \int_{\partial M} \overline{\partial}\langle g, \overline{\partial} f \wedge \partial \overline{f}\rangle_N \wedge \omega^{m-2} \\
&\ge 0.
\end{aligned}$$

Hence the integrand on the left-hand side of (4.1) is zero pointwise. Then the rest of the argument goes as in [JY1] or [S1]. Therefore, the proof of Theorem 4.1 is complete. □

5. Application 2: Holomorphic equivalence of complex manifolds with boundary

We now apply the result on holomorphic extension in the previous section to study the biholomorphic equivalence of two homotopy equivalent compact complex manifolds which have boundaries. One can prove the following theorem by arguments similar to those in [NS]. For completeness, we will give the proof below.

Theorem 5.1. *Let M be a compact astheno-Kähler manifold with nonempty smooth boundary where the trace of the Levi form is nonnegative. Let N be a compact Kähler manifold with nonempty smooth convex boundary and have strongly*

seminegative sectional curvature. Assume that $\dim_C M = \dim_C N \geq 2$. *If there exists a homotopy equivalence* $u : M \to N$ *whose restriction on* ∂M *is a CR-diffeomorphism to* ∂N, *then* M *and* N *are biholomorphically equivalent.*

Proof. By Theorem 4.1, there exists a holomorphic extension f of $u|_{\partial M}$. Since f is homotopic to u, a homotopy equivalence of M and N relative to ∂M, f is onto and of degree 1. We denote the set of points in M where f fails to be a local diffeomorphism by

$$\mathcal{S} = \{p \in M : \det\Big(\frac{\partial f^i}{\partial z^\alpha}\Big) = 0\}.$$

f is of degree one, so f maps $M \backslash f^{-1}(f(\mathcal{S}))$ bijectively onto $f(M)\backslash f(\mathcal{S})$. If $\mathcal{S} \neq \emptyset$, then since $\mathcal{S}$ is a compact complex analytic subvariety of complex codimension 1 in M, $\mathcal{S}$ defines a nonzero homology class $[\mathcal{S}]$ in $H_{2m-2}(M, \mathrm{R})$. On the other hand, since $f : M \to N$ is a proper holomorphic mapping, $f(\mathcal{S})$ is a complex-analytic subvariety of complex codimension at least 2 in N by a theorem of Remmert [R]. Thus $f_*([\mathcal{S}]) = 0$ in $H_{2m-2}(N, \mathrm{R})$. This contradicts the fact that f is a homotopy equivalence of M to N. Hence $\mathcal{S} = \emptyset$. □

There is an interpretation about the CR-diffeomorphism on boundaries in terms of Cartan-Moser chains (see [J2] for the background materials). In [J1] it is shown that any two sufficiently close points on a strictly pseudoconvex CR manifold (of hypersurface type) can be connected by a smooth chain. Moreover, according to a result in [C] any diffoemorphism that preserves chains between two nondegenerate CR manifolds (not necessarily of the same signature type a priori) is either a CR diffeomorphism or a conjugate CR diffeomorphism, which in turn implies that the two CR manifolds have the same signature type. We state a corollary for complex manifolds of complex dimension 2. Recall that if $\dim_C M = 2$, then M is automatically astheno-Kähler.

Corollary 5.2. *Let* M *be a compact complex manifold of complex* 2 *dimension and with smooth strictly pseudoconvex boundary. Let* N *be a compact Kähler manifold with smooth nonempty boundary. If there is a diffeomorphism from* ∂M *to* ∂N *which preserves the Cartan-Moser chains and extends to a homotopy equivalence of* M *and* N, *then* M *is biholomorphic to* N.

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