

THE LOCAL COHOMOLOGY MODULES OF MATLIS REFLEXIVE MODULES ARE ALMOST COFINITE

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ABSTRACT. We show that if M and N are Matlis reflexive modules over a complete Gorenstein local domain R and I is an ideal of R such that the dimension of R/I is one, then the modules $\text{Ext}_R^i(N, H_I^j(M))$ are Matlis reflexive for all i and j if $\text{Supp}(N) \subseteq V(I)$. It follows that the Bass numbers of $H_I^j(M)$ are finite. If R is not a domain, then the same results hold for $M = R$.

1. INTRODUCTION

Let R be a Noetherian local ring with maximal ideal \mathfrak{m} and residue field k . If M is a finitely generated module, then the local cohomology modules $H_{\mathfrak{m}}^j(M)$ are artinian, so $\text{Ext}_R^i(k, H_{\mathfrak{m}}^j(M))$ is finitely generated for every i and j . Grothendieck [G] conjectured the following:

For any ideal I and any finitely generated module M , the module $\text{Hom}_R(R/I, H_I^j(M))$ is finitely generated for all j .

Here, $H_I^j(M)$ denotes the j^{th} local cohomology module of M with support in I . Hartshorne [H2] showed that this is false in general. However, he defined an R module M to be *I -cofinite* if $\text{Supp}(M) \subseteq V(I)$ and $\text{Ext}_R^i(R/I, M)$ is finitely generated for all i and asked:

For which rings R and ideals I are the modules $H_I^j(M)$ I -cofinite for all j , and all finitely generated modules M ?

Hartshorne answered this question for the following two cases:

- (1) If R is a complete regular local ring and I is a nonzero principal ideal, and
- (2) If R is a complete regular local ring and I is a prime ideal with $\dim R/I = 1$.

Hartshorne's work provides motivation for some more recent results on this question. Huneke and Koh extended Hartshorne's second case. Theorem 4.1 of [H-K] shows that if R is a complete Gorenstein domain, I is an ideal of R with $\dim R/I = 1$, and N is finitely generated with support in $V(I)$, then $\text{Ext}_R^i(N, H_I^j(M))$ is finitely generated for all i and j and for all finitely generated modules M .

This result was further extended in [D], Theorem 3. Delfino was able to remove the Gorenstein condition on R as long as R satisfied one of the three conditions (here K denotes a coefficient ring of R , q the uniformizing parameter of K):

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- (i) K is a field;
- (ii) K is not a field and $q \in I$;
- (iii) K is not a field and q is not an element of any minimal prime of R/I .

Delfino and Marley have eliminated the complete domain hypothesis entirely. They prove in [D-M] that if M is a finitely generated module over a commutative Noetherian ring R and I is an ideal such that $\dim R/I = 1$, then the local cohomology modules $H_I^j(M)$ are I -cofinite for all j .

Hartshorne's work also provided motivation for study in the special case when $M = R$. Theorem 2.3 of [H-K] shows that if R is a regular local ring of characteristic $p > 0$, I is any ideal of R , $j > \text{bight}(I)$, and $\text{Hom}(R/I, H_I^j(R))$ is finitely generated, then $H_I^j(R) = 0$. Here, $\text{bight}(I) = \max\{\text{ht } \mathfrak{p} \mid \mathfrak{p} \text{ is a minimal prime ideal of } I\}$.

Huneke and Sharp obtained more information on the structure of $H_I^j(R)$ under the same conditions on R and I . Theorem 2.1 of [H-S] shows that the Bass numbers of $H_I^j(R)$ are finite for all j . As a corollary, they show that $\text{Ass}(H_I^j(R))$ is finite for all j . Using the theory of D -modules, Lyubeznik [L] obtained analogous results for equicharacteristic rings.

The goal of the present paper is to obtain similar results as above, but for a larger class of modules. We first focus our attention to the case when R is a complete Gorenstein domain and I is an ideal of R such that $\dim R/I = 1$. Let $E = E_R(k)$ be the injective hull of the residue field and let $(-)^{\vee}$ denote the functor $\text{Hom}(-, E)$. Recall that a module is *Matlis reflexive* if $M \cong (M^{\vee})^{\vee}$. The class of Matlis reflexive modules over a complete ring includes all finitely generated and artinian modules. We show that if M and N are Matlis reflexive with $\text{Supp}(N) \subseteq V(I)$, then $\text{Ext}_R^i(N, H_I^j(M))$ is Matlis reflexive for all i and j . In fact, $\text{Ext}_R^i(N, H_I^j(M))$ is finitely generated for all i when $j > 0$. We give an example to show that $\text{Ext}_R^i(N, H_I^0(M))$ need not be finitely generated. However, it turns out that this is enough to show that all the Bass numbers of $H_I^j(M)$ are finite.

We then remove the domain assumption on R and study the case when $M = R$ and N is finitely generated. In this situation we show that $\text{Ext}_R^i(N, H_I^j(R))$ is Matlis reflexive. As a corollary, we show that $H_I^j(R)$ has finite Bass numbers for every j .

2. THE DOMAIN CASE

Throughout the rest of the paper, R will denote a Noetherian local ring with maximal ideal \mathfrak{m} and residue field k , I will denote an ideal of R such that $\dim R/I = 1$, E will denote the injective hull of the residue field, and $(-)^{\vee}$ will denote the functor $\text{Hom}_R(-, E)$. For basic facts about Matlis reflexive modules, we refer the reader to section 3.2 of [St]. In particular, we make extensive use of the following fact:

If R is complete, then an R -module M is Matlis reflexive if and only if there is a short exact sequence

$$0 \rightarrow S \rightarrow M \rightarrow A \rightarrow 0$$

with S finitely generated and A artinian (see, for example, [E], Proposition 1.3, or [St], Theorem 3.4.13).

Lemma 1. *If M and N are Matlis reflexive, then $\text{Ext}_R^i(N, H_I^0(M))$ is Matlis reflexive for all i .*

Proof. Since $H_I^0(M) \subseteq M$, this follows from Theorem 3 of [Be]. □

Theorem 2. *If R is a complete Gorenstein domain, M is Matlis reflexive, and N is finitely generated with $\text{Supp}(N) \subseteq V(I)$, then $\text{Ext}_R^i(N, H_I^j(M))$ is Matlis reflexive. In fact, $\text{Ext}_R^i(N, H_I^j(M))$ is finitely generated for all i when $j \geq 1$.*

Proof. Since M is Matlis reflexive, there is a short exact sequence

$$0 \rightarrow S \rightarrow M \rightarrow A \rightarrow 0$$

with S finitely generated and A artinian. Since A is artinian, we obtain from the induced long exact sequence of local cohomology modules the exact sequence

$$0 \rightarrow H_I^0(S) \rightarrow H_I^0(M) \rightarrow A \rightarrow H_I^1(S) \rightarrow H_I^1(M) \rightarrow 0$$

and

$$H_I^j(M) \cong H_I^j(S)$$

for $j \geq 2$.

Let K denote the kernel of the map $H_I^1(S) \rightarrow H_I^1(M)$. Then K is artinian, since A is, so K is Matlis reflexive. Thus, from the short exact sequence

$$0 \rightarrow K \rightarrow H_I^1(S) \rightarrow H_I^1(M) \rightarrow 0$$

we obtain the long exact sequence

$$\cdots \rightarrow \text{Ext}_R^i(N, H_I^1(S)) \rightarrow \text{Ext}_R^i(N, H_I^1(M)) \rightarrow \text{Ext}_R^{i+1}(N, K) \rightarrow \cdots$$

By Theorem 4.1 of [H-K], $\text{Ext}_R^i(N, H_I^1(S))$ is finitely generated, hence $\text{Ext}_R^i(N, H_I^1(M))$ is finitely generated if and only if $\text{Ext}_R^{i+1}(N, K)$ is finitely generated.

Since K is artinian, $\text{Hom}(N, K)$ is also artinian, and since $\text{Hom}(N, K) \subseteq \text{Hom}(N, H_I^1(S))$, $\text{Hom}(N, K)$ has finite length. So $N \otimes K^\vee \cong \text{Hom}(N, K)^\vee$ has finite length and hence $\text{Tor}_i(N, K^\vee)$ has finite length. Thus $\text{Ext}^i(N, K) \cong \text{Tor}_i(N, K^\vee)^\vee$ has finite length for every i . Therefore $\text{Ext}_R^i(N, H_I^1(M))$ is finitely generated, and hence Matlis reflexive, for all i .

By Theorem 1, $\text{Ext}_R^i(N, H_I^0(M))$ is Matlis reflexive. For $j \geq 2$,

$$\text{Ext}_R^i(N, H_I^j(M)) \cong \text{Ext}_R^i(N, H_I^j(S))$$

for all i , and so by Theorem 4.1 of [H-K], $\text{Ext}_R^i(N, H_I^j(M))$ is finitely generated, hence Matlis reflexive. □

We note that $\text{Ext}_R^i(N, H_I^0(M))$ need not be finitely generated. For example, let $N = R/I$, $M = E$, and $i = 0$. Then $\text{Ext}_R^i(N, H_I^0(M)) = (R/I)^\vee$. Since $\dim R/I = 1$, $(R/I)^\vee$ is not finitely generated. However, the following corollary shows that this is not too bad.

Recall that the i^{th} term $E^i(M)$ in the minimal injective resolution of an R -module M is uniquely determined, up to isomorphism, by M . In fact, there is a family \mathcal{F} of prime ideals of R such that $E^i(M) \cong \bigoplus_{\mathfrak{q} \in \mathcal{F}} E(R/\mathfrak{q})$. The i^{th} Bass number with respect to \mathfrak{p} , denoted $\mu_i(\mathfrak{p}, M)$, of M is defined to be the number of copies of the injective hull of R/\mathfrak{p} in this direct sum decomposition of $E^i(M)$. By 2.7 of [B], this is the same as the vector space dimension of $(\text{Ext}_R^i(R/\mathfrak{p}, M))_{\mathfrak{p}}$ over $(R/\mathfrak{p})_{\mathfrak{p}}$. We refer to the numbers $\mu_i(\mathfrak{p}, M)$ for all i and all \mathfrak{p} collectively as the Bass numbers of M .

Corollary 3. *If R is a complete Gorenstein domain and M is a Matlis reflexive R -module, then the Bass numbers of $H_I^j(M)$ are finite for every j .*

Proof. Let k be the residue field of R . Then $\text{Ext}_R^i(k, H_I^j(M))$ is Matlis reflexive by Theorem 2. Since $\text{Ext}_R^i(k, H_I^j(M))$ is also a k vector space, it must be finitely generated. If \mathfrak{p} is any nonmaximal prime, it follows from Proposition 1.3 of [E] that $M_{\mathfrak{p}}$ is finitely generated over $R_{\mathfrak{p}}$. Since $\dim R/I = 1$, we have $(H_I^j(M))_{\mathfrak{p}} \cong H_{\mathfrak{p}R_{\mathfrak{p}}}^j(M_{\mathfrak{p}})$ if $\mathfrak{p} \supseteq I$ or $(H_I^j(M))_{\mathfrak{p}} \cong 0$ if $\mathfrak{p} \not\supseteq I$. In either case, it follows that $(\text{Ext}_R^i(R/\mathfrak{p}, H_I^j(M)))_{\mathfrak{p}}$ is finitely generated over $R_{\mathfrak{p}}$. \square

Lemma 4. *Let R be a complete Gorenstein domain, M a Matlis reflexive module, and N an artinian module. Then $\text{Ext}_R^i(N, H_I^j(M))$ is finitely generated (and thus Matlis reflexive) for all i and j .*

Proof. Since M is Matlis reflexive, $H_I^j(M)$ has finite Bass numbers by Corollary 3. Fix j . Let $0 \rightarrow H_I^j(M) \rightarrow \mathbf{J}^*$ be a minimal injective resolution of $H_I^j(M)$. Then for each t ,

$$\text{Hom}(N, J^t) \cong \bigoplus \text{Hom}(N, E)$$

since N is artinian. Since $H_I^j(M)$ has finite Bass numbers, the direct sum is a finite direct sum. By Matlis duality, $\text{Hom}(N, E)$ is finitely generated. Thus $\text{Hom}(N, J^t)$, and hence $\text{Ext}_R^i(N, H_I^j(M))$, is finitely generated. \square

Theorem 5. *If R is a complete Gorenstein domain and M and N are Matlis reflexive such that $\text{Supp}(N) \subseteq V(I)$, then $\text{Ext}_R^i(N, H_I^j(M))$ is Matlis reflexive for all i and j . In fact, $\text{Ext}_R^i(N, H_I^j(M))$ is finitely generated for all i when $j > 0$.*

Proof. For $j = 0$, this is just Lemma 1. Fix $j > 0$. Since N is Matlis reflexive, there is a short exact sequence

$$0 \rightarrow T \rightarrow N \rightarrow B \rightarrow 0$$

with T finitely generated and B artinian. This induces a long exact sequence

$$\dots \rightarrow \text{Ext}_R^i(B, H_I^j(M)) \rightarrow \text{Ext}_R^i(N, H_I^j(M)) \rightarrow \text{Ext}_R^i(T, H_I^j(M)) \rightarrow \dots$$

By Theorem 2, $\text{Ext}_R^i(T, H_I^j(M))$ is finitely generated. By Lemma 4, $\text{Ext}_R^i(B, H_I^j(M))$ is finitely generated. Thus $\text{Ext}_R^i(N, H_I^j(M))$ is finitely generated, and hence Matlis reflexive for all i when $j > 0$. \square

3. THE NONDOMAIN CASE

In this section we drop the domain requirement on the ring R and obtain similar results for the local cohomology modules of R . As one of our main tools, we use Grothendieck’s local duality theorem. In the special case when R is a Gorenstein ring with $\dim R = d$, local duality states that, for any finitely generated R -module M ,

$$H_{\mathfrak{m}}^i(M) \cong \text{Ext}_R^{d-i}(M, R)^\vee$$

for every i . For details the reader is referred to [H1].

Lemma 6. *If R is a complete Gorenstein ring and N is finitely generated with $\text{Supp}(N) \subseteq V(I)$, then $\text{Ext}_R^i(N, H_I^j(R))$ is Matlis reflexive for all i and j .*

Proof. By Lemma 2 of [D], we may assume that $N = R/I$. Let $0 \rightarrow R \rightarrow \mathbf{D}^*$ be an injective resolution of R . Since R is Gorenstein, $D^t = (\bigoplus E(R/\mathfrak{p}); \text{ht } \mathfrak{p} = t)$, so $H_I^0(\mathbf{D}^*)$ is the complex

$$0 \rightarrow E' \xrightarrow{\psi} E \rightarrow 0$$

where $E' = (\bigoplus E(R/\mathfrak{p}); \text{ht } \mathfrak{p} = d - 1, \mathfrak{p} \supseteq I)$, E is the injective hull of the residue field, and $d = \dim R$. Thus $H_I^j(R) = 0$ for $j \neq d - 1, d$, and there is a short exact sequence

$$0 \rightarrow H_I^{d-1}(R) \rightarrow E' \rightarrow A \rightarrow 0$$

where A is the image of ψ . It follows that there is an exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}(R/I, H_I^{d-1}(R)) &\rightarrow \text{Hom}(R/I, E') \\ &\rightarrow \text{Hom}(R/I, A) \rightarrow \text{Ext}^1(R/I, H_I^{d-1}(R)) \rightarrow 0 \end{aligned}$$

and that $\text{Ext}^i(R/I, H_I^{d-1}(R)) \cong \text{Ext}^{i-1}(R/I, A)$ for $i > 1$. Since A is artinian, it follows that $\text{Ext}^i(R/I, H_I^{d-1}(R))$ is artinian, and hence Matlis reflexive, for $i > 1$.

Consider the complex

$$0 \rightarrow \text{Hom}(R/I, E') \xrightarrow{\text{Hom}(R/I, \psi)} \text{Hom}(R/I, E) \rightarrow 0.$$

Let J denote the image of $\text{Hom}(R/I, \psi)$ and K denote the kernel of $\text{Hom}(R/I, \psi)$. Then $K = \text{Ext}_R^{d-1}(R/I, R)$ and $\text{Hom}(R/I, E)/J = \text{Ext}_R^d(R/I, R)$. Using the left exactness of $\text{Hom}(R/I, -)$ and local duality, we have

$$\text{Hom}(R/I, H_I^{d-1}(R)) \cong K \cong (H_m^1(R/I))^\vee$$

and

$$\text{Ext}^1(R/I, H_I^{d-1}(R)) \cong \text{Hom}(R/I, A)/J \subseteq \text{Hom}(R/I, E)/J \cong (H_m^0(R/I))^\vee.$$

Since R is complete and $H_m^s(R/I)$ is artinian for any s , we have $\text{Ext}^i(R/I, H_I^{d-1}(R))$ is finitely generated for $i = 0, 1$, and hence Matlis reflexive.

Finally, note that $H_I^d(R)$ is a factor of E , hence artinian. Therefore, $\text{Ext}^i(R/I, H_I^d(R))$ is artinian, and hence Matlis reflexive, for all i . \square

Corollary 7. *If R is a complete Gorenstein ring, then $H_I^j(R)$ has finite Bass numbers for every j .*

Proof. By Lemma 6, $\text{Ext}_R^i(k, H_I^j(R))$ is Matlis reflexive. The proof then proceeds as in Corollary 3. \square

Lemma 8. *If R is a complete Gorenstein ring and N is an artinian module, then the module $\text{Ext}_R^i(N, H_I^j(R))$ is finitely generated for all i and j .*

Proof. Since $H_I^j(R)$ has finite Bass numbers, the proof of Lemma 4 may be adapted. \square

Theorem 9. *If R is a complete Gorenstein ring and N is a Matlis reflexive module with $\text{Supp}(N) \subseteq V(I)$, then $\text{Ext}_R^i(N, H_I^j(R))$ is Matlis reflexive for all i and j .*

Proof. Since N is Matlis reflexive, there is a short exact sequence

$$0 \rightarrow T \rightarrow N \rightarrow B \rightarrow 0$$

with T finitely generated and B artinian. This induces a long exact sequence

$$\dots \rightarrow \text{Ext}_R^i(B, H_I^j(R)) \rightarrow \text{Ext}_R^i(N, H_I^j(R)) \rightarrow \text{Ext}_R^i(T, H_I^j(R)) \rightarrow \dots$$

By Lemma 6, $\text{Ext}_R^i(T, H_I^j(R))$ is Matlis reflexive for all j . By Lemma 8, $\text{Ext}_R^i(B, H_I^j(R))$ is finitely generated for all j . Thus $\text{Ext}_R^i(N, H_I^j(R))$ is Matlis reflexive for all i and j . \square

REFERENCES

- [B] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8-28. MR **27**:3669
- [Be] R. Belshoff, *Some change of ring theorems for Matlis reflexive modules*, Comm. Algebra **22** (1994), 3545-3552. CMP 94:13
- [D] D. Delfino, *On the cofiniteness of local cohomology modules*, Math. Proc. Camb. Phil. Soc. **115** (1994), 79-84. MR **94m**:13023
- [D-M] D. Delfino and T. Marley, *Cofinite Modules and Local Cohomology*, preprint.
- [E] E. Enochs, *Flat covers and flat cotorsion modules*, Proc. Amer. Math. Soc. (2) **92** (1984), 179-184. MR **85j**:13016
- [G] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux*. S.G.A. II. North Holland, 1968. MR **57**:16294
- [H1] R. Hartshorne, "Residues and Duality," Lecture Notes in Mathematics, No. 20, Springer-Verlag, 1966. MR **36**:5145
- [H2] R. Hartshorne, *Affine duality and cofiniteness*, Invent. Math. **9** (1970), 145-164. MR **41**:1750
- [H-K] C. Huneke and J. Koh, *Cofiniteness and vanishing of local cohomology modules*, Math. Proc. Camb. Phil. Soc. **110** (1991), 421-429. MR **92g**:13021
- [H-S] C. Huneke and R. Sharp, *Bass numbers of local cohomology modules*, Trans. Amer. Math. Soc. **339** (1993), 765-779. MR **93m**:13008
- [L] G. Lyubeznik, *Finiteness properties of local cohomology modules*, Invent. Math. **113** (1993), 41-55. MR **94e**:13032
- [St] J. R. Strooker, *Homological Questions in Local Algebra*, London Mathematical Society Lecture Notes 145, Cambridge University Press, Cambridge, 1990. MR **91m**:13013

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