

SOME CHARACTERIZATIONS OF $C(\mathcal{M})$

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ABSTRACT. We show that a function f on the unit disk extends continuously to \mathcal{M} , the maximal ideal space of $H^\infty(\mathbb{D})$ iff it is uniformly continuous (in the hyperbolic metric) and close to constant on the complementary components of some Carleson contour.

1. INTRODUCTION

Let $H^\infty(\mathbb{D})$ denote the algebra of bounded holomorphic functions on the unit disk, \mathbb{D} . Let \mathcal{M} denote the maximal ideal space of $H^\infty(\mathbb{D})$ (see [3] for definitions). Hoffman proved that every bounded harmonic function on the disk extends continuously to \mathcal{M} and that such functions generate $C(\mathcal{M})$ as a closed algebra (closed with respect to the sup norm). The purpose of this note is to give a more geometric characterization of the functions on \mathbb{D} which extend continuously to \mathcal{M} . Recall that a positive measure μ on the disk is a Carleson measure if there is a $C > 0$ such that $\mu(D(a, r)) \leq Cr$ for every $a \in \mathbb{T} = \partial\mathbb{D}$ and $r > 0$. A union of curves $\Gamma \subset \mathbb{D}$ is called a Carleson contour if arclength measure on Γ is Carleson.

Theorem 1.1. *For a bounded, continuous function g on \mathbb{D} the following are equivalent.*

- (1) g extends continuously to \mathcal{M} .
- (2) For every $\epsilon > 0$ there is a smooth function φ on \mathbb{D} so that $\|g - \varphi\|_\infty \leq \epsilon$, $\sup_z |\nabla\varphi(z)|(1 - |z|^2) < \infty$ and $|\nabla\varphi|dx dy$ is a Carleson measure.
- (3) g is uniformly continuous with respect to the hyperbolic metric on \mathbb{D} , and for every $\epsilon > 0$ there is a Carleson contour Γ so that g is within ϵ of a constant on each connected component of $\mathbb{D} \setminus \Gamma$.

Recall that a Douglas algebra is any closed algebra $H^\infty(\mathbb{T}) \subset A \subset L^\infty(\mathbb{T})$. Our first application of Theorem 1.1 is

Corollary 1.2. *Suppose f is bounded and continuous on the disk and that $H^\infty(\mathbb{T}) \subseteq A \subseteq L^\infty(\mathbb{T})$ is a Douglas algebra. Let $\mathcal{M}_A \subset \mathcal{M}$ denote the maximal ideal space of A . Then the following are equivalent.*

- (1) f extends continuously to \mathcal{M}_A .

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- (2) For every $\epsilon > 0$ there is an inner function I , such that $\bar{I} \in A$, and a $u \in C(\mathcal{M})$ on \mathbb{D} so that and $|(f(z) - u(z))I(z)| < \epsilon$ on \mathbb{D} .
- (3) For every $\epsilon > 0$ there is an inner function I , $\bar{I} \in A$, and a Carleson contour Γ so that f differs from a constant by at most ϵ on every component $\mathbb{D} \setminus \Gamma$ which hits $\{|I| > 1/2\}$.

At one extreme $A = H^\infty$ and the only allowable I 's are constants. In this case, Corollary 1.2 is just Theorem 1.1. The other extreme occurs for $A = L^\infty$. Axler and Shields [1] showed that any function extending to the maximal ideal space X of L^∞ (X is also the Shilov boundary of H^∞) has nontangential limits almost everywhere and asked if the converse is true. This was proven independently by O.V. Ivanov [5].

Corollary 1.3. *A bounded, continuous function f on \mathbb{D} has a continuous extension to X (the Shilov boundary of H^∞) iff f has nontangential limits a.e. on \mathbb{T} .*

2. PROOF OF THEOREM 1.1

We shall show (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1). The first implication is known, the second is a minor variation of a known technique, but the third seems to be new and depends on a recent result from [2].

Proof of (1) \Rightarrow (2). Let \mathcal{A} denote the set of smooth, bounded functions on the disk described by condition (2). \mathcal{A} is an algebra and by [3, Theorem VIII.6.1], every bounded harmonic function can be uniformly approximated by functions in \mathcal{A} . By Hoffman's result ([4], Lemma 4.4) bounded harmonic functions generate $C(\mathcal{M})$, so the sup norm closure of \mathcal{A} contains $C(\mathcal{M})$.

Proof of (2) \Rightarrow (3). This is a standard "stopping time argument", but since I do not know a reference for this precise application, I will sketch the proof.

Suppose g satisfies condition (2) of Theorem 1.1. First we show g is uniformly continuous with respect to the hyperbolic metric, ρ . Given $\epsilon > 0$ suppose $\|g - \varphi\|_\infty < \epsilon/4$ and φ satisfies the conditions in (2). The estimate $|\nabla\varphi(z)| \leq C(1 - |z|)$ implies φ is uniformly continuous (in fact, Lipschitz) with respect to the ρ , so there is a $\delta = \delta(\epsilon)$ so that $\rho(z, w) \leq \delta$ implies $|\varphi(z) - \varphi(w)| \leq \epsilon/2$. Thus

$$|g(z) - g(w)| \leq |g(z) - \varphi(z)| + |\varphi(z) - \varphi(w)| + |\varphi(w) - g(w)| \leq \epsilon,$$

so g is uniformly continuous with respect to ρ .

Next we will construct a Carleson contour Γ so that φ is within ϵ of a constant on each component of $\mathbb{D} \setminus \Gamma$ (and therefore g is within 2ϵ of a constant on each component). Given $\epsilon > 0$, choose $\delta > 0$ so that φ varies by less than $\epsilon/8$ over any hyperbolic ball of diameter 10δ . It will also be convenient to take $\delta = 2^{-N}$, a power of 2. To simplify the construction we move to the upper half plane. Clearly it is enough to consider what happens in one square, say $Q_0 = [0, 1] \times [0, 1]$. Now divide Q_0 into rectangles of the form

$$Q = \{z = x + iy : j2^{-k} < x < (j+1)2^{-k}, 2^{-k-1} < y < 2^{-k}\}.$$

Subdivide each such Q into $2\delta^{-2}$ equal subsquares of side length $\delta 2^{-k-1}$. Each of these will then have hyperbolic diameter less than 10δ , so φ varies by at most $\epsilon/8$ on each such square.

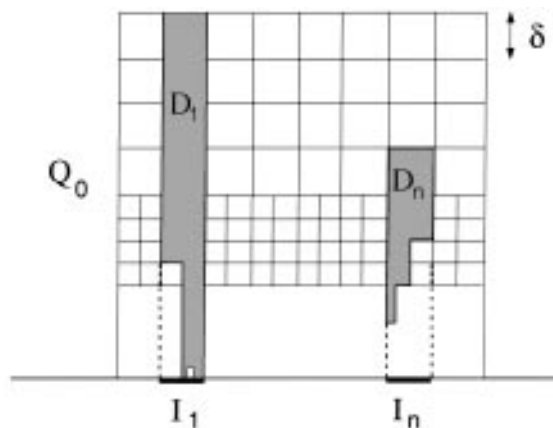


FIGURE 2.1. Stopping time regions in the proof of (2) \Rightarrow (3).

For each square S let z_S denote the center of the square. Each square S in the partition has either 1 or 2 squares adjacent and underneath it; let $G_1(S)$ denote these squares. In general, let

$$G_n(S) = \bigcup \{S'' \in G_1(S') : S' \in G_{n-1}(Q) \text{ and } |\varphi(z_S) - \varphi(z_{S''})| \leq \epsilon/2\}.$$

Let $D_S = S \cup \bigcup_{n=1}^{\infty} \bigcup_{S' \in G_n(S)} S'$.

Choose a square S on the “top row” of Q_0 and let $D_1 = D_S$. In general, let S be a square of maximal size not contained in $\bigcup_{j=1}^n D_j$ and let $D_{n+1} = D(S)$. This gives a countable number of disjoint (except for boundaries) regions which exhaust Q_0 . For each j let I_j be the vertical projection of D_j onto \mathbb{R} , $F_j = \partial D_j \cap \mathbb{R} \subset I_j$, and let z_j denote the center of the top edge of D_j . Let $\Gamma = \bigcup_j \partial D_j$.

Note that for each such region $D = D_j$ with “base” $I = I_j$, $\ell(\partial D) \leq 4(1 + \delta^{-1})|I|$, where ℓ denotes arclength. To see this, note that we can write the region D as $D = R \setminus \bigcup_j R_j$, where R is a $|I| \times 2\delta^{-1}|I|$ vertical rectangle and the R_j are similar rectangles with disjoint bases in I . Thus

$$(2.1) \quad \ell(\partial D) \leq \ell(\partial R_0) + \sum_j \ell(\partial R_j) \leq 4(1 + \delta^{-1})|I|,$$

as desired.

By construction, $|\varphi - \varphi(z_j)| \leq \epsilon$ on D_j , so we need only check that arclength on Γ is a Carleson measure. We start by showing

$$(2.2) \quad \text{length}(\partial D_j) \leq C_1|F_j| + C_2 \iint_{D_j} |\nabla \varphi| dx dy.$$

This is proved by considering two cases. First suppose $|F_j| \geq |I_j|/2$. Then by (2.1) equation (2.2) holds with $C_1 = 8(1 + \delta^{-1})$. On the other hand, if $|F_j| < |I_j|/2$, then we can find a set E of length $\geq \ell(I_j)/2$ along the top edge of D_j so that if we follow any vertical line down from E it eventually enters a different D_k . Consider the segment formed by intersecting the line with D_j . By the definition of D_j (and the fact that φ varies by less than $\epsilon/8$ over each square S) the values of φ at the endpoints of this segment differ by at least $\epsilon/4$. Thus the integral of $|\nabla \varphi|$ along

this segment is at least $\epsilon/4$. Hence,

$$\iint_{D_j} |\nabla\varphi| dx dy \geq \frac{\epsilon}{4}|E| \geq \frac{\epsilon}{8}|I_j|.$$

Thus (2.2) holds with $C_2 = 8/\epsilon$.

Finally we want to deduce arclength on Γ is a Carleson measure. Consider a dyadic Carleson square

$$Q = \{z = x + iy : j2^{-k} < x < (j + 1)2^{-k}, 0 < y < 2^{-k}\},$$

with base $I = [j2^{-k}, (j + 1)2^{-k}]$. It is sufficient to show that $\ell(\Gamma \cap Q) \leq C|I|$, with some absolute C independent of Q .

First note that there are at most $2\delta^{-1}$ regions D which hit Q but are not contained in Q (because any such region must contain a square S from the “top row” of Q and there are only $2\delta^{-1}$ such squares). For each such region, the proof of (2.1) shows $\ell(\partial D \cap Q) \leq 4(1 + \delta^{-1})|I|$. On the other hand, if $D_j \subset Q$, then $F_j \subset I$, so

$$\sum_{j:D_j \subset Q} |F_j| \leq |I|.$$

Furthermore,

$$\sum_{j:D_j \subset Q} \iint_{D_j} |\nabla\varphi(z)| dx dy \leq \iint_Q |\nabla\varphi(z)| dx dy \leq C|I|,$$

by assumption. Thus

$$\begin{aligned} \ell(\Gamma \cap Q) &\leq \sum_{j:D_j \cap Q \neq \emptyset, D_j \not\subset Q} \ell(\partial D_j \cap Q) + \sum_{j:D_j \subset Q} \ell(\partial D_j) \\ &\leq 8\delta^{-1}(1 + \delta^{-1})|I| + \sum_{j:D_j \subset Q} (C_1|F_j| + C_2 \iint_{D_j} |\nabla\varphi| dx dy) \\ &\leq C(\delta, \epsilon)|I|, \end{aligned}$$

as desired.

Proof of (3) \Rightarrow (1). We start with some definitions. For f on the disk, $a \in \mathbb{C}$ and $\delta > 0$, let $\Omega_f(a, \delta) = f^{-1}(D(a, \delta)) = \{z : |f(z) - a| < \delta\}$. For convenience, given an open set $\Omega \subset \mathbb{D}$ we will write

$$\text{dist}(g, H^\infty(\Omega)) = \inf_{G \in H^\infty(\Omega)} \|g|_\Omega - G\|_{L^\infty(\Omega, dx dy)}.$$

If Ω is empty, we interpret the distance as 0. Let $H^\infty(\mathbb{D})[f]$ denote the smallest algebra containing $H^\infty(\mathbb{D}) \cup f$ which is closed with respect to the sup norm on \mathbb{D} . If f is bounded and harmonic, then it is clear that $H^\infty(\mathbb{D})[f] \subset C(\mathcal{M})$.

The following is Theorem 1.1 of [2].

Proposition 2.1. *Suppose $\bar{f} \in H^\infty(\mathbb{D})$ and $g \in L^\infty(\mathbb{D})$. Then*

$$\text{dist}(g, H^\infty(\mathbb{D})[f]) = \inf_{\delta > 0} \sup_{a \in \mathbb{C}} \text{dist}(g, H^\infty(\Omega_f(a, \delta))).$$

(Distance is in the L^∞ norm on \mathbb{D} .)

We will now use this to prove (3) \Rightarrow (1). Suppose g satisfies condition (3) in Theorem 1.1. Fix $\epsilon > 0$ and let Γ be the Carleson contour given by part (3) of the theorem. Choose a sequence $\{z_n\} \subset \Gamma$ so that $\rho(z_n, z_m) \geq 1/4$ for all $n \neq m$ and so that every $z \in \Gamma$ is within hyperbolic distance $1/2$ of one of the z_n 's. Then $\{z_n\}$ is an interpolating sequence because it is hyperbolically separated and satisfies

$$\begin{aligned} \sum_{z_n \in D(x,r)} (1 - |z_n|) &\leq C \sum_{z_n \in D(x,r)} \ell(\Gamma \cap D(z_n, \frac{1}{100}(1 - |z_n|))) \\ &\leq C \ell(\Gamma \cap D(x, 2r)) \\ &\leq Cr \end{aligned}$$

for every disk $D(x, r)$ centered on $\partial\mathbb{D}$. (See [3], Chapter VII.) Let

$$B(z) = \prod_{n=1}^{\infty} \frac{z_n - z}{1 - \bar{z}_n z} \frac{|z_n|}{z_n}$$

be the Blaschke product with zeros $\{z_n\}$. Note that $\Gamma \subset \{z : |B(z)| < 1/2\}$. By Exercise VII.1 of [3], we also have $|B'(z_n)|(1 - |z_n|^2) > C_0$ for all n and some $C_0 > 0$ (depending only on the Carleson norm of arclength on Γ).

We claim that for any δ there is a $\nu(\delta) > 0$ such that any component of $\Omega_B(a, \delta)$ which hits Γ has hyperbolic diameter at most $\nu(\delta)$. Suppose not. Then there is a $\nu_0 > 0$ so that for any $\delta > 0$ we can choose $a \in \mathbb{C}$ and a component Ω of $\Omega_B(a, \delta)$ such that

- (1) $\Omega \cap \Gamma \neq \emptyset$.
- (2) The hyperbolic diameter of Ω is $\geq \nu_0$.

Choose a sequence $\delta_j \rightarrow 0$ and let $\{\Omega_j\}$ be the corresponding components. Each such Ω_j is within hyperbolic distance $1/2$ of some point z_j in our sequence. Renormalize by Möbius transformations so that $z_j = 0$ for all j . We now have a sequence of uniformly bounded functions $\{B_j\}$ on \mathbb{D} such that $|B'_j(0)| > C_0$ and B_j is within δ_j of a constant on a connected set of hyperbolic diameter $\geq \nu_0$ which intersects the disk $D(0, 1/2)$. By normal families there is a subsequence which converges to a constant function uniformly on compacta, but this contradicts the condition $|B'_j(0)| > C_0$. This proves the claim.

We can now apply the proposition. Since g is uniformly continuous with respect to the hyperbolic metric, there is a $\eta = \eta(g, \epsilon)$ so that g is within ϵ of a constant on any set of hyperbolic diameter $\leq \eta$. Choose δ so small that $\nu(\delta) \leq \eta(g, \epsilon)$. Suppose $a \in \mathbb{C}$ and Ω is a connected component of $\Omega_B(a, \delta)$. If Ω has hyperbolic diameter $\leq \eta$, then g is within ϵ of a constant on Ω by our choice of η . If Ω has hyperbolic diameter $\geq \eta$, then Ω cannot meet Γ , so it lies completely within a component of $\mathbb{D} \setminus \Gamma$. Thus g differs by less than ϵ from a constant by hypothesis. Since a piecewise constant function is holomorphic on $\Omega_f(a, \delta)$, we have

$$\text{dist}(g, H^\infty(\Omega_f(a, \delta))) \leq \epsilon,$$

for all a . Hence by the proposition,

$$\text{dist}(g, C(\mathcal{M})) \leq \text{dist}(g, H^\infty(\mathbb{D})[f]) \leq \epsilon.$$

Since g can be uniformly approximated by elements of $C(\mathcal{M})$, we have $g \in C(\mathcal{M})$. This completes the proof of Theorem 1.1. \square

Remark . The proof of the final implication shows that any function $g \in C(\mathcal{M})$ is contained in an closed subalgebra generated by $H^\infty(\mathbb{D})$ and a countable collection

of anti-holomorphic functions. However, I don't know if g must be in a subalgebra generated by $H^\infty(\mathbb{D})$ and a single anti-holomorphic function (even if g itself is harmonic).

3. PROOF OF THE COROLLARIES

Proof of Corollary 1.2. Given Theorem 1.1, Corollary 1.2 is very easy. To prove (1) \Rightarrow (2) apply the Tietze extension theorem (which we can do since \mathcal{M} is a compact Hausdorff space, hence normal) to f restricted to \mathcal{M}_A to get $u \in C(\mathcal{M})$ with $u = f$ on \mathcal{M}_A . By [3], Theorem IX.1.3, and Marshall's theorem ([3], Theorem IX.3.2)

$$\mathcal{M}_A = \bigcap_{I: \bar{I} \in A} \{m \in \mathcal{M} : |I(m)| = 1\},$$

which implies sets of the form $\{|I(m)| > 1 - \epsilon\}$ with $\epsilon > 0$, $\bar{I} \in A$ form a base of neighborhoods of \mathcal{M}_A . Since $f - u$ vanishes on \mathcal{M}_A , we can find an ϵ and I so that $|f - u| \leq \epsilon$ on $\{|I| > 1 - \epsilon\}$. Replacing I by a high power of itself gives (2). Conversely, (2) \Rightarrow (1) because by taking $\epsilon \rightarrow 0$ we can obtain a sequence of continuous functions on \mathcal{M} (and hence continuous on \mathcal{M}_A) which converge uniformly to f on \mathcal{M}_A .

Next we prove (2) \Rightarrow (3). By (2) (applied with ϵ^2) we can find u and I so that $|f - u| < \epsilon$ on $\{|I| > \epsilon\}$. Take the contour corresponding to $g = u$ and ϵ in part (3) of Theorem 1.1. We also wish to insure that any component that intersects $\{|I| > 1/2\}$ is contained in $\{|I| > \epsilon\}$ as long as ϵ is less than some absolute constant ϵ_0 . Carleson's construction (e.g., proof of [3], Theorem VIII.4.1) says there is an $\epsilon_0 > 0$ and a Carleson contour Γ which separates $\{|I| = 1/2\}$ from $\{|I| = \epsilon_0\}$. By adding this contour to the one for u we get the desired property. Thus we have $|f - u| \leq \epsilon$ on any component which hits $\{|I| > 1/2\}$, as required. Conversely, (3) \Rightarrow (2) holds because given the piecewise constant approximation we extend it to the whole disk by setting it equal to zero outside the contours which hit $\{|I| > 1/2\}$ and smoothing it (see e.g., [3], page 357). The resulting function u satisfies (2) if we replace I by a high power of itself. This proves Corollary 1.2. \square

Proof of Corollary 1.3. If f extends continuously to $X = \mathcal{M}_{L^\infty(\mathbb{T})}$, then by part (2) of Corollary 1.2 for every $\epsilon > 0$ we can find some $u \in C(\mathcal{M})$ and some inner function I so that $|f - u| < 2\epsilon$ on $\{|I| > 1/2\}$. Every function in $C(\mathcal{M})$ has nontangential limits almost everywhere (they can all be uniformly approximated by products and sums of bounded harmonic functions) and for any fixed $\theta < \pi$ the set $\{|I| > 1/2\}$ contains cones with vertex angle θ at almost every point of the circle. Thus the nontangential oscillation of f in cones of angle θ is less than ϵ almost everywhere. Taking $\epsilon \rightarrow 0$ and $\theta \rightarrow \pi$ shows f has nontangential limits almost everywhere.

Conversely, suppose f has nontangential limits almost everywhere. By subtracting the harmonic function v with the same limits we may assume f has limit 0 almost everywhere. Since v extends continuously to \mathcal{M} (hence to X), this does not alter the problem. Suppose $\epsilon > 0$ and fix a cone angle, say $\theta = \pi/2$; for each $n = 1, 2, \dots$ let K_n be the set of points x on the circle such that $|f| < \epsilon$ in the cone $W(x, \frac{\pi}{2}, n)$ with vertex x , angle $\frac{\pi}{2}$ and diameter 2^{-n} . Note that $|K_n| \rightarrow 2\pi$ as $n \rightarrow \infty$. Choose compact sets $E_n \subset K_n$ so that $|E_n| \geq |K_n| - \frac{1}{n}$.

Next, choose n_0 so that $|E_{n_0}| > 1/2$ and let $W_0 = \bigcup_{x \in E_{n_0}} W(x, \frac{\pi}{2}, n_0)$. For each complementary interval J of E_{n_0} we can choose an $n = n_J$ such that $|F_J| =$

$|E_n \cap J| > \frac{1}{2}|J|$ and $2^{-n} \leq |J|$. Define $W_J = \bigcup_{x \in F_J} W(x, \frac{\pi}{2}, n_J)$. Then W is a sawtooth domain with boundary length at most $6|J|$.

Continuing in this way we obtain by induction a collection of intervals $\{J_k\}$, sets $F_k \subset J_k$ and sawtooth regions W_k such that

- (1) $|F_k| \geq \frac{1}{2}|J_k|$.
- (2) The $\{F_k\}$ are pairwise disjoint.
- (3) $\partial W_k \cap \partial \mathbb{D} = F_k$.
- (4) $|\bigcup_k F_k| = |\partial \mathbb{D}|$.
- (5) $\ell(\partial W_k) \leq C|J_k| \leq 2C|F_k|$. In particular, $\sum_k \ell(\partial W_k) < \infty$.

Define a sequence $\{z_n\}$ in the disk by placing points along each arc in each ∂W_k at hyperbolic distance ϵ from each other. The sequence is Blaschke, i.e., satisfies $\sum_n (1 - |z_n|) < \infty$ because the sum along any arc of ∂W_k is easily dominated by a multiple of the length of that arc, and we observed above that $\sum \ell(\partial W_k) < \infty$. Now let $I(z)$ be the Blaschke product corresponding to the sequence $\{z_n\}$. Then $|I| \leq \epsilon$ on each ∂W_k (because we are never more than distance ϵ from a zero) and is therefore less than ϵ outside the union of the W_k (because any component of $\mathbb{D} \setminus \bigcup_k \overline{W_k}$ hits the unit circle in zero measure). Therefore f differs from zero by less than ϵ on every component of $\{|I| > \epsilon\}$. Taking u to be the zero function and applying part (2) of Corollary 1.2 completes the proof (since $A = L^\infty$ we automatically get $\bar{I} \in A$). \square

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