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THE CONVERGENCE OF THE BOCHNER-RIESZ MEANS AT THE CRITICAL INDEX

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ABSTRACT. In this paper, we study the pointwise convergence of the Bochner-Riesz means at the critical index on the space $L\log^+ L(Q_n)$. We weaken the hypothesis, "x is a Lebesgue point", which is required on some research results by instead considering the convergence of averages of the function over balls when the radials of the balls approach to 0.

Let Q_n denote the n-dimensional cube $\{x \in R^n|\ |x_i| \leq \pi,\ i = 1, 2, \cdots, n\}$, $B(x, r)$ denote the n-dimensional closed ball with center x and radius r, and $|E|$ denote the Lebesgue measure of a measurable set $E \subset R^n$.

Definition 1. Suppose $f \in L^1(Q_n)$. If

$$\lim_{r\to 0} |B(x_0, r) \cap Q_n|^{-1} \int_{B(x_0,r)\cap Q_n} |f(x) - f(x_0)|dx = 0,$$

then x_0 is called a Lebesgue point of f.

If $f \in L^1(Q_n)$, the partial sum of the Bochner-Riesz mean of its Fourier series at critical index $(n-1)/2$ is defined by

$$S_R(f, x) = \sum_{|l|\leq R} (1 - \frac{|l|^2}{R^2})^{(n-1)/2} a_l e^{il\cdot x}$$

where $\sum a_l e^{il\cdot x}$ is the Fourier series of f.

In 1954 [4], Salem studied the difference between the partial sum operator of Fourier series and the Bochner-Riesz mean of an integrable function on $[-\pi, \pi]$ where the function f satisfies the "Salem condition"

$$\text{(A)} \qquad \int_0^h (f(x+t) - f(x-t))dt = o(|h|/\log|h|^{-1}) \qquad (h \to 0)$$

uniformly for $x \in [-\pi, \pi]$. On the other hand, it is well known that the Bochner-Riesz mean at critical index converges on the Banach space $L^p(Q_n)$ for $1 < p < \infty$ and that there exists a function from $L^1(Q_n)$ such that the Bochner-Riesz mean diverges (see [2]). In [3], S. Lu extended the Salem condition to the higher dimensions (see Definition 2 below) and studied the convergence of the Bochner-Riesz

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means on the "limit" case of $L^p(Q_n)$, $p \to 1$, i.e. $L\log^+ L$ space (see Theorem A below).

Let $F(x)$ denote an indefinite integral of f. Then Salem condition (A) can be written

$$F(x+h) + F(x-h) - 2F(x) = o(|h|/\log|h|^{-1}).$$

From this observation we can define the higher dimensional Salem condition.

Definition 2. Let $f \in L^1(Q_n)$ and define

$$F(x,r) = \int_{B(x,r)\cap Q_n} (f(t) - f(x))dt.$$

If $x_0 \in Q_n$ and there exists a positive real number r_0 such that

$$r^{1-n}\{F(x_0, r+2h) + F(x_0, r) - 2F(x_0, r+h)\} = o(|h|/\log|h|^{-1}) \quad (h \to 0)$$

uniformly for r, $h \le r \le r_0$, then f is said to satisfy the Salem condition at x_0.

Theorem A [3]**.** *Let $f \in L\log^+ L(Q_n)$, $n \ge 2$, and f satisfy the Salem condition at x_0. Then*

$$\lim_{R\to\infty} S_R(f, x_0) = f(x_0)$$

provided x_0 is a Lebesgue point of f.

In one-dimensional Fourier theory, if f is integrable on $[-\pi,\ \pi]$ and satisfies the condition of integral modulus of continuity

$$\sup_{0\le h\le\delta} \Big(\int_{-\pi}^{\pi} |f(x+h) - f(x)|^p dx\Big)^{1/p} = o(\delta^{1/p}),$$

then the Fourier series $S(f)$ converges to $f(x)$ provided x is a Lebesgue point of f. For more detail and deeper results, see [7] page 65, Lebesgue test. On the other hand, there is no completely positive answer for the convergence of Fourier series when x is not a Lebesgue point of f. However, [1] and [4] dealt with the weaker hypothesis about the Lebesgue points. They consider the average of function f at point x_0, $\bar{f}_{x_0}(t)$, taken over the surface of the sphere with center x_0 and radius t. They assumed

$$\int_0^t |\bar{f}_{x_0}(t) - f(x_0)| t^{n-1} dt = o(t^n) \qquad (t \to 0)$$

and studied the convergence of the Bochner-Riesz means. The purpose of this paper is to study the pointwise convergence of the Bochner-Riesz means on the critical index if the point is not a Lebesgue point of f or the point which satisfies the above condition.

Before giving an example to show that the hypothesis "x_0 is a Lebesgue point of f" is not necessary and can be replaced by a weaker condition, we start from a well-known theorem due to E. M. Stein (see [5], Theorem 2).

Theorem B [5]**.** *Let $f \in L\log^+ L(Q_n)$, $n \ge 2$, and $x_0 + Q_n = \{x_0 + x \mid x \in Q_n\}$. Let $g(x) = f(x)$ if $x \in x_0 + Q_n$, and $g(x) = 0$ otherwise, and define*

$$\sigma_R(g, x) = \int_{|\alpha|\le R} \Big(1 - \frac{|y|^2}{R^2}\Big)^{(n-1)/2} \hat{g}(y) e^{iy\cdot x} dy$$

where $\hat{g}$ is the Fourier Transform of g. Then $S_R(f,x) - \sigma_R(g,x) \to 0$ as $R \to \infty$ uniformly in any closed subset of $x_0 + Q_n$.

Let

$$f(x) = \begin{cases} \Omega(x)|x|^{-n}(\log|x|)^{-2-\epsilon}\phi(x), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0 \end{cases}$$

where ϕ is a smooth radial function, $\phi(x) = 1$ if $|x| \leq 1/4$, $\phi(x) = 0$ if $|x| \geq 1/2$ and Ω is a continuous odd function on the $(n-1)$-dimensional unit sphere and homogeneous of degree 0. It is easy to see that $\sigma_R(f, 0) = 0$ for every R since the Fourier transform of an odd function is odd. By Theorem B, one can see $S_R(f, 0) \to 0$ as $R \to \infty$ but $x_0 = 0$ is not a Lebesgue point of f.

Suppose $f \in L^1(Q_n)$. Let us write

$$f_x(t) = |S(x,t) \cap Q_n|^{-1} \int_{S(x,t)\cap Q_n} f(y) d\sigma_t(y)$$

where $S(x,t) = \{y \in R^n | \ |x-y| = t\}$ is the sphere on R^n with radius t and center x and $d\sigma_t(y)$ is the Lebesgue measure on the sphere $S(x,t)$ and $|S(x,t) \cap Q_n|$ is the Lebesgue measure of $S(x,t) \cap Q_n$ with respect to $d\sigma_t(y)$. Define $\bar{f}_{x_0}(t) = f_{x_0}(t) - f(x_0)$. Clearly

$$\bar{f}_{x_0}(t) = |S(x_0,t) \cap Q_n|^{-1} \int_{S(x_0,t)\cap Q_n} (f(y) - f(x_0)) d\sigma_t(y).$$

Definition 3. Let the α-integral

$$F_\alpha(x,r) = \int_{B(x,r)\cap Q_n} (f(y) - f(x))|y-x|^{\alpha-n+1} dx$$

for $\alpha \geq 0$, and suppose $F_\alpha(x,r) < \infty$ for $x \in R^n$ and $r \in R^+$. Let $x_0 \in Q_n$. If there exists $r_0 > 0$ such that

(B)
$$r^{-\alpha}\{F_\alpha(x_0, r+2h) + F_\alpha(x_0, r) - 2F_\alpha(x_0, r+h)\} = o(|h|/\log|h|^{-1}) \quad (h \to 0)$$

uniformly for r, $h \leq r \leq r_0$, then f is said to satisfy the α-Salem condition at x_0. Here is the main theorem of this paper.

Theorem. *Let $f \in L\log^+ L(Q_n)$, $n \geq 2$. Suppose f satisfies the α-Salem condition at x_0 for some $\alpha \geq 0$. Then*

$$\lim_{R\to\infty} S_R(f, x_0) = f(x_0)$$

provided

$$\text{(C)} \qquad \int_{|y|\leq t} (f(y) - f(x_0)) dy = o(t^n) \qquad (t \to 0)$$

and $|f(x_0)| < \infty$.

Remark. 1. The $(n-1)$-Salem condition is equivalent to the Salem condition in Definition 2.

2. If x_0 is a Lebesgue point of f, then (C) is satisfied.

3. The hypothesis (C) is equivalent to

$$\text{(D)} \qquad \int_0^t \tau^{n-1} \bar{f}_{x_0}(\tau) d\tau = o(t^n) \qquad (t \to 0).$$

Lemma 1. *Let k be any positive real number. Then*

$$\int_0^k \{(r+h+\tau)^\alpha \bar{f}_x(r+h+\tau) - (r+h-\tau)^\alpha \bar{f}_x(r+h-\tau)\}d\tau$$
$$= F_\alpha(x, r+h+k) + F_\alpha(x, r+h-k) - 2F_\alpha(x, r+h).$$

Proof. The above integral is equal to

$$\int_{r+h}^{r+h+k} \tau^\alpha \bar{f}_x(\tau)d\tau - \int_{r+h-k}^{r+h} \tau^\alpha \bar{f}_x(\tau)d\tau$$
$$= \int_0^{r+h+k} \tau^\alpha \bar{f}_x(\tau)d\tau - 2\int_0^{r+h} \tau^\alpha \bar{f}_x(\tau)d\tau + \int_0^{r+h-k} \tau^\alpha \bar{f}_x(\tau)d\tau.$$

Lemma 1 is proved.

Lemma 2. *Suppose $f \in L^1(Q_n)$ and f satisfies* (C) *at x_0. Then for any fixed positive real number N and $0 < t \le N\pi/R$, one has*

$$\int_0^t \tau^\alpha \bar{f}_{x_0}(\tau)\cos(R\tau)d\tau = o(R^{-\alpha-1})$$

and

$$\int_0^t \tau^\alpha \bar{f}_{x_0}(\tau)\sin(R\tau)d\tau = o(R^{-\alpha-1})$$

as $R \to \infty$.

Proof. We will prove the first equality since both equalities are similar. Write

$$\int_0^t \tau^\alpha \bar{f}_{x_0}(\tau)\cos(R\tau)d\tau = \int_0^t \tau^{n-1}\bar{f}_{x_0}(\tau)\tau^{\alpha-n+1}\cos(R\tau)d\tau.$$

Then Lemma 2 is proved by applying integration by parts and the hypothesis that f satisfies (C) at x_0 and (D) in the remark.

Lemma 3. *Let $f \in L^1(Q_n)$. Suppose f satisfies* (B) *and* (C) *at x_0. Then*

$$\int_0^t \tau^\alpha \bar{f}_{x_0}(\tau)\cos(R\tau)d\tau = o(R^{-\alpha-1}) + (\log R)^{-1}o(t^{\alpha+1})$$

and

$$\int_0^t \tau^\alpha \bar{f}_{x_0}(\tau)\sin(R\tau)d\tau = o(R^{-\alpha-1}) + (\log R)^{-1}o(t^{\alpha+1})$$

for $\pi/R \le t$ as $t \to 0$.

Proof. By Lemma 2, without loss of generality, we may assume $t \ge 4\pi/R$ and we only prove the first equality. Let us choose and fix r_0 such that $0 < r_0 \le 2\pi/R$ and $R(t-r_0)/\pi - 1$ is even. Denote $m = R(t-r_0)/\pi - 1$ and decompose the integral

$$I = \int_0^t \tau^\alpha \bar{f}_{x_0}(\tau)\cos(R\tau)d\tau = \int_0^{r_0} \cdots + \sum_{j=1}^m \int_{r_0+j\pi/R}^{r_0+(j+1)\pi/R} \cdots .$$

By Lemma 2, the first integral on the right-hand side of the above equality is

$$\int_0^{r_0} \tau^\alpha \bar{f}_{x_0}(\tau)\cos(R\tau)d\tau = o(R^{-\alpha-1}).$$

We write

$$
\begin{aligned}
I &= o(R^{-\alpha-1}) + \sum_{j=1}^{m} \int_{r_0+j\pi/R}^{r_0+(j+1)\pi/R} \tau^{\alpha} \bar{f}_{x_0}(\tau) \cos(R\tau) d\tau \\
&= o(R^{-\alpha-1}) + \sum_{j=1}^{m} \int_{0}^{\pi/R} (\tau + r_0 + j\pi/R)^{\alpha} \\
&\qquad\qquad \cdot \bar{f}_{x_0}(\tau + r_0 + j\pi/R) \cos(R\tau + Rr_0 + j\pi) d\tau \\
&= o(R^{-\alpha-1}) + \sum_{j=1}^{m} (-1)^j \int_{0}^{\pi/R} (\tau + r_0 + j\pi/R)^{\alpha} \\
&\qquad\qquad \cdot \bar{f}_{x_0}(\tau + r_0 + j\pi/R) \cos(R\tau + Rr_0) d\tau \\
&= o(R^{-\alpha-1}) + \sum_{j=1}^{m/2} \int_{0}^{\pi/R} \big[(\tau + r_0 + 2j\pi/R)^{\alpha} \bar{f}_{x_0}(\tau + r_0 + 2j\pi/R) \\
&\qquad - (\tau + r_0 + (2j-1)\pi/R)^{\alpha} \bar{f}_{x_0}(\tau + r_0 + (2j-1)\pi/R)\big] \cos(R\tau + Rr_0) d\tau \\
&\equiv o(R^{-\alpha-1}) + \sum_{j=1}^{m/2} S^{(j)}.
\end{aligned}
$$

By the identity of trigonometric function, $\cos(R\tau + Rr_0)$, $S^{(j)}$ is the sum of

$$
\begin{aligned}
\cos(Rr_0) \int_{0}^{\pi/R} &\big[(\tau + r_0 + 2j\pi/R)^{\alpha} \bar{f}_{x_0}(\tau + r_0 + 2j\pi/R) \\
&- (\tau + r_0 + (2j-1)\pi/R)^{\alpha} \bar{f}_{x_0}(\tau + r_0 + (2j-1)\pi/R)\big] \cos(R\tau) d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
-\sin(Rr_0) \int_{0}^{\pi/R} &\big[(\tau + r_0 + 2j\pi/R)^{\alpha} \bar{f}_{x_0}(\tau + r_0 + 2j\pi/R) \\
&- (\tau + r_0 + (2j-1)\pi/R)^{\alpha} \bar{f}_{x_0}(\tau + r_0 + (2j-1)\pi/R)\big] \sin R\tau d\tau.
\end{aligned}
$$

The estimates of the last two integrals are the same and again we only study the first integral; without using too much notation, we still denote the first integral by $\cos(Rr_0)S^{(j)}$. Let us write

$$
S^{(j)} = \int_{0}^{\pi/(2R)} \cdots + \int_{\pi/(2R)}^{\pi/R} \cdots .
$$

By a change of variable ($\tau \to \pi/R - \tau$) on the second integral of the last equality,

$$\begin{aligned}
S^{(j)} = &\int_0^{\pi/(2R)} \big[(\tau + r_0 + 2j\pi/R)^\alpha \bar{f}_{x_0}(\tau + r_0 + 2j\pi/R) \\
&\quad - (\tau + r_0 + (2j-1)\pi/R)^\alpha \bar{f}_{x_0}(\tau + r_0 + (2j-1)\pi/R) \\
&\quad - (-\tau + r_0 + (2j+1)\pi/R)^\alpha \bar{f}_{x_0}(-\tau + r_0 + (2j+1)\pi/R) \\
&\quad + (-\tau + r_0 + 2j\pi/R)^\alpha \bar{f}_{x_0}(-\tau + r_0 + 2j\pi/R)\big] \cos(R\tau) d\tau \\
= &- \int_0^{\pi/(2R)} \big[((4j+1)\pi/(2R) + r_0 + (\pi/(2R) - \tau))^\alpha \\
&\qquad \cdot \bar{f}_{x_0}((4j+1)\pi/(2R) + r_0 + (\pi/(2R) - \tau)) \\
&\quad - ((4j+1)\pi/(2R) + r_0 - (\pi/(2R) - \tau))^\alpha \\
&\qquad \cdot \bar{f}_{x_0}((4j+1)\pi/(2R) + r_0 - (\pi/(2R) - \tau))\big] \cos(R\tau) d\tau \\
&+ \int_0^{\pi/(2R)} \big[((4j-1)\pi/(2R) + r_0 + (\pi/(2R) - \tau))^\alpha \\
&\qquad \cdot \bar{f}_{x_0}((4j-1)\pi/(2R) + r_0 + (\pi/(2R) - \tau)) \\
&\quad - ((4j-1)\pi/(2R) + r_0 - (\pi/(2R) - \tau))^\alpha \\
&\qquad \cdot \bar{f}_{x_0}((4j-1)\pi/(2R) + r_0 - (\pi/(2R) - \tau))\big] \cos(R\tau) d\tau \\
\equiv &\, S_1^{(j)} + S_2^{(j)}.
\end{aligned}$$

For simplicity, we define $r = 2j\pi/R + r_0$. In $S_1^{(j)}$, by changing variable ($\tau \to \pi/(2R) - \tau$), we have

$$\begin{aligned}
S_1^{(j)} = - \int_0^{\pi/(2R)} &\big[(r + \pi/(2R) + \tau)^\alpha \bar{f}_{x_0}(r + \pi/2R) + \tau) \\
&- (r + \pi/(2R) - \tau)^\alpha \bar{f}_{x_0}(r + \pi/2R) - \tau)\big] \sin(R\tau) d\tau.
\end{aligned}$$

Applying the integration by parts and Lemma 1,

$$\begin{aligned}
S_1^{(j)} = &\int_0^{\pi/(2R)} \big[(r + \pi/(2R) + \tau)^\alpha \bar{f}_{x_0}(r + \pi/2R) + \tau) \\
&\quad - (r + \pi/(2R) - \tau)^\alpha \bar{f}_{x_0}(r + \pi/2R) - \tau)\big] d\tau \\
&- R \int_0^{\pi/(2R)} \int_0^\tau \big[(r + \pi/(2R) + s)^\alpha \bar{f}_{x_0}(r + \pi/(2R) + s) \\
&\quad - (r + \pi/(2R) - s)^\alpha \bar{f}_{x_0}(r + \pi/(2R) - s)\big] ds \cos(R\tau) d\tau \\
= &\, [F_\alpha(x_0, r + \pi/R) + F_\alpha(x_0, r) - 2F_\alpha(x_0, r + \pi/(2R))] \\
&- R \int_0^{\pi/(2R)} [F_\alpha(x_0, r + \pi/(2R) + \tau) + F(x_0, r + \pi/(2R) - \tau) \\
&\qquad\qquad - 2F(x_0, r + \pi/(2R))] \cos(R\tau) d\tau.
\end{aligned}$$

From the hypothesis, f satisfies the α-Salem condition at x_0, the first term on the right-hand side of the last equal sign equals

$$\begin{aligned}
r^\alpha o((R \log R)^{-1}) &= (2j\pi/R + r_0)^\alpha o((R \log R)^{-1}) \\
&= j^\alpha o(R^{-\alpha-1} (\log R)^{-1}) \quad (R \to \infty)
\end{aligned}$$

(recall $0 < r_0 \le 2\pi/R$) and the second term, the integral, can be written as

$$\begin{aligned} &R\int_0^{\pi/(2R)} ((4j+1)\pi/(2R)+r_0-\tau)^\alpha o(\tau/\log\tau^{-1})\cos(R\tau)d\tau \\ &= o\left(R\int_0^{\pi/(2R)} ((4j+1)\pi/(2R)+r_0)^\alpha \tau/\log\tau^{-1}d\tau\right) \\ &= o\left(R^{1-\alpha}j^\alpha\int_0^{\pi/(2R)} \tau/\log\tau^{-1}d\tau\right) = o\left(j^\alpha R^{-\alpha-1}(\log R)^{-1}\right). \end{aligned}$$

Hence

$$S_1^{(j)} = j^\alpha o(R^{-\alpha-1}(\log R)^{-1}).$$

Similarly, we can prove

$$S_2^{(j)} = j^\alpha o(R^{-\alpha-1}(\log R)^{-1}).$$

It implies

$$\begin{aligned} I &= o(R^{-\alpha-1}) + \sum_{j=1}^{m/2} S_1^{(j)} + S_2^{(j)} \\ &= o(R^{-\alpha-1}) + \sum_{j=1}^{m/2} j^\alpha o(R^{-\alpha-1}(\log R)^{-1}) \\ &= o(R^{-\alpha-1}) + (Rt)^{\alpha+1} o(R^{-\alpha-1}(\log R)^{-1}) \\ &= o(R^{-\alpha-1}) + o(t^{\alpha+1}(\log R)^{-1}), \end{aligned}$$

where the second last equality is due to $m/2 = R(t-r_0)/\pi - 1$. Lemma 3 is proved.

Proof of the Theorem. Let $f \in L\log^+ L(Q_n)$, and define $g(x) = f(x)$ if $x \in x_0 + Q_n$ and $g(x) = 0$ otherwise. Clearly, $g(x)$ satisfies (C) and the α-Salem condition at x_0. By Theorem B, to show

$$\lim_{R\to\infty} S_R(f, x_0) = f(x_0),$$

it suffices to show

$$\lim_{R\to\infty} \sigma_R(g, x_0) = g(x_0).$$

It is well known that the inverse Fourier transform of $(1-|\xi|)^{(n-1)/2}$ is

$$c_n|x|^{-n+1/2}J_{n-1/2}(|x|)$$

on the distribution sense. Hence we write

$$
\begin{aligned}
&\sigma_R(g,x_0)-g(x_0)\\
&\quad= CR^{1/2}\int_{R^n}|y|^{-n+1/2}J_{n-1/2}(R|y|)(g(x_0-y)-g(x_0))dy\\
&\quad= CR^{1/2}\int_0^\infty t^{-n+1/2}J_{n-1/2}(Rt)\int_{\xi\in S^{n-1}}(g(x_0-t\xi)-g(x_0))d\sigma_1(\xi)t^{n-1}dt\\
&\quad= CR^{1/2}\int_0^\infty t^{-n+1/2}J_{n-1/2}(Rt)t^{n-1}|S(x_0,t)|^{-1}\int_{S(x_0,t)}(g(\xi)-g(x_0))d\sigma_t(\xi)dt\\
&\quad= CR^{1/2}\int_0^\infty t^{-n+1/2}J_{n-1/2}(Rt)t^{n-1}\bar{g}_{x_0}(t)dt\\
&\quad\equiv CR^n\int_0^\infty V_{n-1/2}(Rt)t^{n-1}\bar{g}_{x_0}(t)dt \qquad (V_{n-1/2}(t)=t^{-n+1/2}J_{n-1/2}(t))\\
&\quad= C\left(R^n\int_0^{\pi/R}\cdots+R^n\int_{\pi/R}^{\eta}\cdots+R^n\int_\eta^\infty\cdots\right)\equiv C(I_1+I_2+I_3),
\end{aligned}
$$

where η is a fixed real number such that $\eta<r_0$ and the r_0 is given in Definition 3.

Employing integration by parts and $D_t(t^{-k}J_k(t))=-t^{-k}J_{k+1}(t)$,

$$
\begin{aligned}
I_1 &= CR^nV_{n-1/2}(\pi)\int_0^{\pi/R}t^{n-1}\bar{g}_{x_0}(t)dt\\
&\quad+CR^{n+2}\int_0^{\pi/R}uV_{n+1/2}(Ru)\int_0^u t^{n-1}\bar{g}_{x_0}(t)dtdu.
\end{aligned}
$$

From hypothesis (C) and (D) in the remark,

$$
I_1=o(1)+o(R^{n+2}\int_0^{\pi/R}u^{n+1}du)=o(1)\quad(R\to\infty).
$$

By the Lemma in [6] (Lemma 3.4, page 158), one writes

$$
\text{(E)}\qquad J_{n-1/2}(r)=\sqrt{2(\pi r)^{-1}}\cos(r-n\pi/2)+O(r^{-3/2})
$$

for $r\geq\pi$. Therefore, applying (E),

$$
\begin{aligned}
I_3 &= CR^n\int_\eta^\infty V_{n-1/2}(Rt)t^{n-1}\bar{g}_{x_0}(t)dt\\
&= CR^{1/2}\int_\eta^\infty J_{n-1/2}(Rt)t^{-1/2}\bar{g}_{x_0}(t)dt\\
&= C\int_\eta^\infty t^{-1}\cos(Rt-n\pi/2)\bar{g}_{x_0}(t)dt+C/R\int_\eta^\infty O(t^{-2})\bar{g}_{x_0}(t)dt\\
&\equiv I_3^{(1)}+I_3^{(2)}.
\end{aligned}
$$

For the second integral, $I_3^{(2)}$, we write

$$\begin{aligned}|I_3^{(2)}| &\le \frac{C}{R}\int_\eta^\infty t^{-2}\Big(|S(x_0,t)|^{-1}\int_{S(x_0,t)} g(y)d\sigma_t(y) - g(x_0)\Big)dt\\ &= \frac{C}{R}\int_\eta^\infty t^{-n-1}\int_{S(x_0,t)} g(y)d\sigma_t(y)dt - C\eta^{-1}R^{-1}g(x_0)\\ &\le \frac{C}{R}\eta^{-n-1}\|g\|_{L^1(R)} - C\eta^{-1}R^{-1}g(x_0) \le CR^{-1} = o(1) \qquad (R\to\infty).\end{aligned}$$

Let us estimate $I_3^{(1)}$. Write

$$\begin{aligned}I_3^{(1)} &= C\int_\eta^\infty t^{-1}\cos(Rt-n\pi/2)|S(x_0,t)|^{-1}\int_{S(x_0,t)} g(y)d\sigma_t(y)dt\\ &\quad + Cg(x_0)\int_\eta^\infty t^{-1}\cos(Rt-n\pi/2)dt\\ &= C\int_\eta^\infty \cos(Rt-n\pi/2)t^{-n}\int_{S(x_0,t)} g(y)d\sigma_t(y)dt\\ &\quad + Cg(x_0)\int_{R\eta}^\infty \cos(t-n\pi/2)\frac{dt}{t}\\ &= o(1)\end{aligned}$$

where the first integral in the right-hand side of the second last equal sign equals $o(1)$ since the function

$$t^{-n}\int_{S(x_0,t)} g(y)d\sigma_t(y)\chi_{t\ge\eta}$$

is in $L^1(0,\infty)$ and the Riemann-Lebesgue Theorem. That the second integral equals $o(1)$ is basic on the convergence of the integral

$$\int_1^\infty \cos(t-n\pi/2)\frac{dt}{t}.$$

Finally we estimate the term I_2 as $R\to\infty$. Again, plugging (E) in the term I_2, one has

$$\begin{aligned}I_2 &= C\int_{\pi/R}^\eta t^{-\alpha-1}t^\alpha\cos(Rt-n\pi/2)\bar g_{x_0}(t)dt + O(R^{-1}\int_{\pi/R}^\eta t^{-2}\bar g_{x_0}(t)dt)\\ &\equiv I_2^{(1)} + I_2^{(2)}.\end{aligned}$$

By integration by parts,

$$\begin{aligned}I_2^{(1)} &= Ct^{-\alpha-1}\int_0^t s^\alpha\cos(Rs-n\pi/2)\bar g_{x_0}(s)ds\Big|_{\pi/R}^\eta\\ &\quad + C'\int_{\pi/R}^\eta t^{-\alpha-2}\int_0^t s^\alpha\cos(Rs-n\pi/2)\bar g_{x_0}(s)dsdt.\end{aligned}$$

Applying Lemma 3,

$$\begin{aligned}I_2^{(1)} &= CR^{\alpha+1}o(R^{-\alpha-1})(\log R)^{-1} + \int_{\pi/R}^\eta t^{-\alpha-2}(o(R^{-\alpha-1}) + o(t^{\alpha+1})(\log R)^{-1})dt\\ &= o(1) \qquad (R\to\infty).\end{aligned}$$

For the $I_2^{(2)}$, by integration by parts again,

$$\begin{aligned} I_2^{(2)} &= O\Big(\frac{1}{R}\int_{\pi/R}^{\eta} t^{-n-1}\int_{S(x_0,s)}(g(y)-g(x_0))d\sigma_s(y)dt\Big) \\ &= O\Big(\frac{1}{Rt^{n+1}}\int_0^t\int_{S(x_0,s)}(g(y)-g(x_0))d\sigma_s(y)ds\Big|_{t=\pi/R}^{t=\eta}\Big) \\ &\quad + C\frac{1}{R}\int_{\pi/R}^{\eta}\frac{1}{t^{n+2}}\int_0^t\int_{S(x_0,s)}(g(y)-g(x_0))d\sigma_s(y)dsdt. \end{aligned}$$

Then by (C) and (D), the last equality of the above equation is equal to

$$R^{-1}o(\eta^{-1}-R)+CR^{-1}\int_{\pi/R}^{\eta}o(t^{-2})dt = o(1) \qquad (R\to\infty).$$

The Theorem is proved.

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