A COHERENT FAMILY OF PARTIAL FUNCTIONS ON $\mathbb{N}$

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(Communicated by Andreas R. Blass)

Abstract. We prove that there is a family of partial functions $f_\alpha : A_\alpha \to \alpha$ ($\alpha \to \omega_1$, $A_\alpha$ is a tower in $P(\omega)/\text{Fin}$) such that every surjection $g : \omega_1 \to \{0, 1\}$ is associated to a cohomologically different Hausdorff gap (see Talayco). This improves a result of Talayco.

We prove a generalization of the classical result of Hausdorff about gaps in $P(\omega)/\text{Fin}$ which, when formulated in terms of characteristic functions of sets, states that there is a family $f_\alpha : A_\alpha \to 2$ ($\alpha < \omega_1$) of coherent functions supported by a tower $A_\alpha$ ($\alpha < \omega_1$) of infinite subsets of $\omega$ which is nontrivial in the sense that there is no single function $f : \omega \to 2$ inducing (modulo finite) all $f_\alpha$'s. The strengthening we give allows the $f_\alpha$'s to have all countable ordinals as possible values rather than just 0 or 1. Moreover, the nontriviality condition itself is strengthened; its more precise definition is given in section 1 below. The result also strengthens a more recent result of [Talayco] who proved a similar strengthening of Hausdorff’s theorem with $f_\alpha$’s having ranges in $\omega$ rather than $\omega_1$. We should also note that [Talayco] also proves a version of our full result using some additional axioms of set theory.

The paper has three sections. In the first one we give basic definitions. In the second we prove that the statement “there is a nontrivial coherent family supported by an $\omega_1$-tower $A_\alpha$" is consistent with ZFC. This is accomplished by choosing a tower $A_\alpha$ ($\alpha < \omega_1$) and then defining a poset that generically adds the appropriate family of functions, while preserving $\omega_1$. The third section is devoted to proving that the statement “there is a nontrivial coherent family supported by an $\omega_1$-tower $A_\alpha$" is absolute for models of ZFC.

I would like to thank Professor Stevo Todorčević for mentioning this problem and the idea for its solution to me, and for many valuable discussions afterwards. I would also like to thank the referee for a careful reading of this paper and for giving many valuable suggestions.

1. Definitions

Definition 1. A family $A_\alpha$ ($\alpha < \omega_1$) of subsets of $\omega$ is a tower iff $A_\alpha \subseteq^* A_\beta$ for all $\alpha < \beta < \omega_1$.

Definition 2. A family $B_\alpha$ ($\alpha < \omega_1$) is a nontrivial coherent subtower of the tower $A_\alpha$ iff
(G1) \( B_\alpha \subseteq A_\alpha \) for all \( \alpha < \omega_1 \),
(G2) \( B_\beta \cap A_\alpha = {}^* B_\alpha \) for all \( \alpha < \beta < \omega_1 \),
(G3) there is no \( B \subseteq \omega \) such that \( B \cap A_\alpha = {}^* B_\alpha \) for all \( \alpha < \omega_1 \).

Notice that \( A_\alpha \setminus B_\alpha \), \( B_\alpha \) forms an \((\omega_1, \omega^*_1)\)-gap in \((\omega^\omega, \subseteq^*)\) in the usual sense; this is why we also say that the family \( B_\alpha \) is a gap inside a tower \( A_\alpha \). A family satisfying (G1) and (G2) is said to be a coherent subtower (or just subtower) of \( A_\alpha \). In this case \( A_\alpha \setminus B_\alpha \), \( B_\alpha \) forms what is usually called a pre-gap.

Note that this is equivalent to saying that there is a family of partial functions \( f_\alpha \) (\( \alpha < \omega_1 \)) such that:

\begin{align*}
(G'1) \quad & f_\alpha : A_\alpha \to 2, \\
(G'2) \quad & f_\alpha \upharpoonright A_\beta = {}^* f_\beta \upharpoonright A_\alpha \text{ for all } \beta < \alpha < \omega_1, \\
(G'3) \quad & \text{there is no } f : \omega \to 2 \text{ such that } f \upharpoonright A_\alpha = {}^* f_\alpha \text{ for all } \alpha < \omega_1.
\end{align*}

This is a reformulation due to Todorčević of the classical Hausdorff Gap Theorem for the purpose of giving an alternate proof of this result (see [Bekkali, pp. 96–98]), as well as for the purpose of connecting the problem about the existence of gaps in \( P(\omega)/\text{Fin} \) with a problem from homology theory (see [Dow-Simon-Vaughan, section 4]). Note that the original Hausdorff condition of providing (G3) in the present context reads as follows:

\[(G''3) \quad \text{for every } \beta < \omega_1 \text{ and all } n < \omega \text{ the set } \{ \alpha < \beta : f_\beta^{-1}(1) \cap f_\alpha^{-1}(0) \subseteq n \} \text{ is finite.} \]

[This condition is in general stronger than (G'3), because it is preserved under any forcing which does not collapse \( \aleph_1 \), while if as in (G'3) can sometimes be added by a ccc forcing. This absoluteness of (G''3) appears as the consequence of the fact that it is a first-order statement (unlike (G'3), where we quantify over all subsets of \( \omega \)).]

In the definition below, indexes run from \( \omega \) to \( \omega_1 \) for some technical reasons.

**Definition 3.** A nontrivial coherent family supported by an \( \omega_1 \)-tower \( A_\alpha \) is a family of functions \( f_\alpha \), \( \omega \leq \alpha < \omega_1 \), such that for all \( \alpha < \omega_1 \):

\begin{align*}
(A1) \quad & f_\alpha : A_\alpha \to \alpha, \\
(A2) \quad & f_\alpha \upharpoonright A_\beta = {}^* f_\beta \upharpoonright A_\alpha \text{ for all } \beta < \alpha, \\
(A3) \quad & \text{for all } \xi < \eta < \alpha \text{ and for all } n \in \omega, \text{ the set } \\
& \{ \beta \in [\eta, \alpha) : f_\beta^{-1}(\{\xi\}) \cap f_\alpha^{-1}(\{\eta\}) \subseteq n \}
\end{align*}

is finite.

Notice that (A3) corresponds to (G''3) and not to (G'3) (compare with the remark after (G''3)). It would probably be more appropriate to say that a “nontrivial coherent family supported by an \( \omega_1 \)-tower \( A_\alpha \)” is a family \( B^E_\alpha \) which satisfies the conclusion of Lemma 1 below (which is the natural generalization to Definition 2), but then we would need another name for the object of our study. In particular, “\( \aleph_1 \) Gap Theorem” of [Talayco] uses a different combinatorial approach to an object satisfying the conclusion of Lemma 1, so it is not clear if this object satisfies our Definition 3. This is why we have to re-prove this consistency result.

In Definition 1 we did not require that the tower \( A_\alpha \) is not eventually constant, i.e., we are allowing the existence of a set \( A \subseteq \omega \) such that \( A_\alpha = {}^* A \) for all large enough \( \alpha \). On the other hand, if some tower \( A_\alpha \) has a nontrivial coherent subtower (or if there is a nontrivial coherent family supported by it), then it obviously can...
not be eventually constant; therefore there is a cofinal subset $C$ of $\omega_1$ such that $A_{\alpha} \setminus A_{\beta}$ is infinite whenever $\alpha > \beta$ are in $C$.

**Definition 4.** For a family of functions as in Definition 3 define families of subtowers $B_\alpha^X (0 \neq \xi < \alpha < \omega_1$ and $\omega \leq \alpha)$ and $B_\alpha^X (\omega \leq \alpha < \omega_1$ and $X \subseteq \omega_1)$ by:

\[
B_\alpha^X = \{ k \in \omega : f_\alpha(k) = \xi \}, \quad \text{and} \quad B_\alpha^X = \{ k \in \omega : f_\alpha(k) \in X \}.
\]

\[(*)\]

Notice that, while (A2) implies that $B_\alpha^X \cap A_{\beta} = \varnothing B_\beta^X$ for all $\beta < \alpha$, (A3) moreover implies that the set $\{ \xi < \alpha : B_\alpha^X \cap A_{\beta} \neq \varnothing B_\beta^X \}$ is finite for all $\beta < \alpha$.

**Fact.** If $B_{\alpha}, C_{\alpha} (\alpha < \omega_1)$ is a gap and $B_{\alpha}', C_{\alpha}'$ is a pre-gap such that $B_{\alpha} \supseteq B_{\alpha}$ and $C_{\alpha} \supseteq C_{\alpha}$ for all $\alpha < \omega_1$, then $B_{\alpha}', C_{\alpha}'$ is also a gap.

**Lemma 1.** Suppose that there is a nontrivial coherent family supported by an $\omega_1$-tower $A_\alpha$ and that $B_\alpha^X$ are as in Definition 4. Then

a) For all $\xi < \omega_1$, the family $B_\alpha^X (\alpha < \omega_1)$ forms a gap inside a tower $A_\alpha$.

b) For all $\xi < \eta < \omega_1$, families $B_\alpha^X$ and $B_\eta^X (\alpha < \omega_1)$ form a Hausdorff gap.

c) For every nonempty $X \subseteq \omega_1$, the family $B_\alpha^X$ is a subtower of $A_\alpha$.

d) For every $X \subseteq \omega_1$ such that both $X \setminus \{0\}, (\omega_1 \setminus X) \setminus \{0\}$ are nonempty, the family $B_\alpha^X (\alpha < \omega_1)$ forms a gap inside a tower $A_\alpha$.

**Proof.** a) and b) are immediate because (A3) implies (G''3) for $B_\alpha^X$ and $B_\eta^X$. c) is implied by the remark after Definition 4. d) follows from b), c) and the Fact above.

**Remark.** If $\omega_1 = \bigcup_{i \in I} X_i$ is a partition of $\omega_1$, then in the family $\langle B_\alpha^{X_i} : i \in I, \alpha < \omega_1 \rangle$ of subtowers of $A_\alpha$, every pair $B_{\alpha}^{X_i}, B_{\alpha}^{X_j} (\alpha < \omega_1)$ is a gap (if $i \neq j$). These gaps are moreover cohomologically different (for a definition see [Talayco]).

### 2. FORCING

For a fixed tower $A_\alpha (\alpha < \omega_1)$ in $[\omega]^{<\omega}$ (and such that $A_\alpha \setminus A_{\beta}$ is infinite whenever $\alpha > \beta$—see the paragraph before Definition 4), define a poset $P$ as follows: A typical $p \in P$ is $p = (X, Y, N, f)$, where

\[(B1)\] $X, Y \in [\omega_1]^{<\omega}$, $\min X \geq \omega, 0 \notin Y$,

\[(B2)\] $N \in [\omega]^{<\omega}$,

\[(B3)\] $f : X \times N \rightarrow Y \cup \{0\}$,

\[(B4)\] $f(\alpha, n) \neq 0$ if $n \in A_\alpha$,

\[(B5)\] $f(\alpha, n) < \alpha$ for all $\alpha$.

We shall add the subscript “$p$” to elements of $p$ when needed (like in (C1)–(C5) below). Define an ordering on $P$ by letting $p \leq q$ iff:

\[(C1)\] $X_p \supseteq X_q, Y_p \supseteq Y_q$,

\[(C2)\] $N_p \supseteq N_q$,

\[(C3)\] $f_p | X_q \times N_q = f_q$,

\[(C4)\] for all $\xi < \eta < \beta < \alpha$ such that $\xi, \eta \in Y_q, \alpha \in X_q, \beta \in X_p \setminus X_q$, there is an $n \in N_p \setminus N_q$ such that $f_p(\alpha, n) = \eta$ and $f_p(\beta, n) = \xi$, and

\[(C5)\] $f_p(\alpha, k) = f_p(\beta, k)$ for all $\alpha < \beta \in X_q$ and all $k \in (N_p \setminus N_q) \cap A_\alpha \setminus A_{\beta}$.

So $f_p(\alpha, \cdot)$ is a finite approximation to $f_\alpha$ as in Definition 3, where (C4) and (C5) are supposed to assure (A3) and (A2) respectively (see also Theorem 1).
**Definition 5.** For a finite subset $F$ of $\omega_1$ and a countable ordinal $\beta$ define the set

$$D_{F, \beta} = \bigcap_{\gamma \in F \setminus \beta} A_\gamma \setminus \bigcup_{\gamma \in F \cap \beta} A_\gamma.$$  

Notice that $D_{F, \beta}$ is infinite iff $F \setminus \beta \neq \emptyset$.

If $X$ and $Y$ are finite sets of ordinals, then $X < Y$ means that $\max X < \min Y$.

**Lemma 2.** If $r = \langle X_\Delta \cup Z_r, Y_r, N, f_r \rangle, q = \langle X_\Delta \cup Z_q, Y_q, N, f_q \rangle, X_\Delta < Z_r < Z_q$, and $f_r \upharpoonright X_\Delta \times N = f_q \upharpoonright X_\Delta \times N$, then there is a $p \leq r, q$ in $\mathcal{P}$.

**Proof.** Set $X_p = X_\Delta \cup Z_r \cup Z_q, Y_p = Y_r \cup Y_q$, and $f_p \upharpoonright X_p \times N = f_r \upharpoonright f_q$. First note that it is much easier to get $p \leq r$ than $p \leq q$, because the condition (C4) will be vacuously true. The major part of this proof is devoted to assuring (C4) for $\alpha \in Z_q$ and $\beta \in Z_r$, while preserving (C5) for all $k \in N_p \setminus N$.

**Case 1.** If either $Z_q$ or $Z_r$ is empty, then let $p = \langle X_p, Y_p, N, f_p \rangle$ as above.

**Case 2.** Otherwise, enumerate $\{Y_q\}^2 = \{\{\xi, \eta_i\} : i < \eta\}$ so that

1. $\xi_i < \eta_i$ for all $i < \eta$,
2. $\eta_i \leq \eta_j$ for all $i < j < \eta$.

**Case 2A.** If $Z_r \setminus (\eta_0 + 1) = \emptyset$, then set $p = \langle X_p, Y_p, N, f_p \rangle$ as above.

**Case 2B.** If not, then let $k = \max\{i < \eta : Z_r \setminus (\eta_i + 1) \neq \emptyset\}$. We address (C4) for $\xi_0 < \eta_0 < \beta_0$ and all necessary $\alpha$’s. Define $\beta_0 = \min(Z_r \setminus (\eta_0 + 1)), n_0 = \min(D_{X_p, \beta_0} \setminus N), N_1 = N \cup \{n_0\},$ and

$$f_p(\alpha, n_0) = \begin{cases} 
\eta_0, & \alpha \in Z_q, \\
\xi_0, & \alpha \in Z_r \setminus \beta_0, \\
0, & \alpha \in X_p \cap \beta_0.
\end{cases}$$

For all $i < k$, recursively define $\beta_{i+1} = \min(Z_r \setminus (\eta_i + 1)), n_{i+1} = \min(D_{X_p, \beta_i} \setminus N_i), N_{i+1} = N_i \cup \{n_i\},$ and

$$f_p(\alpha, n_i) = \begin{cases} 
\eta_i, & \alpha \in Z_q, \\
\xi_i, & \alpha \in Z_r \setminus \beta_i, \\
0, & \alpha \in X_p \cap \beta_i.
\end{cases}$$

At the end, set $p = \langle X_p, Y_p, N_k, f_p \rangle$. This construction has taken care of (C4) for all quadruples $\xi < \eta < \beta < \alpha$.

Now check that the construction preserves (C5): if $\alpha, \beta \in X_\Delta \cup Z_q$ and $k \in N_p \setminus N$, then $k = n_i$ for some $i < \eta$. Hence, if $k \in A_\alpha \cap A_\beta$, by the construction of $n_i$’s this means that both $\alpha$ and $\beta$ are in $Z_q$, and $f_p(\alpha, k) = f_p(\beta, k)$. One similarly checks the case when $\alpha, \beta \in X_\Delta \cup Z_r$.

**Lemma 3.** For $\beta \in [\omega, \omega_1], \xi < \omega_1$, and $l < \omega$, the sets $D_\beta = \{p \in \mathcal{P} : \beta \in X_p\}, E_\xi = \{p \in \mathcal{P} : \xi \in Y_p\}$, and $F_l = \{p \in \mathcal{P} : l \in N_p\}$ are dense in $\mathcal{P}$.

**Proof.** We fix a condition $q = \langle X, Y, N, f \rangle$, and find $p \leq q$ in the set that is in question.

$E_\xi$ is dense: Let $p = \langle X, Y \cup \{\xi\}, N, f \rangle$.

$D_\beta$ is dense for all $\beta \geq \omega$: We distinguish the following three cases:

**Case 0.** If $\beta \in X$, set $p = q$. 

Case 1. If $\beta > \max X$, then set $X_p = X \cup \{\beta\}$, choose some $\xi \in [1, \beta)$, and let

$$f_p(\beta, k) = \begin{cases} 
\xi, & k \in A_\beta, \\
0, & k \notin A_\beta,
\end{cases} \quad \text{for } k \in N.$$

Let $p = \langle X \cup \{\beta\}, Y \cup \{\xi\}, N, f_p \rangle$. Now notice that $p$ satisfies (B1)–(B5), that (C1)–(C3) are satisfied by the construction, and that (C5) is vacuously satisfied. By $\beta > \max X$ there is no $\alpha > \beta$ in $X$, so (C4) is also vacuously true.

Case 2. If $X \setminus (\beta + 1) \neq \emptyset$, choose $\xi < \beta$ such that $\xi \notin Y$—this is possible because $\beta \geq \omega$ and $Y$ is finite. Define:

$$X_\Delta = X \cap \beta, \quad Z_r = \{\beta\}, \quad Z_q = X \setminus X_\Delta,$$

$$Y_r = (Y \cap \beta) \cup \{\xi\}, \quad Y_q = Y, \quad f_q = f,$$

$$f_r \upharpoonright X_\Delta \times N = f \upharpoonright X_\Delta \times N, \quad f_r(\beta, k) = \begin{cases} 
\xi, & k \in A_\beta, \\
0, & k \notin A_\beta,
\end{cases} \quad \text{for } k \in N.$$

Then conditions $r$ and $q$ satisfy the assumptions of Lemma 2; hence there is a $p \leq r, q$, i.e., a $p \leq q$ such that $p \in D_\beta$.

By Lemma 4, any tower $A$ is dense; assume $l \notin N$. By the above we may assume without loss of generality that $X$ is nonempty; set $X_p = X$ and notice that with this value (C4) will be vacuous. Let $N_p = N \cup \{l\}$, choose any $\xi \in [1, \min X_p)$, e.g., $\xi = 1$ will do, and define $f_p$ to match $f$ on its domain while for $\beta \in X_p$ let

$$f_p(\beta, l) = \begin{cases} 
\xi, & l \in A_\beta, \\
0, & l \notin A_\beta.
\end{cases} \quad \text{This makes (C5) true. Finally, let } Y_p = Y \cup \{\xi\}. \text{ Then } p = \langle X_p, Y \cup \{\xi\}, N \cup \{l\}, f_p \rangle \text{ is in } F_l \text{ and } p \leq q.$$

This finishes the proof of Lemma 3. \hfill \Box

Lemma 4. $P$ is ccc.

Proof. Suppose that $\alpha < \omega_1$ is a family of conditions in $P$; without loss of generality, by using a standard $\Delta$-system and counting arguments we can suppose that:

1. $X_\alpha$ is a $\Delta$-system with root $X_\Delta$.
2. $Y_\alpha$ is a $\Delta$-system with root $Y_\Delta$.
3. $N_\alpha = N$ for some fixed $N \in [\omega]^{<\omega}$.
4. $g_\alpha \upharpoonright X_\Delta \times N = g$ for some fixed $g$.

[We can get (E4) because $\max(f_\alpha^+ X_\Delta) < \max X_\Delta$ for all $\alpha$.] By Lemma 2, conditions in this family are pairwise compatible. \hfill \Box

Theorem 1. If $f = f_G = \bigcup_{p \in G} f_p$ for a $P$-generic set $G$, then $f_\alpha$ (defined by $f_\alpha(k) = f(\alpha, k)$) is a nontrivial coherent family supported by an $\omega_1$-tower $A_\alpha$.

Proof. By Lemma 4, $\omega_1$ is preserved by forcing by $P$. Checking (A1)–(A3): (A1) is true, by (B4) and (B5). By (C5)

$$(D1) \quad q \vdash \{k \in A_\alpha : f_\alpha(k) \neq f_\beta(k)\} \subseteq N_q \cup (A_\alpha \setminus A_\beta) \text{ for all } \alpha < \beta \in X_q,$$

and this assures (A2). By Lemma 3 applied $n$ times the set $\{q \in P : n \subseteq N_q\}$ is dense for all $n < \omega$, and by (C4)
(D2) $q \vdash \{ \beta \in [\eta, \alpha] : f_{\beta}^{-1}(\{\xi\}) \cap f_{\alpha}^{-1}(\{\eta\}) \subseteq N_q \} \subseteq X_q$ for all $\xi < \eta < \alpha$ and $\alpha \in X_q$ such that $\xi < \eta < \alpha$, so this assures (A3).

3. Absolute

In this section we shall use some results from Model Theory to prove that our main result, the existence of a nontrivial coherent family supported by an $\omega_1$-tower $A_\alpha$, is absolute for transitive models of ZFC.

A bit of notation: $\mathfrak{M}$ will always denote a model of a first-order theory, while $M$ will denote its domain. Now we shall briefly describe the logic $L^\omega(Q)$; for more details and proofs see [Keisler]. The variables will be denoted by the same letters that we were using for the objects that they denote: $\alpha, \beta$ for ordinals, $m, n$ for integers, etc.

**Definition 6. Syntax:** $L(Q)$ is an extension of the first-order logic. Its language is the language of a predicate calculus with equality. It has one additional rule for building formulas: if $\phi$ is a formula of $L(Q)$, then $Qx\phi$ is a formula of $L(Q)$. There is a simple set of axioms that makes this logic complete (see [Keisler]; rules of inference are the usual ones).

The logic $L^\omega(Q)$ is a further extension of $L(Q)$ with the added unary predicate $N$ and constant symbols $0, 1, \ldots$. Its additional axioms are $N(0), N(1), \ldots$, and it has an additional rule of inference, called the $\omega$-rule: for any formula $\phi(x)$, from $\phi(0), \phi(1), \phi(2), \ldots$ conclude $\forall x (N(x) \rightarrow \phi(x))$.

**Definition 7. Semantics:** A standard model for $L(Q)$ is a structure $\mathfrak{M}$ where the satisfaction relation “$\models$” is defined in the usual way, with the addition of the rule: $\mathfrak{M} \models Qx\phi(x)$ iff the set $\{ a \in M : \mathfrak{M} \models \phi[a] \}$ is uncountable. A weak model for $L(Q)$ is a structure $(\mathfrak{M}, q)$, where $q$ is a family of subsets of $M$ and the satisfaction relation “$\models$” is defined in the usual way, with the addition of the rule: $(\mathfrak{M}, q) \models Qx\phi(x)$ iff the set $\{ a \in M : (\mathfrak{M}, q) \models \phi[a] \}$ is in $q$. We also require that all axioms for $L(Q)$ are valid in $(\mathfrak{M}, q)$.

An $\omega$-model is a model $\mathfrak{M}$ of the language of $L^\omega$ with the property: $\mathfrak{M} \models N[a]$ iff there is an $n \in \omega$ such that $\mathfrak{M} \models a = n$ for all $a \in M$. A (standard, weak) model for $L^\omega(Q)$ is a (standard, weak) model for $L(Q)$ that is also an $\omega$-model.

**Theorem 2** (Completeness of $L^\omega(Q)$, Keisler). The following are equivalent for any countable set $\Phi$ of $L^\omega(Q)$-sentences:

1. $\Phi$ is consistent in $L^\omega(Q)$,
2. $\Phi$ has a weak $\omega$-model,
3. $\Phi$ has a standard $\omega$-model of size $\aleph_1$.

**Corollary.** If a countable set $\Phi$ of $L^\omega(Q)$-sentences has an $\omega$-model in a forcing extension $V^P$ of the universe $V$, then in fact $\Phi$ has an $\omega$-model in $V$.

**Proof.** If $\Phi$ is inconsistent in $V$, then it would remain such in $V^P$. 

**Remark.** A precise version of this Corollary would say that the ZFC statement “$\Phi$ has an $\omega$-model” is absolute between transitive models of a sufficiently large portion of ZFC.

**Lemma 5.** There is a countable theory $\Phi$ in $L^\omega(Q)$ which has an $\omega$-model iff there is a nontrivial coherent family supported by an $\omega_1$-tower $A_\alpha$. 

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Proof. Our language is the language of $L^\omega(Q)$ plus the following:

(F1) binary relational symbol “$<$”,
(F2) binary function symbol “$e$”,
(F3) binary relational symbol “$A$”,
(F4) binary function symbol “$\mathsf{f}$”.

We shall express “$k \in A_\alpha$” as “$A(\alpha,k)$” and “$f_\alpha(k) = \xi$” as “$\mathsf{f}(\alpha,k) = \xi$” (almost—see (*) below). Here “$<$” is meant to be the usual ordering on ordinals and $e$ is a function with domain $\{ (\alpha, \beta) : \alpha < \beta \}$ such that $e(\cdot, \alpha)$ is a 1-1 mapping from $\alpha$ into $\omega$ for all $\alpha < \omega_1$. So $X \subseteq \alpha$ is finite iff $\{ e(\xi, \alpha) : \xi \in X \}$ is a bounded subset of $\omega$.

Define $\Phi$ as the set of universal closures of the following statements (using the common shortcuts such as “$\forall \alpha < \beta$” or “$\forall n \in N$”):

(F1) $Qx(x = x)$,
(F2) “$<$” is a linear ordering and $(\forall \alpha)(\neg \mathcal{Q} \beta(\beta < \alpha)$,
(F3) $\alpha > \beta \& N(\alpha) \rightarrow N(\beta)$, and $0 < 1 < \frac{1}{2} < \ldots$,
(F4) $\forall k(A(\alpha,k) \rightarrow N(k))$, and $\alpha < \beta \rightarrow (\exists n \in N)(\forall k > n)(A(k,b) \rightarrow A(\alpha,k))$,
(F5) $f(\alpha, k) \neq 0 \rightarrow A(\alpha, k)$, and $f(\alpha, k) < \alpha$,
(F6) $\alpha > \beta \rightarrow (\exists n)(\forall k)(A(\alpha,k) \& A(\beta,k) \& f(\alpha,k) \neq f(\beta,k) \rightarrow k < n)$,
(F7) $(\forall \xi < \eta < \zeta)(\forall n)(\exists m \in N)(\forall \beta \in [n, \zeta))(e(\beta, \alpha) \geq m \rightarrow (\exists k > n, k \in N)(f(\beta,k) = \xi k \& f(\alpha, k) = \eta))$.

Hence, (F6) is (A1), (F7) is (A2), and (F8) is (A3). In the latter two we are using the fact that, for a subset $X$ of $\omega$, “$\text{being bounded}$” is equivalent to “$\text{being finite}$”—it is essential here that we have an $\omega$-model.

We shall now prove that an $\omega$-model for $\Phi$ is a good enough approximation to a nontrivial coherent family supported by an $\omega_1$-tower $A_\alpha$. Notice that every nontrivial coherent family supported by an $\omega_1$-tower $A_\alpha$, together with any function $e$ with the described properties, defines a model for $\Phi$. The other direction is a little bit less trivial. We are using the same symbols for the elements of our language and for their interpretations. Suppose that we have a standard $\omega$-model $\mathfrak{N} \models \Phi$ of the size $\aleph_1$. It need not be well-founded, but the cofinality of the set $M$ must be $\omega_1$ as a consequence of (F2). Hence, we can choose a cofinal increasing sequence $D = \{ a_\alpha : \alpha < \omega_1 \}$ of its elements so that the first $\omega$ members are $0, 1, \ldots$. Now we have some problems with the function $f$, whose range is not a subset of $D$, even if $f$ is restricted to $D \times N$. This is why we define

(*)

$$f_\alpha(k) = \begin{cases} \beta, & \text{if } f(\alpha, k) = a_\beta \in D, \\ 0, & \text{if } f(\alpha, k) \notin D, \end{cases}$$

and $A_\alpha = \text{supp } f_\alpha = \{ k : f_\alpha(k) \neq 0 \}$.

So we are shrinking $A_\alpha$’s and “forgetting” about those subtowers whose indexes fall outside of $D$. By (F5) the family $A_\alpha$ is increasing with respect to $\subseteq^*$ (and this suffices—see the paragraph before Definition 4). We have to check that $f_\alpha, A_\alpha$ satisfy Definition 3 with the interval $[1, \omega_1)$ in place of $\omega_1$ as the range space:

(A1) follows from (F6) and (*).

(A2) By (*), $f_\alpha(k) \neq f_\beta(k)$ implies $f(\alpha, k) \neq f(\alpha, k) \neq f(\beta, k) \neq f(\beta, k)$ so (A2) follows from (F7).
(A3) Note first that by (*) $n \in f^{-1}_\alpha(\{\xi\})$ iff $n \in \bigcap_{\alpha} f^{-1}_\alpha(\{a_\xi\})$ for all $\xi < \alpha < \omega_1$ and all $n < \omega$. Pick $0 < \xi < \eta < \alpha < \omega_1$ and $n < \omega$. If $\beta \in [\eta, \alpha)$ and $f^{-1}_\beta(\{\xi\}) \cap f^{-1}_\alpha(\{\eta\}) \subseteq n$, then we also have $\bigcap_{\alpha} f^{-1}_\beta(\{a_\xi\}) \cap \bigcap_{\alpha} f^{-1}_\alpha(\{a_\eta\}) \subseteq n$, and by (Φ8) there are only finitely many such $\beta$’s.

This finishes the proof. □

By putting together results in this section with those in previous sections, we get our main theorem:

**Theorem 3.** There is a nontrivial coherent family of functions $f_\alpha : A_\alpha \to \omega_1$ supported by an $\omega_1$-tower $A_\alpha$.

**References**


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