PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 9, September 1996

A COHERENT FAMILY OF PARTIAL FUNCTIONS ON $\mathbb{N}$

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(Communicated by Andreas R. Blass)

ABSTRACT. We prove that there is a family of partial functions $f_\alpha : A_\alpha \to \alpha$ ($\alpha \to \omega_1$, A_α is a tower in $P(\omega)/\operatorname{Fin}$) such that every surjection $g : \omega_1 \to \{0, 1\}$ is associated to a cohomologically different Hausdorff gap (see Talayco). This improves a result of Talayco.

We prove a generalization of the classical result of Hausdorff about gaps in $P(\omega)/\operatorname{Fin}$ which, when formulated in terms of characteristic functions of sets, states that there is a family $f_\alpha : A_\alpha \to 2$ ($\alpha < \omega_1$) of coherent functions supported by a tower A_α ($\alpha < \omega_1$) of infinite subsets of ω which is nontrivial in the sense that there is no single function $f : \omega \to 2$ inducing (modulo finite) all f_α's. The strengthening we give allows the f_α's to have all countable ordinals as possible values rather than just 0 or 1. Moreover, the nontriviality condition itself is strengthened; its more precise definition is given in section 1 below. The result also strengthens a more recent result of [Talayco] who proved a similar strengthening of Hausdorff's theorem with f_α's having ranges in ω rather than ω_1. We should also note that [Talayco] also proves a version of our full result using some additional axioms of set theory.

The paper has three sections. In the first one we give basic definitions. In the second we prove that the statement "there is a nontrivial coherent family supported by an ω_1-tower A_α" is consistent with ZFC. This is accomplished by choosing a tower A_α ($\alpha < \omega_1$) and then defining a poset that generically adds the appropriate family of functions, while preserving ω_1. The third section is devoted to proving that the statement "there is a nontrivial coherent family supported by an ω_1-tower A_α" is absolute for models of ZFC.

I would like to thank Professor Stevo Todorčević for mentioning this problem and the idea for its solution to me, and for many valuable discussions afterwards. I would also like to thank the referee for a careful reading of this paper and for giving many valuable suggestions.

1. Definitions

Definition 1. A family A_α ($\alpha < \omega_1$) of subsets of ω is a *tower* iff $A_\alpha \subseteq^* A_\beta$ for all $\alpha < \beta < \omega_1$.

Definition 2. A family B_α ($\alpha < \omega_1$) is a *nontrivial coherent subtower of the tower* A_α iff

Received by the editors June 20, 1994 and, in revised form, March 20, 1995.
1991 *Mathematics Subject Classification.* Primary 03C80, 03E40, 04A20.
Research supported by the Science Fund of Serbia grant number 0401A.

(G1) $B_\alpha \subseteq A_\alpha$ for all $\alpha < \omega_1$,
(G2) $B_\beta \cap A_\alpha =^* B_\alpha$ for all $\alpha < \beta < \omega_1$,
(G3) there is no $B \subseteq \omega$ such that $B \cap A_\alpha =^* B_\alpha$ for all $\alpha < \omega_1$.

Notice that $A_\alpha \setminus B_\alpha$, B_α forms an (ω_1, ω_1^*)-gap in $([\omega]^\omega, \subseteq^*)$ in the usual sense; this is why we also say that the family B_α is a *gap inside a tower* A_α. A family satisfying (G1) and (G2) is said to be a *coherent subtower* (or just subtower) of A_α. In this case $A_\alpha \setminus B_\alpha$, B_α forms what is usually called a *pre-gap*.

Note that this is equivalent to saying that there is a family of partial functions f_α ($\alpha < \omega_1$) such that:

(G′1) $f_\alpha : A_\alpha \to 2$,
(G′2) $f_\alpha \restriction A_\beta =^* f_\beta \restriction A_\alpha$ for all $\beta < \alpha < \omega_1$,
(G′3) there is no $f : \omega \to 2$ such that $f \restriction A_\alpha =^* f_\alpha$ for all $\alpha < \omega_1$.

This is a reformulation due to Todorčević of the classical Hausdorff Gap Theorem for the purpose of giving an alternate proof of this result (see [Bekkali, pp. 96–98]), as well as for the purpose of connecting the problem about the existence of gaps in $P(\omega)/\operatorname{Fin}$ with a problem from homology theory (see [Dow-Simon-Vaughan, section 4]). Note that the original Hausdorff condition of providing (G3) in the present context reads as follows:

(G″3) for every $\beta < \omega_1$ and all $n < \omega$ the set $\{\alpha < \beta : f_\beta^{-1}(1) \cap f_\alpha^{-1}(0) \subseteq n\}$ is finite.

[This condition is in general stronger than (G′3), because it is preserved under any forcing which does not collapse $\aleph_1$, while f as in (G′3) can sometimes be added by a ccc forcing. This absoluteness of (G″3) appears as the consequence of the fact that it is a *first-order* statement (unlike (G′3), where we quantify over all subsets of ω).]

In the definition below, indexes run from ω to ω_1 for some technical reasons.

Definition 3. A nontrivial coherent family supported by an ω_1-tower A_α is a family of functions f_α, $\omega \leq \alpha < \omega_1$, such that for all $\alpha < \omega_1$:

(A1) $f_\alpha : A_\alpha \to \alpha$,
(A2) $f_\alpha \restriction A_\beta =^* f_\beta \restriction A_\alpha$ for all $\beta < \alpha$,
(A3) for all $\xi < \eta < \alpha$ and for all $n \in \omega$, the set
$$\{\beta \in [\eta, \alpha) : f_\beta^{-1}(\{\xi\}) \cap f_\alpha^{-1}(\{\eta\}) \subseteq n\}$$
is finite.

Notice that (A3) corresponds to (G″3) and not to (G′3) (compare with the remark after (G″3)). It would probably be more appropriate to say that a "nontrivial coherent family supported by an ω_1-tower A_α" is a family B_a^ξ which satisfies the conclusion of Lemma 1 below (which is the natural generalization to Definition 2), but then we would need another name for the object of our study. In particular, "$\aleph_1$ Gap Theorem" of [Talayco] uses a different combinatorial approach to an object satisfying the conclusion of Lemma 1, so it is not clear if this object satisfies our Definition 3. This is why we have to re-prove this consistency result.

In Definition 1 we did not require that the tower A_α is not *eventually constant*, i.e., we are allowing the existence of a set $A \subseteq \omega$ such that $A_\alpha =^* A$ for all large enough α. On the other hand, if some tower A_α has a nontrivial coherent subtower (or if there is a nontrivial coherent family supported by it), then it obviously can not be eventually constant; therefore there is a cofinal subset C of ω_1 such that $A_\alpha \setminus A_\beta$ is infinite whenever $\alpha > \beta$ are in C.

Definition 4. For a family of functions as in Definition 3 define families of subtowers B^ξ_α ($0 \neq \xi < \alpha < \omega_1$ and $\omega \leq \alpha$) and B^X_α ($\omega \leq \alpha < \omega_1$ and $X \subseteq \omega_1$) by:

$$(*) \qquad \begin{aligned} B^\xi_\alpha &= \{k \in \omega : f_\alpha(k) = \xi\}, \quad \text{and} \\ B^X_\alpha &= \{k \in \omega : f_\alpha(k) \in X\}. \end{aligned}$$

[Notice that, while (A2) implies that $B^\xi_\alpha \cap A_\beta =^* B^\xi_\beta$ for all $\beta < \alpha$, (A3) moreover implies that the set $\{\xi < \alpha : B^\xi_\alpha \cap A_\beta \neq B^\xi_\beta\}$ is finite for all $\beta < \alpha$.]

Fact. If B_α, C_α ($\alpha < \omega_1$) is a gap and B'_α, C'_α is a pre-gap such that $B'_\alpha \supseteq B_\alpha$ and $C'_\alpha \supseteq C_\alpha$ for all $\alpha < \omega_1$, then B'_α, C'_α is also a gap. □

Lemma 1. *Suppose that there is a nontrivial coherent family supported by an ω_1-tower A_α and that B^ξ_α are as in Definition 4. Then*

a) *For all $\xi < \omega_1$, the family B^ξ_α ($\alpha < \omega_1$) forms a gap inside a tower A_α.*

b) *For all $\xi < \eta < \omega_1$, families B^ξ_α and B^η_α ($\alpha < \omega_1$) form a Hausdorff gap.*

c) *For every nonempty $X \subseteq \omega_1$, the family B^X_α is a subtower of A_α.*

d) *For every $X \subseteq \omega_1$ such that both $X \setminus \{0\}, (\omega_1 \setminus X) \setminus \{0\}$ are nonempty, the family B^X_α ($\alpha < \omega_1$) forms a gap inside a tower A_α.*

Proof. a) and b) are immediate because (A3) implies (G″3) for B^ξ_α and B^η_α. c) is implied by the remark after Definition 4. d) follows from b), c) and the Fact above. □

Remark. If $\omega_1 = \bigcup_{i \in I} X_i$ is a partition of ω_1, then in the family $\langle B^{X_i}_\alpha : i \in I, \alpha < \omega_1\rangle$ of subtowers of A_α, every pair $B^{X_i}_\alpha, B^{X_j}_\alpha$ ($\alpha < \omega_1$) is a gap (if $i \neq j$). These gaps are moreover cohomologically different (for a definition see [Talayco]).

2. Forcing

For a fixed tower A_α ($\alpha < \omega_1$) in $[\omega]^\omega$ (and such that $A_\alpha \setminus A_\beta$ is infinite whenever $\alpha > \beta$—see the paragraph before Definition 4), define a poset $\mathcal{P}$ as follows: A typical $p \in \mathcal{P}$ is $p = \langle X, Y, N, f\rangle$, where

(B1) $X, Y \in [\omega_1]^{<\omega}, \min X \geq \omega, 0 \notin Y$,
(B2) $N \in [\omega]^{<\omega}$,
(B3) $f : X \times N \to Y \cup \{0\}$,
(B4) $f(\alpha, n) \neq 0$ iff $n \in A_\alpha$,
(B5) $f(\alpha, n) < \alpha$ for all α.

We shall add the subscript "p" to elements of p when needed (like in (C1)–(C5) below). Define an ordering on $\mathcal{P}$ by letting $p \leq q$ iff:

(C1) $X_p \supseteq X_q, Y_p \supseteq Y_q$,
(C2) $N_p \supseteq N_q$,
(C3) $f_p \restriction X_q \times N_q = f_q$,
(C4) for all $\xi < \eta < \beta < \alpha$ such that $\xi, \eta \in Y_q, \alpha \in X_q$, and $\beta \in X_p \setminus X_q$, there is an $n \in N_p \setminus N_q$ such that $f_p(\alpha, n) = \eta$ and $f_p(\beta, n) = \xi$, and
(C5) $f_p(\alpha, k) = f_p(\beta, k)$ for all $\alpha < \beta \in X_q$ and all $k \in (N_p \setminus N_q) \cap A_\alpha \cap A_\beta$.

So $f_p(\alpha, \cdot)$ is a finite approximation to f_α as in Definition 3, where (C4) and (C5) are supposed to assure (A3) and (A2) respectively (see also Theorem 1).

Definition 5. For a finite subset F of ω_1 and a countable ordinal β define the set

$$D_{F,\beta} = \bigcap_{\gamma \in F \setminus \beta} A_\gamma \setminus \bigcup_{\gamma \in F \cap \beta} A_\gamma.$$

Notice that $D_{F,\beta}$ is infinite iff $F \setminus \beta \neq \varnothing$.

If X and Y are finite sets of ordinals, then $X < Y$ means that $\max X < \min Y$.

Lemma 2. *If* $r = \langle X_\Delta \cup Z_r, Y_r, N, f_r \rangle, q = \langle X_\Delta \cup Z_q, Y_q, N, f_q \rangle, X_\Delta < Z_r < Z_q$, *and* $f_r \restriction X_\Delta \times N = f_q \restriction X_\Delta \times N$, *then there is a* $p \leq r, q$ *in* $\mathcal{P}$.

Proof. Set $X_p = X_\Delta \cup Z_r \cup Z_q$, $Y_p = Y_r \cup Y_q$, and $f_p \restriction X_p \times N = f_r \cup f_q$. First note that it is much easier to get $p \leq r$ than $p \leq q$, because the condition (C4) will be vacuously true. The major part of this proof is devoted to assuring (C4) for $\alpha \in Z_q$ and $\beta \in Z_r$, while preserving (C5) for all $k \in N_p \setminus N$.

Case 1. If either Z_q or Z_r is empty, then let $p = \langle X_p, Y_p, N, f_p \rangle$ as above.

Case 2. Otherwise, enumerate $[Y_q]^2 = \{\{\xi_i, \eta_i\} : i < l\}$ so that

(1) $\xi_i < \eta_i$ for all $i < l$, and
(2) $\eta_i \leq \eta_j$ for all $i < j < l$.

Case 2A. If $Z_r \setminus (\eta_0 + 1) = \varnothing$, then set $p = \langle X_p, Y_p, N, f_p \rangle$ as above.

Case 2B. If not, then let $k = \max\{i < l : Z_r \setminus (\eta_i + 1) \neq \varnothing\}$. We address (C4) for $\xi_0 < \eta_0 < \beta_0$ and all necessary α's. Define $\beta_0 = \min(Z_r \setminus (\eta_0 + 1))$, $n_0 = \min(D_{X_p,\beta_0} \setminus N)$, $N_1 = N \cup \{n_0\}$, and

$$f_p(\alpha, n_0) = \begin{cases} \eta_0, & \alpha \in Z_q, \\ \xi_0, & \alpha \in Z_r \setminus \beta_0, \\ 0, & \alpha \in X_p \cap \beta_0. \end{cases}$$

For all $i < k$, recursively define $\beta_{i+1} = \min(Z_r \setminus (\eta_i + 1)), n_{i+1} = \min(D_{X_p,\beta_i} \setminus N_i)$, $N_{i+1} = N_i \cup \{n_i\}$, and

$$f_p(\alpha, n_i) = \begin{cases} \eta_i, & \alpha \in Z_q, \\ \xi_i, & \alpha \in Z_r \setminus \beta_i, \\ 0, & \alpha \in X_p \cap \beta_i. \end{cases}$$

At the end, set $p = \langle X_p, Y_p, N_k, f_p \rangle$. This construction has taken care of (C4) for all quadruples $\xi < \eta < \beta < \alpha$.

Now check that the construction preserves (C5): if $\alpha, \beta \in X_\Delta \cup Z_q$ and $k \in N_p \setminus N$, then $k = n_i$ for some $i < l$. Hence, if $k \in A_\alpha \cap A_\beta$, by the construction of n_i's this means that both α and β are in Z_q, and $f_p(\alpha, k) = f_p(\beta, k)$. One similarly checks the case when $\alpha, \beta \in X_\Delta \cup Z_r$. □

Lemma 3. *For* $\beta \in [\omega, \omega_1), \xi < \omega_1$, *and* $l < \omega$, *the sets* $D_\beta = \{p \in \mathcal{P} : \beta \in X_p\}$, $E_\xi = \{p \in \mathcal{P} : \xi \in Y_p\}$, *and* $F_l = \{p \in \mathcal{P} : l \in N_p\}$ *are dense in* $\mathcal{P}$.

Proof. We fix a condition $q = \langle X, Y, N, f \rangle$, and find $p \leq q$ in the set that is in question.

E_ξ **is dense**: Let $p = \langle X, Y \cup \{\xi\}, N, f \rangle$.

D_β **is dense for all** $\beta \geq \omega$: We distinguish the following three cases:

Case 0. If $\beta \in X$, set $p = q$.

Case 1. If $\beta > \max X$, then set $X_p = X \cup \{\beta\}$, choose some $\xi \in [1, \beta)$, and let

$$f_p(\beta, k) = \begin{cases} \xi, & k \in A_\beta, \\ 0, & k \notin A_\beta, \end{cases} \qquad \text{for } k \in N.$$

Let $p = \langle X \cup \{\beta\}, Y \cup \{\xi\}, N, f_p\rangle$. Now notice that p satisfies (B1)–(B5), that (C1)–(C3) are satisfied by the construction, and that (C5) is vacuously satisfied. By $\beta > \max X$ there is no $\alpha > \beta$ in X, so (C4) is also vacuously true.

Case 2. If $X \setminus (\beta + 1) \neq \varnothing$, choose $\xi < \beta$ such that $\xi \notin Y$—this is possible because $\beta \geq \omega$ and Y is finite. Define:

$$X_\Delta = X \cap \beta, \qquad Z_r = \{\beta\}, \qquad Z_q = X \setminus X_\Delta,$$
$$Y_r = (Y \cap \beta) \cup \{\xi\}, \qquad Y_q = Y, \qquad f_q = f,$$

$$f_r \restriction X_\Delta \times N = f \restriction X_\Delta \times N, \qquad f_r(\beta, k) = \begin{cases} \xi, & k \in A_\beta, \\ 0, & k \notin A_\beta, \end{cases} \quad \text{for } k \in N.$$

Then conditions r and q satisfy the assumptions of Lemma 2; hence there is a $p \leq r, q$, i.e., a $p \leq q$ such that $p \in D_\beta$.

F_l **is dense**: Assume $l \notin N$. By the above we may assume without loss of generality that X is nonempty; set $X_p = X$ and notice that with this value (C4) will be vacuous. Let $N_p = N \cup \{l\}$, choose any $\xi \in [1, \min X_p)$, e.g., $\xi = 1$ will do, and define f_p to match f on its domain while for $\beta \in X_p$ let

$$f_p(\beta, l) = \begin{cases} \xi, & l \in A_\beta, \\ 0, & l \notin A_\beta. \end{cases}$$

This makes (C5) true. Finally, let $Y_p = Y \cup \{\xi\}$. Then $p = \langle X_p, Y \cup \{\xi\}, N \cup \{l\}, f_p\rangle$ is in F_l and $p \leq q$.

This finishes the proof of Lemma 3. □

Lemma 4. $\mathcal{P}$ *is ccc.*

Proof. Suppose that $p_\alpha = \langle X_\alpha, Y_\alpha, N_\alpha, g_\alpha\rangle$ $(\alpha < \omega_1)$ is a family of conditions in $\mathcal{P}$; without loss of generality, by using a standard Δ-system and counting arguments we can suppose that:

(E1) X_α $(\alpha < \omega_1)$ is a Δ-system with root X_Δ,
(E2) Y_α $(\alpha < \omega_1)$ is a Δ-system with root Y_Δ,
(E3) $N_\alpha = N$ for some fixed $N \in [\omega]^{<\omega}$,
(E4) $g_\alpha \restriction X_\Delta \times N = g$ for some fixed g.

[We can get (E4) because $\max(f''_\alpha X_\Delta) < \max X_\Delta$ for all α.] By Lemma 2, conditions in this family are pairwise compatible. □

Theorem 1. *If* $f = f_G = \bigcup_{p\in G} f_p$ *for a* $\mathcal{P}$*-generic set* G*, then* f_α *(defined by* $f_\alpha(k) = f(\alpha, k)$*) is a nontrivial coherent family supported by an* ω_1*-tower* A_α*.*

Proof. By Lemma 4, ω_1 is preserved by forcing by $\mathcal{P}$. Checking (A1)–(A3): (A1) is true, by (B4) and (B5). By (C5)

(D1) $q \Vdash \{k \in A_\alpha : f_\alpha(k) \neq f_\beta(k)\} \subseteq N_q \cup (A_\alpha \setminus A_\beta)$ for all $\alpha < \beta \in X_q$,

and this assures (A2). By Lemma 3 applied n times the set $\{q \in \mathcal{P} : n \subseteq N_q\}$ is dense for all $n < \omega$, and by (C4)

(D2) $q \Vdash \{\beta \in [\eta, \alpha) : f_\beta^{-1}(\{\xi\}) \cap f_\alpha^{-1}(\{\eta\}) \subseteq N_q\} \subseteq X_q$ for all $\xi < \eta \in Y_q$ and $\alpha \in X_q$ such that $\xi < \eta < \alpha$,

so this assures (A3). □

3. Absoluteness

In this section we shall use some results from Model Theory to prove that our main result, the existence of a nontrivial coherent family supported by an ω_1-tower A_α, is absolute for transitive models of ZFC.

A bit of notation: $\mathfrak{M}$ will always denote a model of a first-order theory, while M will denote its domain. Now we shall briefly describe the logic $L^\omega(Q)$; for more details and proofs see [Keisler]. The variables will be denoted by the same letters that we were using for the objects that they denote: α, β for ordinals, m, n for integers, etc.

Definition 6. *Syntax*: $L(Q)$ is an extension of the first-order logic. Its language is the language of a predicate calculus with equality. It has one additional rule for building formulas: if ϕ is a formula of $L(Q)$, then $Qx\phi$ is a formula of $L(Q)$. There is a simple set of axioms that makes this logic complete (see [Keisler]; rules of inference are the usual ones).

The logic $L^\omega(Q)$ is a further extension of $L(Q)$ with the added unary predicate N and constant symbols $\underline{0}, \underline{1}, \ldots$. Its additional axioms are $N(\underline{0}), N(\underline{1}), \ldots$, and it has an additional rule of inference, called the ω-rule: for any formula $\phi(x)$, from $\phi(\underline{0}), \phi(\underline{1}), \phi(\underline{2}), \ldots$ conclude $\forall x(N(x) \to \phi(x))$.

Definition 7. *Semantics*: A *standard model* for $L(Q)$ is a structure $\mathfrak{M}$ where the satisfaction relation "$\models$" is defined in the usual way, with the addition of the rule: $\mathfrak{M} \models Qx\phi(x)$ iff the set $\{a \in M : \mathfrak{M} \models \phi[a]\}$ is uncountable. A *weak model* for $L(Q)$ is a structure $(\mathfrak{M}, q)$, where q is a family of subsets of M and the satisfaction relation "$\models$" is defined in the usual way, with the addition of the rule: $(\mathfrak{M}, q) \models Qx\phi(x)$ iff the set $\{a \in M : (\mathfrak{M}, q) \models \phi[a]\}$ is in q. We also require that all axioms for $L(Q)$ are valid in $(\mathfrak{M}, q)$.

An ω-model is a model $\mathfrak{M}$ of the language of L^ω with the property: $\mathfrak{M} \models N[a]$ iff there is an $n \in \omega$ such that $\mathfrak{M} \models a = \underline{n}$ for all $a \in M$. A (standard, weak) model for $L^\omega(Q)$ is a (standard, weak) model for $L(Q)$ that is also an ω-model.

Theorem 2 (Completeness of $L^\omega(Q)$, Keisler)**.** *The following are equivalent for any countable set Φ of $L^\omega(Q)$-sentences:*

(1) *Φ is consistent in $L^\omega(Q)$,*
(2) *Φ has a weak ω-model,*
(3) *Φ has a standard ω-model of size $\aleph_1$.* □

Corollary. *If a countable set Φ of $L^\omega(Q)$-sentences has an ω-model in a forcing extension $V^{\mathcal{P}}$ of the universe V, then in fact Φ has an ω-model in V.*

Proof. If Φ is inconsistent in V, then it would remain such in $V^{\mathcal{P}}$. □

Remark. A precise version of this Corollary would say that the ZFC statement "Φ has an ω-model" is absolute between transitive models of a sufficiently large portion of ZFC.

Lemma 5. *There is a countable theory Φ in $L^\omega(Q)$ which has an ω-model iff there is a nontrivial coherent family supported by an ω_1-tower A_α.*

Proof. Our language is the language of $L^\omega(Q)$ plus the following:

(F1) binary relational symbol "$<$",
(F2) binary function symbol "e",
(F3) binary relational symbol "A",
(F4) binary function symbol "$\underline{f}$".

We shall express "$k \in A_\alpha$" as "$A(\alpha, k)$" and "$f_\alpha(k) = \xi$" as "$\underline{f}(\alpha, k) = \xi$" (almost—see (*) below). Here "$<$" is meant to be the usual ordering on ordinals and e is a function with domain $\{\langle\alpha, \beta\rangle : \alpha < \beta\}$ such that $e(\cdot, \alpha)$ is a 1-1 mapping from α into ω for all $\alpha < \omega_1$. So $X \subseteq \alpha$ is finite iff $\{e(\xi, \alpha) : \xi \in X\}$ is a bounded subset of ω.

Define Φ as the set of universal closures of the following statements (using the common shortcuts such as "$\forall\alpha < \beta$" or "$\forall n \in N$"):

(Φ1) $Qx(x = x)$,
(Φ2) "$<$" is a linear ordering and $(\forall\alpha)(\neg Q\beta)(\beta < \alpha)$,
(Φ3) $(\alpha > \beta \& N(\alpha)) \to N(\beta)$, and $\underline{0} < \underline{1} < \underline{2} < \dots$,
(Φ4) $\alpha > \beta > \gamma \to e(\gamma, \alpha) \neq e(\beta, \alpha)$, and $\alpha > \beta \to N(e(\beta, \alpha))$,
(Φ5) $\forall k(A(\alpha, k) \to N(k))$, and $\alpha < \beta \to (\exists n \in N)(\forall k > n)(A(\beta, k) \to A(\alpha, k))$,
(Φ6) $\underline{f}(\alpha, k) \neq 0 \leftrightarrow A(\alpha, k)$, and $\underline{f}(\alpha, k) < \alpha$,
(Φ7) $\alpha > \beta \to (\exists n)(\forall k)(A(\alpha, k) \& A(\beta, k) \& \underline{f}(\alpha, k) \neq \underline{f}(\beta, k) \to k < n)$,
(Φ8) $(\forall\xi < \eta < \alpha)(\forall n \in N)(\exists m \in N)(\forall\beta \in [\eta, \alpha))$
$(e(\beta, \alpha) \geq m \to (\exists k > n, k \in N)(\underline{f}(\beta, k) = \xi \& \underline{f}(\alpha, k) = \eta)$.

Hence, (Φ6) is (A1), (Φ7) is (A2), and (Φ8) is (A3). In the latter two we are using the fact that, for a subset X of ω, "being bounded" is equivalent to "being finite"—it is essential here that we have an ω-model.

We shall now prove that an ω-model for Φ is a good enough approximation to a nontrivial coherent family supported by an ω_1-tower A_α. Notice that every nontrivial coherent family supported by an ω_1-tower A_α, together with any function e with the described properties, defines a model for Φ. The other direction is a little bit less trivial. We are using the same symbols for the elements of our language and for their interpretations. Suppose that we have a standard ω-model $\mathfrak{M} \vDash \Phi$ of the size $\aleph_1$. It need not be well-founded, but the cofinality of the set M must be ω_1 as a consequence of (Φ2). Hence, we can choose a cofinal increasing sequence $D = \{a_\alpha : \alpha < \omega_1\}$ of its elements so that the first ω members are $\underline{0}, \underline{1}, \dots$. Now we have some problems with the function $\underline{f}$, whose range is not a subset of D, even if $\underline{f}$ is restricted to $D \times N$. This is why we define

$$(*) \qquad f_\alpha(k) = \begin{cases} \beta, & \text{if } \underline{f}(a_\alpha, \underline{k}) = a_\beta \in D, \\ 0, & \text{if } \underline{f}(a_\alpha, \underline{k}) \notin D, \end{cases}$$
$$\text{and } A_\alpha = \operatorname{supp} f_\alpha = \{k : f_\alpha(k) \neq 0\}.$$

So we are shrinking A_α's and "forgetting" about those subtowers whose indexes fall outside of D. By (Φ5) the family A_α is increasing with respect to $\subseteq^*$ (and this suffices—see the paragraph before Definition 4). We have to check that f_α, A_α satisfy Definition 3 with the interval $[1, \omega_1)$ in place of ω_1 as the range space:

(A1) follows from (Φ6) and (*).

(A2) By (*), $f_\alpha(k) \neq f_\beta(k)$ implies $\underline{f}(a_\alpha, k) \neq \underline{f}(a_\beta, \underline{k}) \neq \underline{f}(a_\beta, \underline{k})$ so (A2) follows from (Φ7).

(A3) Note first that by (*) $n \in f_\alpha^{-1}(\{\xi\})$ iff $\underline{n} \in \underline{f}_{a_\alpha}^{-1}(\{a_\xi\})$ for all $\xi < \alpha < \omega_1$ and all $n < \omega$. Pick $0 < \xi < \eta < \alpha < \omega_1$ and $n < \omega$. If $\beta \in [\eta, \alpha)$ and $f_\beta^{-1}(\{\xi\}) \cap f_\alpha^{-1}(\{\eta\}) \subseteq n$, then we also have $\underline{f}_{a_\beta}^{-1}(\{a_\xi\}) \cap \underline{f}_{a_\alpha}^{-1}(\{a_\eta\}) \subseteq \underline{n}$, and by ($\Phi$8) there are only finitely many such β's.

This finishes the proof. □

By putting together results in this section with those in previous sections, we get our main theorem:

Theorem 3. *There is a nontrivial coherent family of functions* $f_\alpha : A_\alpha \to \omega_1$ *supported by an* ω_1*-tower* A_α.

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