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CONSTANT-TO-ONE EXTENSIONS OF SHIFTS OF FINITE TYPE

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ABSTRACT. Any transitive shift of finite type has a transitive constant-to-one extension which is not of finite type.

Searching for an analog of Rudolph's result in ergodic theory ("any weakly mixing constant-to-1 extension of a Bernoulli system is also Bernoulli" [R2]), F. Blanchard posed the question of whether any transitive constant-to-1 extension of a shift of finite type is also of finite type [B]. So far, such extensions have been shown to be coded [B], and to be of finite type if one assumes the extension to be sofic [BH]. We answer the original question and show that any shift of finite type has a transitive constant-to-1 extension which is not of finite type (and thus not even sofic).

We introduce some notation and recall some definitions. Let A be a finite set. A *subshift* S is a closed shift invariant subset of the compact set $A^{\mathbf{Z}}$ together with the (left) shift map restricted to S. For a point $s = (s_i)_{i\in\mathbf{Z}} \in S$ let $s[n,m]$ denote the subblock $s_n s_{n+1} \cdots s_m$ of s ($-\infty \le n \le m \le \infty$). An S-*block* is a finite subblock of some point in S. The set of elements in A which are S-blocks is the *symbol set* of S. A subshift S is *transitive* if for any pair of S-blocks, say a and c, there is an S-block abc connecting them. A *shift of finite type* is a subshift S which is defined by excluding a finite set of blocks, say B, that is, a point $s \in A^{\mathbf{Z}}$ belongs to S if and only if no subblock of s is an element of B.

Let S and T be subshifts. A *factor map* $\pi : S \to T$ is a shift commuting continuous map from S onto T. The subshift S is then said to be an *extension* of T. The map π is *constant-to-*1 if all fibers $\pi^{-1}(t)$, $t \in T$, have the same finite cardinality.

Theorem. *Let T be an infinite transitive shift of finite type. Let $d \ge 2$ be an integer. Then there is a transitive subshift S which is not a shift of finite type and a factor map $\pi : S \to T$ such that all fibers have cardinality d.*

Proof. We need some preparations. By recoding we may assume that T is given by a finite excluding block system where all blocks have length 2 [DGS]. Then a block is a T-block if and only if all its subblocks of length 2 are T-blocks. Let Σ be the symbol set of T. Fix a shortest T-block $b_0 \cdots b_n$ with $n \ge 1$ and $b_0 = b_n$. Since T is not finite, there is some i, $1 \le i \le n$, and a symbol $\sigma \in \Sigma$ with $\sigma \ne b_{i-1}$ and for which σb_i is a T-block. We may assume that $i = 1$ (if necessary replace $b_0 \cdots b_n$ by $b_{i-1} \cdots b_n b_1 \cdots b_i$). Fix a shortest T-block $a_0 \cdots a_k$ with $k \ge 1$, $a_0 = b_n$

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and $a_k = \sigma$. Let $A = a_1 \cdots a_k$ and $B = b_1 b_2 \cdots b_n$. A and B satisfy the following properties:

–Any finite concatenation of the blocks AB and B is a T-block.

–σ does not occur in B and b_n does not occur in A, thus A and B cannot overlap.

–Let $t \in T$. If for some $-|B| < i \leq 0$ it holds that $t[i, i + |B| - 1] = B$, then the index i is unique. The analogue holds for the block A.

Let Σ' be a copy of Σ and fix a bijection $f : \Sigma \to \Sigma'$. We extend f to a map on blocks with all symbols in Σ, which we call again f, by $f(u_1 \cdots u_n) := f(u_1) \cdots f(u_n)$.

Let C, D, E be T-blocks, $t \in T$ and $n \in \mathbf{Z}$. We say t_n *lies in the block* D if for some $n - |D| < i \leq n$ it holds that $t[i, i + |D| - 1] = D$. We say t_n *lies in* D *within the context* (C, E) if there is some $n - |D| < i \leq n$ such that $t(-\infty, i - 1]$ ends with C and $t[i, \infty)$ begins with DE. Thus, if t_n lies in D within the context (C, E), then t_n lies in CDE.

Observe that the notion "t_n lies in D" does not only mean that the nth symbol of t is a subblock of D but also that D is a subblock of $t[n - |D| + 1, n + |D| - 1]$. The same applies to the notion of "t_n lies in D within the context (C, E)".

For each $k \geq 1$ let $L(k) := AB^k$ and $R(k) = B^k A$. We define a shift commuting (non-continuous) map $\Phi : T \to (\Sigma' \cup \Sigma)^{\mathbf{Z}}$ by defining $\Phi(t)_0$ for each $t \in T$. Let $t \in T$.

Case 1: t_0 lies in B. If t_0 lies in B^n within the context $(L(n)L(n-1) \cdots L(1)A, AR(1)R(2) \cdots R(n))$ for some $n \geq 100$, then let $\Phi(t)_0 = f(t_0)$ and let $\Phi(t)_0 = t_0$ otherwise.

Case 2: If t_0 lies in A and, for some $n \geq 100$, t_0 lies in $L(n-2)L(n-3) \cdots L(2)L(1)A$ within the context $(L(n)L(n-1), B^n)$, then let $\Phi(t)_0 = t_0$.

Case 3: If t_0 lies in A and, for some $n \geq 100, t_0$ lies in $AR(1)R(2) \cdots R(n-3)R(n-2)$ within the context $(B^n, R(n-1)R(n))$, then let $\Phi(t)_0 = t_0$.

Case 4: None of the above cases apply. Then let $\Phi(t)_0 = f(t_0)$.

By the non-overlapping properties of the blocks A and B the map Φ is well defined.

Fix $d-1$ copies of Σ, say $\Sigma^{(i)}$, $1 \leq i < d$, and let $\Sigma^{(0)} = \Sigma$. For each $1 \leq i < d$ fix a bijection from Σ to $\Sigma^{(i)}$, and denote the image of $u \in \Sigma$ by $u^{(i)}$. For $u \in \Sigma$ we also write $u^{(0)}$. We consider the symbols $f(u) \in \Sigma'$ $(u \in \Sigma)$ appearing in points in the image of Φ as variables where $f(u)$ can be replaced by any of the $u^{(i)}$, $0 \leq i < d$, with the convention that $u^{(0)} = u$. We do these replacements according to the following rules. (Recall that $\sigma \in \Sigma$ is the last symbol in the block A and that σ does not occur in B.)

Let xwy be a block appearing in some point of $\Phi(T)$, where $x, y \in \Sigma'$ and w is either the empty block or consists only of symbols in Σ. Then $x = f(u)$ and $y = f(v)$ for unique $u, v \in \Sigma$. The replacement $u^{(i)} w v^{(j)}$ is allowed if and only if $(u \neq \sigma$ and $i = j)$ or $(u = \sigma$ and $j = i + 1 \bmod d)$.

Now we replace all those $f(u) \in \Sigma'$ in points of $\Phi(T)$ according to the described rule. Observe that we get from each point $y \in \Phi(T)$, which sees a symbol from Σ', exactly d points in $(\Sigma^{(0)} \cup \cdots \cup \Sigma^{(d-1)})^{\mathbf{Z}}$. A point $y \in \Phi(T)$, which does not see a symbol from Σ', will be fixed by the replacements. Thus we have assigned to each point $t \in T$ a set $\widehat{\Phi(t)} \subset (\Sigma^{(0)} \cup \cdots \cup \Sigma^{(d-1)})^{\mathbf{Z}}$ of cardinality either d or 1. Let

$$\widehat{\Phi(T)} = \{x \in (\Sigma^{(0)} \cup \cdots \cup \Sigma^{(d-1)})^{\mathbf{Z}} | x \in \widehat{\Phi(t)} \text{ for some } t \in T\}.$$

Let S be the closure of $\widehat{\Phi(T)}$ in $(\Sigma^{(0)} \cup \cdots \cup \Sigma^{(d-1)})^{\mathbf{Z}}$. Let $\pi : S \to T$ be the 1-block map which erases the superscripts, i.e. $\pi(s)_0 = u$ if $s_0 = u^{(i)} \in \Sigma^{(i)}$ for some i (recall that $u^{(0)} = u$).

We will see that $\Phi : T \to (\Sigma' \cup \Sigma)^{\mathbf{Z}}$ is almost continuous (Claim $1 + 2$), π is a factor map from S onto T (Claim 3), π is constantly d-to-1 (Claim 5), and S is transitive (Claim 6) and not a shift of finite type (Claim 7). Recall that $L(k) = AB^k$ and $R(k) = B^kA$, $k \geq 1$. □

Claim 1. *Let $t \in T$ such that t_0 lies in AB^kA for some $k \geq 1$. Then $\Phi(t)_0$ is determined by $t[-n,n]$ for some $n = n(k)$.*

Proof. If t_0 lies in B^k, then Case 1 applies and $\Phi(t)_0$ is determined by $t[-n,n]$, for $n = |L(k)\cdots L(2)L(1)A| + |B^k|$. If t_0 lies in the first A, then Case 2 applies if t_0 lies in $L(K-2)L(K-3)\cdots L(2)L(1)A$ within the context $(L(K)L(K-1), B^K)$, where $K = \max(100, k+2)$ and Case 3 applies if t_0 lies in $AR(1)R(2)\cdots R(K-3)R(K-2)$ within the context $(B^K, R(K-1)R(K))$, where $K = \max(100, k+1)$. If neither Case 2 nor Case 3, then Case 4 applies. Thus $\Phi(t)_0$ is determined by $t[-n,n]$, for $n = |B^KAR(1)R(2)\cdots R(K-1)R(K)|$, where $K = \max(100, k+2)$. The case that t_0 lies in the second A is symmetric. □

We specify four T-orbits. Let B^∞ denote the periodic T-orbit obtained from bi-infinite concatenations of B. Let $B^\infty AB^\infty$ denote the orbit of the point t with $t(-\infty,-1] = \cdots BBB$ and $t[0,\infty) = ABBB\cdots$. Let $B^\infty AR(1)R(2)R(3)\cdots$ denote the orbit of t with $t(-\infty,-1] = \cdots BBB$ and $t[0,\infty) = AR(1)R(2)R(3)\cdots$, similarly $\cdots L(3)L(2)L(1)AB^\infty$.

Claim 2. *Let*

$$D = T - \{B^\infty, B^\infty AB^\infty, B^\infty AR(1)R(2)R(3)\cdots, \cdots L(3)L(2)L(1)AB^\infty\}.$$

Then for all $t \in D$ and all $k \geq 0$ there is some $n = n(t,k) \geq 0$ such that for all $t' \in T$ with $t'[-n,n] = t[-n,n]$ we have $\Phi(t')[-k,k] = \Phi(t)[-k,k]$.

Proof. Since D is shift invariant it suffices to prove the claim for $k = 0$. Let R be the shift of finite type in T in which all points are bi-infinite concatenations of the blocks AB and B. Then $T - R$ is a subset of D. Let $t \in T - R$. Then there is $m \geq 0$ such that $t[-m,m]$ is not an R-block. Thus, if t_0 satisfies Case 1, Case 2 or Case 3, then for some n with $n < m$. Therefore $\Phi(t)_0$ is determined by $t[-n,n]$ for large enough n. Now let $t \in R \cap D$. It remains to consider the case where Claim 1 does not apply. Then if t_0 lies in A we have that t lies in $B^\infty AB^\infty$, a contradiction. Thus t_0 lies in B. Since Claim 1 does not apply, t_n lies in B for all $n \leq 0$ or for all $n \geq 0$. Assume that t_n lies in B for all $n \leq 0$ (the other case is similar). Let p be the least integer for which t_p lies in A. Since t is not in the orbit $B^\infty AR(1)R(2)R(3)\cdots$, there is some N such that t_p does not lie in A within the context $(B^N, R(1)R(2)\cdots R(N))$. Then, $\Phi(t)_0$ is determined by $t[-L, p+L]$, where $L = |AR(1)R(2)\cdots R(N)|$. □

Claim 3. *Let $s \in \widehat{\Phi(t)}$ for some $t \in T$. Then $\pi(s) = t$. Thus, since $\widehat{\Phi(T)}$ is dense in S and π is continuous, $\pi : S \to T$ is an onto factor map.*

Proof. s is obtained from $\Phi(t)$ by replacing the symbols $f(u) \in \Sigma'$ by some of the $u^{(i)}$ and leaving symbols from Σ fixed. $s_n \in \{u^{(i)}, 0 \leq i < d\}$ only if $t_n = u$. Thus $\pi(s) = t$. □

Claim 4. *Let $t \in D$. Then $\Phi(t)$ sees a symbol from Σ'.*

Proof. If t never sees the block $L(100)L(99)\cdots L(2)L(1)AB^{100}$ and never sees the block $B^{100}AR(1)R(2)\cdots R(99)R(100)$, then $\Phi(t)_n = f(t_n) \in \Sigma'$ for some n. In fact, since t does not lie in the orbit B^∞ and the block B cannot overlap itself, there is some n such that t_n does not lie in B. Therefore, Case 4 applies for t_n. Now assume that, for some n, t_n lies in $L(99)\cdots L(2)L(1)A$ within the context $(L(100), B^{100})$ (the other case is similar). Since t is not in the orbit $\cdots L(3)L(2)L(1)AB^\infty$, there is a maximal $k \geq 100$ such that t_n lies in $L(99)\cdots L(2)L(1)A$ within the context $(L(k)L(k-1)\cdots L(100), B^k)$. Let $m \leq n$ be maximal such that t_m lies in A within the context $(L(k), B^{k-1}A)$. By the maximality of k, Case 4 applies for t_m and thus $\Phi(t)_m = f(t_m) \in \Sigma'$. □

Claim 5. *$\pi : S \to T$ is constantly d-to-1.*

Proof. First we show that any point in T has at most d preimages under π. Assume there is a point $t \in T$ with at least $d+1$ preimages, say $s^{(1)}, s^{(2)}, \ldots, s^{(d+1)}$. Let $k \geq 0$ be so large that $s^{(i)}[-k,k]$, $1 \leq i \leq d+1$, are pairwise distinct blocks. Then for $n \geq 0$ and $1 \leq i \leq d+1$ there is $t^{(i)} = t^{(i,n)} \in T$ such that there is $y^{(i)} = y^{(i,n)} \in \widehat{\Phi(t^{(i)})}$ with $y^{(i)}[-n,n] = s^{(i)}[-n,n]$. By Claim 3 it holds that $\pi(y^{(i)}) = t^{(i)}$ and thus $t^{(i)}[-n,n] = \pi(s^{(i)})[-n,n] = t[-n,n]$. Now we consider different possibilities for t and choose in each case a suitable n to get a contradiction.

Assume first that $t \in D$. Choose $n = n(t,k)$ from Claim 2. Then

$$\Phi(t^{(1)})[-k,k] = \Phi(t^{(2)})[-k,k] = \cdots = \Phi(t^{(d+1)})[-k,k].$$

Thus, by the replacement rules, $\operatorname{card}\{y^{(i)}[-k,k] | 1 \leq i \leq d+1\} \leq d$, contradicting the distinctness of $s^{(i)}[-k,k], 1 \leq i \leq d+1$.

If t lies in the orbit B^∞, then if s and s' are preimages of t and for some $0 \leq i < d$ we have $s_0 \in \Sigma^{(i)}$ and $s'_0 \in \Sigma^{(i)}$, then $s_0 = s'_0$. Thus, for some $s = s^{(m)}$ it holds that $s[p,p+1] = u^{(i)}v^{(j)}$ for some p, and some $0 \leq i,j < d$ with $i \neq j$. Let $n = |p+1| + |B|$ and $t' = t^{(m)}$. Then Case 1 coding applies to t'_p and t'_{p+1}; thus $\Phi(t')[p,p+1] = f(u)f(v)$ or $\Phi(t')[p,p+1] = uv$. Since $s[p,p+1] = u^{(i)}v^{(j)}$ with $i \neq j$, we have $\Phi(t')[p,p+1] = f(u)f(v)$. Thus by the replacement rules, $u = \sigma$, a contradiction, since u is a symbol of B.

If $t(-\infty,-1] = B^\infty$ and $t[0,\infty) = AB^\infty$, let $n = k + |ABB|$. Then $\Phi(t^{(i)})[-k,k] = t[-k,k]$ or $\Phi(t^{(i)})[-k,k] = t[-k,-1]f(A)t[|A|,k]$, for each $1 \leq i \leq d+1$. Thus, since A sees σ only as its last symbol, $\operatorname{card}\{y^{(i)}[-k,k] | 1 \leq i \leq d+1\} \leq d$, a contradiction.

If $t(-\infty,-1] = B^\infty$ and $t[0,\infty) = AR(1)R(2)\cdots$, w.l.o.g.

$$k = |AR(1)R(2)\cdots R(m)| - 1 \quad \text{for some } m \geq 2.$$

Let $n = k + n(1) + \cdots + n(m)$, where the $n(i)$ are from Claim 1. Then $\Phi(t^{(1)})[0,k] = \Phi(t^{(2)})[0,k] = \cdots = \Phi(t^{(d+1)})[0,k] = \Phi(t)[0,k] = t[0,k]$. Since B does not see the symbol σ, the replacement rules imply $\operatorname{card}\{y^{(i)}[-k,k] | 1 \leq i \leq d+1\} \leq d$, a contradiction.

The case that t is in the orbit $\cdots L(2)L(1)AB^\infty$ is symmetric to the last case. This proves that π is at most d-to-1. We now show that any point has at least d preimages.

First let $t \in D$. By Claim 4 $\Phi(t)$ sees a symbol from Σ'. Thus $\widehat{\Phi(t)}$ has cardinality d. All points of $\widehat{\Phi(t)}$ belong to S by definition.

If t lies in the orbit $B^\infty AB^\infty$, then $\Phi(t)$ lies in the orbit $B^\infty f(A)B^\infty$ (Case 1 and Case 4). By the replacement rules t has d preimages.

For every $n \geq 100$ there is a point $u \in T$ such that u_0 lies in B^n within the context $(L(n)L(n-1)\cdots L(2)L(1)A, AR(1)R(2)\cdots R(n-1)R(n))$. Applying Cases 1–3 shows that $\Phi(u)_0$ lies in $f(B)^n$ within the context $(L(n-2)L(n-3)\cdots L(2)L(1)A, AR(1)R(2)\cdots R(n-3)R(n-2))$. Since B does not see the symbol σ, the replacement rule shows that for each $0 \leq i < d$ there is an S-block $L(n-2)L(n-3)\cdots L(2)L(1)A(B^{(i)})^n AR(1)R(2)\cdots R(n-3)R(n-2)$, where $B^{(i)}$ has all symbols in $\Sigma^{(i)}$ and is mapped to B. By compactness of S this proves that any point in one of the orbits B^∞, $B^\infty AR(1)R(2)\cdots$ and $\cdots L(2)L(1)AB^\infty$ has at least d preimages. □

Claim 6. *S is transitive.*

Proof. The last argument used in the proof of Claim 5 actually proved that points in the orbits B^∞, $B^\infty AR(1)R(2)\cdots$ and $\cdots L(2)L(1)AB^\infty$ can be approximated arbitrarily well by points from $\{x \in (\Sigma^{(0)} \cup \cdots \cup \Sigma^{(d-1)})^{\mathbf{Z}} | x \in \widehat{\Phi(t)}$ for some $t \in D\}$. An analogous argument shows that there are S-points which are mapped to the orbits $(B^nA)^\infty$, $n \geq 1$, accumulating at $B^\infty AB^\infty$. Thus S is the closure of $\{x \in (\Sigma^{(0)} \cup \cdots \cup \Sigma^{(d-1)})^{\mathbf{Z}} | x \in \widehat{\Phi(t)}$ for some $t \in D\}$. Now let a and b be two S-blocks. Choose $t \in D$ such that there is $y \in \widehat{\Phi(t)}$ with $y[1, |a|] = a$. By Claim 2 we can choose p so large that $t' \in T$, $t[-p,p] = t'[-p,p]$ implies $\Phi(t)[1,|a|] = \Phi(t')[1,|a|]$. Now choose a point $u \in D$ such that $u[-p,p] = t[-p,p]$, $u[N,\infty) = ABABAB\cdots$ and $u(-\infty,-M] = \cdots ABABAB$ for some $M, N > 0$. By Claim 2 we can choose $m, n \geq 0$ so large that $u' \in T, u'[-m,n] = u[-m,n]$, then $\Phi(u')[-M,N] = \Phi(u)[-M,N]$. We may assume that $u[-m,n]$ begins with A and ends with B. Let $u^{(a)} = u$, $m^{(a)} = m$ and $n^{(a)} = n$. Do the same for the block b to obtain $u^{(b)}, m^{(b)}$ and $n^{(b)}$. Then or every p, there is a point $x^{(p)} \in T$ with $x^{(p)}(-\infty,-1] = u^{(a)}(-\infty, n^{(a)}]$ and $x[0,\infty) = (AB)^p u^{(b)}[-m^{(b)},\infty)$. We have by the choice of $u^{(a)}, u^{(b)}, n^{(a)}$ and $m^{(b)}$ that $\Phi(x^{(p)})(-\infty,-1] = \Phi(u^{(a)})(-\infty, n^{(a)}]$ and $\Phi(x^{(p)})[0,\infty) = (f(A)B)^p \Phi(u^{(b)})[-m^{(b)},\infty)$. By the definition of $u^{(a)}$ and $u^{(b)}$ and since A sees the symbol σ exactly once, there is a p such that one of the points in $\widehat{\Phi(x^{(p)})}$ sees first the block a and after a while the block b. □

Claim 7. *S is not a shift of finite type.*

Proof. Let $y^{(1)}$ be the unique preimage of B^∞ with symbols in $\Sigma^{(1)}$, and let $\pi(y^{(1)})[0, |B|-1] = B$. Let $B^{(1)} = y^{(1)}[0, |B|-1]$. If S were a shift of finite type, then there would be an orbit of the form $(B^{(1)})^\infty w (B^{(1)})^\infty$ which is not the orbit of $(B^{(1)})^\infty$. Choose s from this orbit with $s(-\infty,-1] = (B^{(1)})^\infty$ and with the property that $s[0,\infty)$ does not begin with $B^{(1)}$. Choose $n \geq 2$ so large that $|AR(1)R(2)\cdots R(n-1)| \geq |w|$. Then choose m so large that $m|B| \geq |AR(1)R(2)\cdots R(n)|$. There is some $t \in T$ such that $\Phi(t)[-n|B|, |w| + m|B|] = f(B)^n u f(B)^m$, where $|u| = |w|$. Thus, $t[0,\infty)$ begins with $AR(1)R(2)\cdots R(n)$ and $t[|w|,\infty)$ begins with B^m. By the choice of n and m this implies that A is a subblock of B^m, a contradiction. □

Remark. S is a synchronized system but does not have specification. We think that non-synchronized constant-to-1 extensions can be obtained by a more elaborate variant of the above construction.

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