MEAN THEORETIC APPROACH
TO THE GRAND FURUTA INEQUALITY

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Dedicated to Professor Tsuyoshi Ando, the originator of the theory of operator means, on his retirement from Hokkaido University

Abstract. Very recently, Furuta obtained the grand Furuta inequality which is a parametric formula interpolating the Furuta inequality and the Ando-Hiai inequality as follows: If $A \geq B \geq 0$ and $A$ is invertible, then for each $t \in [0,1],$

$$F_{p,t}(A, B, r, s) = A^{-r/2}(A^{-1/2}B A^{-1/2})^r A^{-r/2}$$

is a decreasing function of both $r$ and $s$ for all $r \geq t$, $p \geq 1$ and $s \geq 1$. In this note, we employ a mean theoretic approach to the grand Furuta inequality. Consequently we propose a basic inequality, by which we present a simple proof of the grand Furuta inequality.

1. Introduction

Throughout this note, we consider bounded linear operators acting on a Hilbert space, simply operators. An operator $A$ is positive if $(A x, x) \geq 0$ for all $x \in H$. The L"owner-Heinz inequality says that the function $t \rightarrow t^\alpha$ on $[0,\infty)$ operator monotone for $0 \leq \alpha \leq 1$, i.e.,

$$A \geq B \geq 0 \quad \text{implies} \quad A^\alpha \geq B^\alpha$$

(cf. [13] and [15]). Furuta [7] gave it an ingenious extension which is called the Furuta inequality: If $A \geq B \geq 0$, then

$$(A^r B A^r)^{1/q} \geq (A^p B A^p)^{1/q}$$

holds for $r \geq 0$, $p \geq 0$ and $q \geq 1$ with $(1 + 2r)q \geq p + 2r$; see [8] for an elementary proof.

Recently Ando and Hiai [2] discussed the log-majorization for positive operators and obtained the following fundamental inequality [2, Theorem 3.5], which is equivalent to their main log-majorization theorem [2, Theorem 2.1]. If $A \geq B \geq 0$ and $A$ is invertible, then the following holds for $p, r \geq 1:$$$

A^r \geq \{A^{r/2}(A^{-1/2}B A^{-1/2})^r A^{-r/2}\}^{1/p}.$$
Very recently, Furuta [12] obtained a parameteric formula interpolating the Furuta inequality (2) and the Ando-Hiai inequality (3) in the following manner:

**The grand Furuta inequality** ([12]). If \( A \geq B \geq 0 \) and \( A \) is invertible, then for each \( t \in [0, 1] \),

\[
F_{p,t}(A, B, r, s) = A^{-r/2} \left\{ A^{r/2} (A^{-t/2} B^p A^{-t/2})^s A^{r/2} \right\}^{ \frac{1-t}{p-t+s-r}} A^{-r/2}
\]

is a decreasing function of both \( r \) and \( s \) for all \( r \geq t \), \( p \geq 1 \) and \( s \geq 1 \).

In particular, the inequality

\[
A^{1-t} = F_{p,t}(A, A, r, s) \geq F_{p,t}(A, B, r, s)
\]

holds for \( r \geq t \), \( p \geq 1 \) and \( s \geq 1 \).

As a matter of fact, (2) and (3) appear in the grand Furuta inequality as the extremal cases \( t = 0 \) and \( t = 1 \) with \( r = s \) respectively. Therefore we call it the grand Furuta inequality. We note that the original proof in [12] is quite elementary but somewhat technical.

In this note, we employ a mean theoretic approach to the grand Furuta inequality as has been done for the Furuta inequality (see [3], [4], [5], [6], [9], [14], [15]). Thus we propose a basic inequality, by which we present a simple proof of the grand Furuta inequality.

### 2. Means of operators

The theory of operator means was established by Kubo and Ando [16], whose heart is the correspondence between operator monotone functions \( f \) and means \( m \) given by

\[
f(t) = 1 m t \quad (t > 0).
\]

In connection with the Löwner-Heinz inequality (1), they exhibit means \( \sharp_s \) for \( 0 \leq s \leq 1 \) such that \( 1 \sharp_s t = t^s \) \((t > 0)\), more precisely

\[
A \sharp_s B = A^{1/2}(A^{-1/2} B A^{-1/2})^s A^{1/2}
\]

for positive invertible operators \( A \) and \( B \).

In [10], Furuta proved the monotonicity of the function

\[
F(p) = (B^p A^p B^p)^{ \frac{r+2p}{p+2r} } \quad \text{for} \quad p \geq 1
\]

associated with his inequality under the assumption \( A \geq B \geq 0 \) and \( r \geq 0 \). Noting that the monotonicity of \( F(p) \) can be rephrased in terms of the function

\[
M(p, r) = B^{-2r} \sharp_{ \frac{r+2p}{p+2r} } A^p,
\]

we showed that \( M(p, r) \) is an increasing function of both \( p \) and \( r \) for all \( p \geq 1 \) and \( r \geq 0 \) [4, Theorem 1]; see Lemma 5 below. Moreover, we pointed out that its modification includes Ando’s theorem on the geometric mean in [1].

We now rewrite (4) in a mean theoretic form, using the same technique that we used to rewrite (7) as (8). For the sake of convenience, we define \( \sharp_s \) for \( s \in \mathbb{R} \) as in [12] as an extension of \( \sharp_s \) for \( 0 \leq s \leq 1 \),

\[
A \sharp_s B = A^{1/2}(A^{-1/2} B A^{-1/2})^s A^{1/2}
\]

(6')
for positive invertible operators $A$ and $B$. For given $A \geq B \geq 0$, $p \geq 1$ and $t \in [0, 1]$, we let

$$F(r, s) = A^{-r+t} \left[ \frac{t}{t-r} \right]^{s-1} (A^t \sharp_s B^p)$$

for $r \geq t$ and $s \geq 1$. It is easily checked that $F(r, s) = A^{1/2} F_{p,t}(A, B, r, s) A^{1/2}$ and so the monotonicity of $F_{p,t}(A, B, r, s)$ in (4) is equivalent to that of $F(r, s)$. Similarly, (5) can be rewritten in the form

$$F(r, s) \leq A \quad \text{for} \quad r \geq t \quad \text{and} \quad s \geq 1$$

if $A \geq B \geq 0$ and $A$ is invertible.

Concluding this section, we note that $\sharp_s$ is multiplicative in the sense that

$$A \sharp_s B = A \sharp_s (A \sharp_s B)$$

for all $a$ and $b$.

3. The grand Furuta inequality

We begin by stating two simple lemmas by Furuta in [10] and [12]. For the sake of convenience, we give them short proofs.

**Lemma 0** ([10]). For positive operators $A$ and $B$,

$$(ABA)^s = AB^{1/2} (B^{1/2} A^2 B^{1/2})^{s-1} B^{1/2} A$$

holds for $s \geq 1$.

**Proof.** Let $AB^{1/2} = UH$ be the polar decomposition of $AB^{1/2}$. Then we have for $s \geq 0$

$$(ABA)^{1+s} = (UH^2 U^*)^{1+s} = UH^{2+2s} U^*$$

$$= UHH^{2s} HU^* = AB^{1/2} (B^{1/2} A^2 B^{1/2})^s B^{1/2} A.$$

**Lemma 1** ([12]). If $A \geq B \geq 0$ and $A$ is invertible, then

$$A^{t} \sharp_s B^p \leq B^{(p-t)s+t}$$

for $p \geq 1$, $1 \leq s \leq 2$ and $0 \leq t \leq 1$.

**Proof.** It follows from (1) that $A^{-t} \leq B^{-t}$ and moreover

$$A^{t} \sharp_s B^p = A^{t/2} (A^{-t/2} B^p A^{-t/2})^s A^{t/2}$$

$$= A^{t/2} A^{-t/2} B^{p/2} (B^{p/2} A^{-t} B^{p/2})^{s-1} B^{p/2} A^{-t/2} A^{t/2} \quad \text{by Lemma 0}$$

$$\leq B^{p/2} (B^{p/2} B^{-t} B^{p/2})^{s-1} B^{p/2}$$

$$= B^{(p-t)s+t}.$$

We use Lemma 1 to prove the following basic inequality which will work well in a proof of the grand Furuta inequality.

**Theorem 2.** If $A \geq B \geq 0$ and $A$ is invertible, then

$$(A^{t} \sharp_s B^p)^{1/(s(p-t)+t)} \leq B$$

for $p \geq 1$, $s \geq 1$ and $0 \leq t \leq 1$. 

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Proof. It suffices to show that
\[(A^t \zeta_{2s} B^p)^{1/(2^k s(p-t)+t)} \leq B\]
for \(1 \leq s \leq 2\) and \(k = 1, 2, \ldots\). Lemma 1 says that (12) holds for \(k = 0\). So we put 
\(p_1 = s(p-t) + t\), \(B_1 = (A^t \zeta_{s} B^p)^{1/p_1}\) and inductively
\[p_{k+1} = 2^k s(p-t) + t\quad \text{and} \quad B_{k+1} = (A^t \zeta_{2s} B^p)^{1/p_{k+1}}\]
for \(k = 1, 2, \ldots\). Then we have \(p_{k+1} = 2(p_k - t) + t\) and by (11)
\[B_{k+1} = (A^t \zeta_{2} B_{k}^p)^{1/p_{k+1}} = (A^t \zeta_{2k-1} B^p)^{1/p_{k+1}}\]
for \(k = 1, 2, \ldots\).

It therefore suffices to prove that \(B_{k+1} \leq B_k\). For a fixed \(k\), we can assume that \(B_k \leq A\) because \(B_1 \leq B \leq A\). Hence we apply Lemma 1 to \(B_k \leq A\), \(p = p_k\) and \(s = 2\). It implies that
\[(A^t \zeta_{2} B_k^p)^{1/(2^k p_k - t)} \leq B_k\]
Finally, since the left-hand side of this is just \(B_{k+1}\) by the above remark, we obtain the conclusion \(B_{k+1} \leq B_k\).

The second tool is a special case in the Furuta inequality (2).

**Lemma 3.** If \(A \geq B \geq 0\), then
\[A \geq (A^{b/2} B^{p} A^{b/2})^{1/(p+b)}\]
for \(p \geq 1\) and \(b \geq 0\).

**Proof.** We take \(r = b/2\) and \(q = p + b\) in (2). Since \(p, q\) and \(r\) satisfy the condition which ensures (2), we have the desired inequality.

Based on Theorem 2 and Lemma 3, we give a simple proof of the statement that 
\(F(r, s)\) is a decreasing function of \(r\). The proof we give is similar to that of [4, Theorem 1].

**Lemma 4.** If \(A \geq B \geq 0\), then \(F(r, s)\) is a decreasing function of \(r\) for all \(r \geq t\).

**Proof.** First of all, we put \(B_1 = (A^t \zeta_{s} B^p)^{1/(p-s+t)}\) as in Theorem 2 and \((s) = \frac{1-r+s}{(p-t)s+t}\) for given \(s \geq 1\), \(p\), \(r\) and \(t\). Since \(B_1 \leq B \leq A\) by Theorem 2, Lemma 3 implies that
\[D = (A^{(r-t+d)/2} B_1^{(p-t+s+t)} A^{(r-t+d)/2})^{1/(p-t)s+r+d} \leq A,\]
so that \(D^d \leq A^d\) by (1) for \(0 < d < 1\). Therefore we have
\[F(r, s) = A^{-r+t} \zeta_{(s)} B_1^{(p-t)s+t} A^{-t} \zeta_{(s)} B_1^{(p-t)s+t} A^{(r-d-t)/2} D^d B_1^{(p-t)s+t} A^{(r-d-t)/2} A^{(r-d-t)/2} A^{-t} \zeta_{(s)} B_1^{(p-t)s+t} A^{(r-d-t)/2} \]
\[= A^{-r-d-t/2} D^d B_1^{(p-t)s+t} A^{(r-d-t)/2} A^{(r-d-t)/2} A^{(r-d-t)/2} D^d B_1^{(p-t)s+t} A^{(r-d-t)/2} A^{-t} \zeta_{(s)} B_1^{(p-t)s+t} A^{(r-d-t)/2} \]
\[= A^{-r-d-t/2} D^d B_1^{(p-t)s+t} A^{(r-d-t)/2} A^{-t} \zeta_{(s)} B_1^{(p-t)s+t} A^{(r-d-t)/2} \]
so the proof is complete.
To prove that $F(r, s)$ is a decreasing function of $s$, we need the following result equivalent to [4, Theorem 3]; cf. [11].

**Lemma 5.** If $A \geq B \geq 0$ and $\gamma \geq 0$ is given, then the operator function
\[ f(\alpha, \beta) = A^{-\alpha} \varphi_{\frac{\alpha + \gamma}{\beta + \gamma}}^B \]
is a decreasing function of both $\alpha$ and $\beta$ for all $\alpha \geq 0$ and $\beta \geq \gamma$.

**Proof.** In [4, Theorem 3], we showed that if $C \geq D \geq 0$ and $\gamma \geq 0$ is given, then
\[ g(\alpha, \beta) = D^{-\alpha} \varphi_{\frac{\alpha + \gamma}{\beta + \gamma}}^C \]
is an increasing function of both $\alpha$ and $\beta$ for all $\alpha \geq 0$ and $\beta \geq \gamma$. Since $B^{-1} \geq A^{-1} \geq 0$, it implies that
\[ h(\alpha, \beta) = (A^{-1})^{-\alpha} \varphi_{\frac{\alpha + \gamma}{\beta + \gamma}}^{(B^{-1})^\beta} = A^\alpha \varphi_{\frac{\alpha + \gamma}{\beta + \gamma}}^{B^{-\beta}} \]
is also an increasing function of both $\alpha$ and $\beta$ for all $\alpha \geq 0$ and $\beta \geq \gamma$. Taking the inverse, we obtain that
\[ f(\alpha, \beta) = A^{-\alpha} \varphi_{\frac{\alpha + \gamma}{\beta + \gamma}}^B = (A^\alpha \varphi_{\frac{\alpha + \gamma}{\beta + \gamma}}^{B^{-\beta}})^{-1} = (h(\alpha, \beta))^{-1} \]
is a decreasing function of both $\alpha$ and $\beta$ for all $\alpha \geq 0$ and $\beta \geq \gamma$, as required.

**Lemma 6.** If $A \geq B \geq 0$, then $F(r, s)$ is a decreasing function of $s$ for all $s \geq 1$.

**Proof.** Since $B_1 \leq A$ by Theorem 2, we can apply Lemma 1 to $B_1$ and $A$. Namely we have, for $1 \leq s_1 \leq 2$
\[ A^t \varphi_{s_1} B_1^{p_1} \leq B_1^{(p_1-\ell)t} \]
for $p_1 \geq 1$. Taking $p_1 = (p - t)s + t \geq 1$ in particular, we have
\[ A^t \varphi_{s_1} B_1^{(p-t)s_1+t} \leq B_1^{(p-t)s_1+t}. \]
On the other hand, since the left-hand side above is of the form
\[ A^t \varphi_{s_1} B_1^{(p-t)s_1+t} = A^t \varphi_{s_1} (A^t \varphi_{s} B^p) = A^t \varphi_{ss_1} B^p, \]
it follows that
\[ A^t \varphi_{ss_1} B^p \leq B_1^{(p-t)s_1+t}. \]
Hence the monotonicity of operator means implies that
\[ A^{-r+t} \varphi_{\frac{(p-t)s_1+t}{(p-t)s_1+1+t}} (A^t \varphi_{ss_1} B^p) \leq A^{-r+t} \varphi_{\frac{(p-t)s_1+t}{(p-t)s_1+1+t}} B_1^{(p-t)s_1+t}. \]
Now we apply Lemma 5 to $A \geq B_1$ for $\alpha = r - t, \beta = (p - t)s_1 + t$ and $\gamma = (p - t)s + t$. Then $\gamma \leq \beta$ by $1 \leq s_1$. That is, we have
\[ A^{-r+t} \varphi_{\frac{(p-t)s_1+t}{(p-t)s_1+1+t}} B_1^{(p-t)s_1+t} \leq B_1^{(p-t)s_1+t} = A^t \varphi_{s} B^p. \]
Combining (13) with (14), we obtain that
\[ A^{-r+t} \varphi_{\frac{(p-t)s_1+t}{(p-t)s_1+1+t}} (A^t \varphi_{ss_1} B^p) \leq A^t \varphi_{s} B^p. \]
Finally it implies that
\[ F(r, ss_1) = A^{-r+t} (A_{ss_1}^{1-t} B^p) \]
\[ = A^{-r+t} (A^{1-t} B^p) \]
\[ \leq A^{-r+t} (A^{1} B^p) \]
\[ = F(r, s), \]
so the proof is complete.

Consequently the grand Furuta inequality is proved by Lemmas 4 and 6, in the form of (9).

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