

## MEAN THEORETIC APPROACH TO THE GRAND FURUTA INEQUALITY

MASATOSHI FUJII AND EIZABURO KAMEI

(Communicated by Palle E. T. Jorgensen)

*Dedicated to Professor Tsuyoshi Ando, the originator of the theory  
of operator means, on his retirement from Hokkaido University*

ABSTRACT. Very recently, Furuta obtained the grand Furuta inequality which is a parametric formula interpolating the Furuta inequality and the Ando-Hiai inequality as follows : If  $A \geq B \geq 0$  and  $A$  is invertible, then for each  $t \in [0, 1]$ ,

$$F_{p,t}(A, B, r, s) = A^{-r/2} \{A^{r/2} (A^{-t/2} B^p A^{-t/2})^s A^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}} A^{-r/2}$$

is a decreasing function of both  $r$  and  $s$  for all  $r \geq t$ ,  $p \geq 1$  and  $s \geq 1$ . In this note, we employ a mean theoretic approach to the grand Furuta inequality. Consequently we propose a basic inequality, by which we present a simple proof of the grand Furuta inequality.

### 1. INTRODUCTION

Throughout this note, we consider bounded linear operators acting on a Hilbert space, simply operators. An operator  $A$  is positive if  $(Ax, x) \geq 0$  for all  $x \in H$ . The Löwner-Heinz inequality says that the function  $t \rightarrow t^\alpha$  on  $[0, \infty)$  is operator monotone for  $0 \leq \alpha \leq 1$ , i.e.,

$$(1) \quad A \geq B \geq 0 \quad \text{implies} \quad A^\alpha \geq B^\alpha$$

(cf. [13] and [15]). Furuta [7] gave it an ingenious extension which is called the Furuta inequality : If  $A \geq B \geq 0$ , then

$$(2) \quad (A^r A^p A^r)^{1/q} \geq (A^r B^p A^r)^{1/q}$$

holds for  $r \geq 0$ ,  $p \geq 0$  and  $q \geq 1$  with  $(1 + 2r)q \geq p + 2r$ ; see [8] for an elementary proof.

Recently Ando and Hiai [2] discussed the log-majorization for positive operators and obtained the following fundamental inequality [2, Theorem 3.5], which is equivalent to their main log-majorization theorem [2, Theorem 2.1]. If  $A \geq B \geq 0$  and  $A$  is invertible, then the following holds for  $p, r \geq 1$ :

$$(3) \quad A^r \geq \{A^{r/2} (A^{-1/2} B^p A^{-1/2})^r A^{r/2}\}^{1/p}.$$

---

Received by the editors November 28, 1994 and, in revised form, March 6, 1995.

1991 *Mathematics Subject Classification.* Primary 47A63, 47B15.

*Key words and phrases.* Positive operators, Löwner-Heinz inequality, Furuta inequality, Ando-Hiai inequality, grand Furuta inequality.

Very recently, Furuta [12] obtained a parameteric formula interpolating the Furuta inequality (2) and the Ando-Hiai inequality (3) in the following manner:

**The grand Furuta inequality** ([12]). *If  $A \geq B \geq 0$  and  $A$  is invertible, then for each  $t \in [0, 1]$ ,*

$$(4) \quad F_{p,t}(A, B, r, s) = A^{-r/2} \{A^{r/2} (A^{-t/2} B^p A^{-t/2})^s A^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}} A^{-r/2}$$

*is a decreasing function of both  $r$  and  $s$  for all  $r \geq t$ ,  $p \geq 1$  and  $s \geq 1$ .*

*In particular, the inequality*

$$(5) \quad A^{1-t} = F_{p,t}(A, A, r, s) \geq F_{p,t}(A, B, r, s)$$

*holds for  $r \geq t$ ,  $p \geq 1$  and  $s \geq 1$ .*

As a matter of fact, (2) and (3) appear in the grand Furuta inequality as the extremal cases  $t = 0$  and  $t = 1$  with  $r = s$  respectively. Therefore we call it the grand Furuta inequality. We note that the original proof in [12] is quite elementary but somewhat technical.

In this note, we employ a mean theoretic approach to the grand Furuta inequality as has been done for the Furuta inequality (see [3], [4], [5], [6], [9], [14], [15]). Thus we propose a basic inequality, by which we present a simple proof of the grand Furuta inequality.

## 2. MEANS OF OPERATORS

The theory of operator means was established by Kubo and Ando [16], whose heart is the correspondence between operator monotone functions  $f$  and means  $m$  given by

$$f(t) = 1 \ m \ t \quad (t > 0).$$

In connection with the Löwner-Heinz inequality (1), they exhibit means  $\sharp_s$  for  $0 \leq s \leq 1$  such that  $1 \ \sharp_s \ t = t^s$  ( $t > 0$ ), more precisely

$$(6) \quad A \ \sharp_s \ B = A^{1/2} (A^{-1/2} B A^{-1/2})^s A^{1/2}$$

for positive invertible operators  $A$  and  $B$ .

In [10], Furuta proved the monotonicity of the function

$$(7) \quad F(p) = (B^r A^p B^r)^{\frac{1+2r}{p+2r}} \quad \text{for } p \geq 1$$

associated with his inequality under the assumption  $A \geq B \geq 0$  and  $r \geq 0$ . Noting that the monotonicity of  $F(p)$  can be rephrased in terms of the function

$$(8) \quad M(p, r) = B^{-2r} \ \sharp_{\frac{1+2r}{p+2r}} \ A^p,$$

we showed that  $M(p, r)$  is an increasing function of both  $p$  and  $r$  for all  $p \geq 1$  and  $r \geq 0$  [4, Theorem 1]; see Lemma 5 below. Moreover, we pointed out that its modification includes Ando's theorem on the geometric mean in [1].

We now rewrite (4) in a mean theoretic form, using the same technique that we used to rewrite (7) as (8). For the sake of convenience, we define  $\natural_s$  for  $s \in \mathbb{R}$  as in [12] as an extension of  $\sharp_s$  for  $0 \leq s \leq 1$ ,

$$(6') \quad A \ \natural_s \ B = A^{1/2} (A^{-1/2} B A^{-1/2})^s A^{1/2}$$

for positive invertible operators  $A$  and  $B$ . For given  $A \geq B \geq 0$ ,  $p \geq 1$  and  $t \in [0, 1]$ , we let

$$(9) \quad F(r, s) = A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p)$$

for  $r \geq t$  and  $s \geq 1$ . It is easily checked that  $F(r, s) = A^{t/2} F_{p,t}(A, B, r, s) A^{t/2}$  and so the monotonicity of  $F_{p,t}(A, B, r, s)$  in (4) is equivalent to that of  $F(r, s)$ . Similarly, (5) can be rewritten in the form

$$(10) \quad F(r, s) \leq A \quad \text{for } r \geq t \quad \text{and } s \geq 1$$

if  $A \geq B \geq 0$  and  $A$  is invertible.

Concluding this section, we note that  $\natural_s$  is multiplicative in the sense that

$$(11) \quad A \natural_{ab} B = A \natural_a (A \natural_b B)$$

for all  $a$  and  $b$ .

### 3. THE GRAND FURUTA INEQUALITY

We begin by stating two simple lemmas by Furuta in [10] and [12]. For the sake of convenience, we give them short proofs.

**Lemma 0** ([10]). *For positive operators  $A$  and  $B$ ,*

$$(ABA)^s = AB^{1/2}(B^{1/2}A^2B^{1/2})^{s-1}B^{1/2}A$$

*holds for  $s \geq 1$ .*

*Proof.* Let  $AB^{\frac{1}{2}} = UH$  be the polar decomposition of  $AB^{\frac{1}{2}}$ . Then we have for  $s \geq 0$

$$\begin{aligned} (ABA)^{1+s} &= (UH^2U^*)^{1+s} = UH^{2+2s}U^* \\ &= UHH^{2s}HU^* = AB^{1/2}(B^{1/2}A^2B^{1/2})^s B^{1/2}A. \end{aligned}$$

**Lemma 1** ([12]). *If  $A \geq B \geq 0$  and  $A$  is invertible, then*

$$A^t \natural_s B^p \leq B^{(p-t)s+t}$$

*for  $p \geq 1$ ,  $1 \leq s \leq 2$  and  $0 \leq t \leq 1$ .*

*Proof.* It follows from (1) that  $A^{-t} \leq B^{-t}$  and moreover

$$\begin{aligned} A^t \natural_s B^p &= A^{t/2}(A^{-t/2}B^pA^{-t/2})^s A^{t/2} \\ &= A^{t/2}A^{-t/2}B^{p/2}(B^{p/2}A^{-t}B^{p/2})^{s-1}B^{p/2}A^{-t/2}A^{t/2} \text{ by Lemma 0} \\ &\leq B^{p/2}(B^{p/2}B^{-t}B^{p/2})^{s-1}B^{p/2} \\ &= B^{(p-t)s+t}. \end{aligned}$$

We use Lemma 1 to prove the following basic inequality which will work well in a proof of the grand Furuta inequality.

**Theorem 2.** *If  $A \geq B \geq 0$  and  $A$  is invertible, then*

$$(A^t \natural_s B^p)^{1/(s(p-t)+t)} \leq B$$

*for  $p \geq 1$ ,  $s \geq 1$  and  $0 \leq t \leq 1$ .*

*Proof.* It suffices to show that

$$(12) \quad (A^t \natural_{2^k s} B^p)^{1/(2^k s(p-t)+t)} \leq B$$

for  $1 \leq s \leq 2$  and  $k = 1, 2, \dots$ . Lemma 1 says that (12) holds for  $k = 0$ . So we put  $p_1 = s(p-t) + t$ ,  $B_1 = (A^t \natural_s B^p)^{1/p_1}$  and inductively

$$p_{k+1} = 2^k s(p-t) + t \quad \text{and} \quad B_{k+1} = (A^t \natural_{2^k s} B^p)^{1/p_{k+1}}$$

for  $k = 1, 2, \dots$ . Then we have  $p_{k+1} = 2(p_k - t) + t$  and by (11)

$$B_{k+1} = (A^t \natural_2 (A^t \natural_{2^{k-1} s} B^p))^{1/p_{k+1}} = (A^t \natural_2 B_k^{p_k})^{1/p_{k+1}}$$

for  $k = 1, 2, \dots$ .

It therefore suffices to prove that  $B_{k+1} \leq B_k$ . For a fixed  $k$ , we can assume that  $B_k \leq A$  because  $B_1 \leq B \leq A$ . Hence we apply Lemma 1 to  $B_k \leq A$ ,  $p = p_k$  and  $s = 2$ . It implies that

$$(A^t \natural_2 B_k^{p_k})^{1/(2(p_k-t)+t)} \leq B_k.$$

Finally, since the left-hand side of this is just  $B_{k+1}$  by the above remark, we obtain the conclusion  $B_{k+1} \leq B_k$ .

The second tool is a special case in the Furuta inequality (2).

**Lemma 3.** *If  $A \geq B \geq 0$ , then*

$$A \geq (A^{b/2} B^p A^{b/2})^{1/(p+b)}$$

for  $p \geq 1$  and  $b \geq 0$ .

*Proof.* We take  $r = b/2$  and  $q = p + b$  in (2). Since  $p, q$  and  $r$  satisfy the condition which ensures (2), we have the desired inequality.

Based on Theorem 2 and Lemma 3, we give a simple proof of the statement that  $F(r, s)$  is a decreasing function of  $r$ . The proof we give is similar to that of [4, Theorem 1].

**Lemma 4.** *If  $A \geq B \geq 0$ , then  $F(r, s)$  is a decreasing function of  $r$  for all  $r \geq t$ .*

*Proof.* First of all, we put  $B_1 = (A^t \natural_s B^p)^{1/((p-t)s+t)}$  as in Theorem 2 and  $\langle s \rangle = \frac{1-t+r}{(p-t)s+r}$  for given  $s \geq 1$ ,  $p$ ,  $r$  and  $t$ . Since  $B_1 \leq B \leq A$  by Theorem 2, Lemma 3 implies that

$$D = (A^{(r-t+d)/2} B_1^{(p-t)s+t} A^{(r-t+d)/2})^{1/((p-t)s+r+d)} \leq A,$$

so that  $D^d \leq A^d$  by (1) for  $0 < d < 1$ . Therefore we have

$$\begin{aligned} F(r, s) &= A^{-r+t} \natural_{\langle s \rangle} B_1^{(p-t)s+t} \\ &= A^{-(r+d-t)/2} (A^d \natural_{\langle s \rangle} A^{(r+d-t)/2} B_1^{(p-t)s+t} A^{(r+d-t)/2}) A^{-(r+d-t)/2} \\ &\geq A^{-(r+d-t)/2} (D^d \natural_{\langle s \rangle} A^{(r+d-t)/2} B_1^{(p-t)s+t} A^{(r+d-t)/2}) A^{-(r+d-t)/2} \\ &= A^{-(r+d-t)/2} (D^d \natural_{\langle s \rangle} D^{(p-t)s+r+d}) A^{-(r+d-t)/2} \\ &= A^{-(r+d-t)/2} D^{1-t+r+d} A^{-(r+d-t)/2} \\ &= A^{-(r+d)+t} \natural_{\frac{1-t+r+d}{(p-t)s+r+d}} B_1^{(p-t)s+t} \\ &= F(r+d, s), \end{aligned}$$

so the proof is complete.

To prove that  $F(r, s)$  is a decreasing function of  $s$ , we need the following result equivalent to [4, Theorem 3]; cf. [11].

**Lemma 5.** *If  $A \geq B \geq 0$  and  $\gamma \geq 0$  is given, then the operator function*

$$f(\alpha, \beta) = A^{-\alpha} \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} B^\beta$$

*is a decreasing function of both  $\alpha$  and  $\beta$  for all  $\alpha \geq 0$  and  $\beta \geq \gamma$ .*

*Proof.* In [4, Theorem 3], we showed that if  $C \geq D \geq 0$  and  $\gamma \geq 0$  is given, then

$$g(\alpha, \beta) = D^{-\alpha} \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} C^\beta$$

is an increasing function of both  $\alpha$  and  $\beta$  for all  $\alpha \geq 0$  and  $\beta \geq \gamma$ . Since  $B^{-1} \geq A^{-1} \geq 0$ , it implies that

$$h(\alpha, \beta) = (A^{-1})^{-\alpha} \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} (B^{-1})^\beta = A^\alpha \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} B^{-\beta}$$

is also an increasing function of both  $\alpha$  and  $\beta$  for all  $\alpha \geq 0$  and  $\beta \geq \gamma$ . Taking the inverse, we obtain that

$$f(\alpha, \beta) = A^{-\alpha} \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} B^\beta = (A^\alpha \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} B^{-\beta})^{-1} = (h(\alpha, \beta))^{-1}$$

is a decreasing function of both  $\alpha$  and  $\beta$  for all  $\alpha \geq 0$  and  $\beta \geq \gamma$ , as required.

**Lemma 6.** *If  $A \geq B \geq 0$ , then  $F(r, s)$  is a decreasing function of  $s$  for all  $s \geq 1$ .*

*Proof.* Since  $B_1 \leq A$  by Theorem 2, we can apply Lemma 1 to  $B_1$  and  $A$ . Namely we have, for  $1 \leq s_1 \leq 2$

$$A^t \natural_{s_1} B_1^{p_1} \leq B_1^{(p_1-t)s_1+t}$$

for  $p_1 \geq 1$ . Taking  $p_1 = (p-t)s + t \geq 1$  in particular, we have

$$A^t \natural_{s_1} B_1^{(p-t)s+t} \leq B_1^{(p-t)ss_1+t}.$$

On the other hand, since the left-hand side above is of the form

$$A^t \natural_{s_1} B_1^{(p-t)s+t} = A^t \natural_{s_1} (A^t \natural_s B^p) = A^t \natural_{ss_1} B^p,$$

it follows that

$$A^t \natural_{ss_1} B^p \leq B_1^{(p-t)ss_1+t}.$$

Hence the monotonicity of operator means implies that

$$(13) \quad A^{-r+t} \sharp_{\frac{(p-t)s+r}{(p-t)ss_1+r}} (A^t \natural_{ss_1} B^p) \leq A^{-r+t} \sharp_{\frac{(p-t)s+r}{(p-t)ss_1+r}} B_1^{(p-t)ss_1+t}.$$

Now we apply Lemma 5 to  $A \geq B_1$  for  $\alpha = r-t, \beta = (p-t)ss_1+t$  and  $\gamma = (p-t)s+t$ . Then  $\gamma \leq \beta$  by  $1 \leq s_1$ . That is, we have

$$(14) \quad A^{-r+t} \sharp_{\frac{(p-t)s+r}{(p-t)ss_1+r}} B_1^{(p-t)ss_1+t} \leq B_1^{(p-t)s+t} = A^t \natural_s B^p.$$

Combining (13) with (14), we obtain that

$$A^{-r+t} \sharp_{\frac{(p-t)s+r}{(p-t)ss_1+r}} (A^t \natural_{ss_1} B^p) \leq A^t \natural_s B^p.$$

Finally it implies that

$$\begin{aligned}
 F(r, s s_1) &= A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)ss_1+r}} (A^t \natural_{ss_1} B^p) \\
 &= A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^{-r+t} \sharp_{\frac{(p-t)s+r}{(p-t)ss_1+r}} (A^t \natural_{ss_1} B^p)) \\
 &\leq A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \\
 &= F(r, s),
 \end{aligned}$$

so the proof is complete.

Consequently the grand Furuta inequality is proved by Lemmas 4 and 6, in the form of (9).

#### ACKNOWLEDGEMENT

The authors would like to express their hearty thanks to Professor T. Furuta for his fruitful and critical discussion and the referee for his warm and kind suggestions.

#### REFERENCES

1. T.Ando, *On some operator inequalities*, Math. Ann., **279** (1987), 157–159. MR **89c**:47019
2. T.Ando and F.Hiai, *Log-majorization and complementary Golden-Thompson type inequalities*, Linear Alg. and its Appl., **197/198** (1994), 113–131. MR **95d**:15006
3. M.Fujii, *Furuta's inequality and its mean theoretic approach*, J. Operator theory, **23** (1990), 67–72. MR **91g**:47012
4. M.Fujii, T.Furuta and E.Kamei, *Furuta's inequality and its application to Ando's Theorem*, Linear Alg. and its Appl., **149** (1991), 91–96. MR **92e**:47023
5. M.Fujii, T.Furuta and E.Kamei, *Operator functions associated with Furuta's inequality*, Linear Alg. and its Appl., **179** (1993), 161–169. MR **93j**:47026
6. M.Fujii and E.Kamei, *Furuta's inequality and a generalization of Ando's Theorem*, Proc. Amer. Math. Soc., **115** (1992), 409–413. MR **92i**:47108
7. T.Furuta,  *$A \geq B \geq 0$  assures  $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$  for  $r \geq 0, p \geq 0, q \geq 1$  with  $(1+2r)q \geq p+2r$* , Proc. Amer. Math. Soc., **101** (1987), 85–88. MR **89b**:47028
8. T.Furuta, *Elementary proof of an order preserving inequality*, Proc. Japan Acad., **65** (1989), 126. MR **90g**:47029
9. T.Furuta, *A proof via operator means of an order preserving inequality*, Linear Alg. and its Appl., **113** (1989), 12–130. MR **89k**:47023
10. T.Furuta, *Two operator functions with monotone property*, Proc. Amer. Math. Soc., **111** (1991), 511–516. MR **91f**:47023
11. T.Furuta, *Application of an order preserving inequalities*, Operator theory : Advances and Applications, Birkhouser Verlag, Basel, **59** (1992), 180–190. MR **94m**:47033
12. T.Furuta, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Alg. and its Appl., **219** (1995), 139–155.
13. E.Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann., **123** (1951), 415–438. MR **13**:471f
14. E.Kamei, *Furuta's inequality via operator means*, Math. Japon., **33** (1988), 737–739. MR **90c**:47037
15. E.Kamei, *A satellite to Furuta's inequality*, Math. Japon., **33** (1988), 883–886. MR **89m**:47011
16. F.Kubo and T.Ando, *Means of positive linear operators*, Math. Ann., **246** (1980), 205–224. MR **84d**:47028
17. K.Löwner, *Über monotone Matrixfunktionen*, Math. Z., **38** (1934), 177–216.

DEPARTMENT OF MATHEMATICS, OSAKA KYOIKU UNIVERSITY, ASAHIGAOKA, KASHIWARA, OSAKA 582, JAPAN

*E-mail address*: mfujii@cc.osaka-kyoiku.ac.jp

MOMODANI SENIOR HIGH SCHOOL, IKUNO, OSAKA 544, JAPAN