

TRANSFORMATIONS CONJUGATE TO THEIR INVERSES HAVE EVEN ESSENTIAL VALUES

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ABSTRACT. Let T be an ergodic automorphism defined on a standard Borel probability space for which T and T^{-1} are isomorphic. We study the structure of the conjugating automorphisms and attempt to gain information about the structure of T . It was shown in *Ergodic transformations conjugate to their inverses by involutions* by Goodson et al. (Ergodic Theory and Dynamical Systems **16** (1996), 97–124) that if T is ergodic having simple spectrum and isomorphic to its inverse, and if S is a conjugation between T and T^{-1} (i.e. S satisfies $TS = ST^{-1}$), then $S^2 = I$, the identity automorphism. We give a new proof of this result which shows even more, namely that for such a conjugation S , the unitary operator induced by T on $L^2(X, \mu)$ must have a multiplicity function whose essential values on the ortho-complement of the subspace $\{f \in L^2(X, \mu) : f(S^2) = f\}$ are always even. In particular, we see that S can be weakly mixing, so the corresponding T must have even maximal spectral multiplicity (regarding ∞ as an even number).

0. INTRODUCTION

Let T be an ergodic transformation defined on a standard Borel probability space (X, \mathcal{F}, μ) having simple spectrum. It was shown in [2] that if T is isomorphic to its inverse T^{-1} and the conjugating automorphism is S , i.e. $TS = ST^{-1}$, then $S^2 = I$, the identity automorphism. It follows that if T has a conjugating automorphism S for which $S^2 \neq I$, then T has non-simple spectrum. Our aim is to obtain more information about the nature of the spectrum in this situation. In this paper we show that in fact the spectrum of T must have even multiplicity in the orthogonal complement of the subspace $\{f \in L^2(X, \mu) : f(S^2) = f\}$ (we regard ∞ as an even number). Of course this ortho-complement is trivial if T has simple spectrum, and it consists of everything except the constant functions if S is weak mixing. Specifically we prove

Theorem 1. *Suppose that $ST = T^{-1}S$ for some automorphism S ; then the essential values of the multiplicity function of T , restricted to the ortho-complement of the subspace*

$$\{f \in L^2(X, \mu) : f(S^2) = f\},$$

are even.

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This theorem, which is proved in section 3, actually follows from a more general result regarding unitary operators on L^2 -spaces which preserve real-valued functions and are unitarily equivalent to their inverses.

In section 2 we show that if S^2 and T with $ST = T^{-1}S$ are both ergodic, then they both must be weak mixing.

Finally, in the last section we give a number of examples illustrating the types of situations which may arise. In particular, examples for which S is weak mixing are given.

1. PRELIMINARIES

Throughout, $T : (X, \mathcal{F}, \mu) \rightarrow (X, \mathcal{F}, \mu)$ will denote an ergodic automorphism defined on a non-atomic standard Borel probability space. Throughout the identity automorphism will be denoted by I . The group of all automorphisms $\mathcal{U}(X)$ of (X, \mathcal{F}, μ) becomes a completely metrizable topological group when endowed with the weak convergence of transformations ($T_n \rightarrow T$ if for all $A \in \mathcal{F}$, $\mu(T_n^{-1}(A)\Delta T^{-1}(A)) + \mu(T_n(A)\Delta T(A)) \rightarrow 0$ as $n \rightarrow \infty$). Denote by $C(T)$ the commutant of T , i.e. those automorphisms of (X, \mathcal{F}, μ) which commute with T . Since we are assuming that the members of $C(T)$ are invertible, $C(T)$ is a group. Suppose that X is an atomic space; then, if $T : X \rightarrow X$ is ergodic, T is a rotation on a finite point set and so is automatically isomorphic to T^{-1} by an involution. Here the theory reduces to the theory of cyclic permutations, and consequently we always assume that X is a non-atomic probability space.

Related to the commutant is the set

$$\mathcal{B}(T) = \{S \in \mathcal{U}(X) : TS = ST^{-1}\}.$$

If S is measure preserving and $TS = ST^{-1}$, then $S^2T = ST^{-1}S = TS^2$. This shows that $\{S^2 : S \in \mathcal{B}(T)\} \subseteq C(T)$.

The spectral properties of T are those of the induced unitary operator defined by

$$U_T : L^2(X, \mu) \rightarrow L^2(X, \mu); \quad U_T f(x) = f(Tx), \quad f \in L^2(X, \mu).$$

Generally a unitary operator $U : H \rightarrow H$ on a separable Hilbert space H is said to have simple spectrum if there exists $h \in H$ such that $Z(h) = H$, where $Z(h)$ is the closed linear span of the vectors $U^n h$, $n \in \mathbb{Z}$. U is determined up to unitary equivalence by a spectral measure class σ and a $\{1, 2, \dots, \infty\}$ -valued multiplicity function M defined on the circle S^1 . The set \mathcal{M}_U of essential spectral multiplicities of U is the set of all σ -essential values of M . The maximal spectral multiplicity (or just multiplicity) of U is $\text{msm}(U) = \sup \mathcal{M}_U$. U has simple spectrum if $\text{msm}(U) = 1$, otherwise U has non-simple spectrum. See Queffelec [7] for a detailed discussion of the spectral properties of dynamical systems.

2. BASIC RESULTS

In this section, we show that if $S \in \mathcal{B}(T)$ where both T and S^2 are ergodic, then they must both be weak mixing. This leads to a simple classification of the members of $\mathcal{B}(T)$. The first part of the next proposition was given in [2]. The second part is well known, consequently we omit their proofs, both being elementary and easy.

Proposition 1. *Let $T : X \rightarrow X$ be an ergodic automorphism.*

- (i) *If $S \in \mathcal{B}(T)$, then $S^{2n} = I$ for some $n > 0$, or S is aperiodic.*

(ii) If $TS = ST$ where T is weak mixing and S is ergodic, then S is also weak mixing.

Ryzhikov [8] improved on results of Fathi [1] by showing that any automorphism T on a standard Borel probability space (X, \mathcal{F}, μ) is the product of three involutions (i.e. $T : X \rightarrow X$ can be written $T = U \circ V \circ W$ where $U^2 = V^2 = W^2 = I$). This is best possible (it was mentioned in [2], and it is an elementary fact that T is isomorphic to its inverse via an involution S , if and only if T is the product of two involutions).

In this direction, below we give a simple characterization of transformations which are isomorphic to their inverses.

Proposition 2. *An automorphism $T : X \rightarrow X$ is isomorphic to its inverse if and only if there exist automorphisms S_1 and S_2 for which $T = S_1 \circ S_2^{-1}$ and satisfying $S_1^2 = S_2^2$.*

Proof. First suppose that T is isomorphic to its inverse; then there exists $S_1 \in \mathcal{B}(T)$ satisfying $S_1 T = T^{-1} S_1$. Let $S_2 = T^{-1} \circ S_1$; then $S_2 \in \mathcal{B}(T)$ and

$$S_1 \circ S_2^{-1} = S_1 \circ (T^{-1} \circ S_1)^{-1} = T,$$

and

$$S_2^2 = (T^{-1} \circ S_1)^2 = T^{-1} \circ S_1 \circ T^{-1} \circ S_1 = S_1 \circ T \circ T^{-1} \circ S_1 = S_1^2.$$

Conversely, suppose that $T = S_1 \circ S_2^{-1}$, where $S_1^2 = S_2^2$; then

$$S_1 \circ T = S_1 \circ (S_1 \circ S_2^{-1}) = S_1^2 \circ S_2^{-1} = S_2,$$

and similarly, $T^{-1} \circ S_1 = S_2$, so that $S_1 \in \mathcal{B}(T)$ and T is isomorphic to its inverse. \square

The following simple corollary shows that if T is ergodic, the members of $\mathcal{B}(T)$ never commute, and if $S \in \mathcal{B}(T)$, then S^2 has non-simple spectrum. This is mainly of interest when S is also ergodic.

Corollary 1. *If T is ergodic and $S \in \mathcal{B}(T)$, then $C(S^2)$, the commutant of S^2 , is non-abelian. In particular, S^2 has a non-simple spectrum.*

Proof. Suppose there exists $S_1 \in \mathcal{B}(T)$ for which $C(S_1^2)$ is abelian. If we define $S_2 = S_1 \circ T$, we see that $S_1^2 = S_2^2$.

Thus the map $R = S_1^2 = S_2^2$ has 2 distinct square roots S_1, S_2 , which therefore belong to $C(R)$ and hence commute.

Now $T = S_1^{-1} \circ S_2$, so that

$$T^2 = S_1^{-1} \circ S_2 \circ S_1^{-1} \circ S_2 = S_1^{-2} \circ S_2^2 = I,$$

contradicting T being ergodic. \square

We now show that if T is ergodic and $S \in \mathcal{B}(T)$ where S^2 is ergodic, then T has to be weak mixing, which in turn implies that S is weak mixing. In a later section we shall give a number of examples of automorphisms conjugated to their inverses by weak mixing transformations.

Theorem 2. *Suppose that $T : X \rightarrow X$ and $S : X \rightarrow X$ are automorphisms for which T and S^2 are ergodic and $ST = T^{-1}S$. Then both S and T are weak mixing.*

Proof. Suppose that $f(Tx) = \lambda f(x)$ for some measurable function f and constant $\lambda \in S^1$. Then

$$f(TSx) = \lambda f(Sx), \text{ so that } f(ST^{-1}x) = \lambda f(Sx).$$

This implies that $\bar{\lambda}f(Sx) = f(STx)$, or that $f \circ S(Tx) = \bar{\lambda}f \circ S(x)$.

Thus $\bar{\lambda}$ is an eigenvalue of T corresponding to the eigenfunction $f \circ S$. In particular

$$\bar{f} \circ S(Tx) = \lambda \bar{f} \circ S(x).$$

The ergodicity of T implies that $\bar{f} \circ S = c \cdot f$ for some constant c , with $|c| = 1$. Then

$$f(S^2x) = f \circ S(Sx) = \bar{c}\bar{f}(Sx) = \bar{c} \cdot cf(x) = f(x).$$

But S^2 is ergodic, so $f = \text{constant}$ and $\lambda = 1$, so that T is weak mixing. Proposition 1 now implies that S^2 and hence S are weak mixing. \square

Corollary 2. *Let $S \in \mathcal{B}(T)$ for some ergodic T .*

(i) *If f is an eigenfunction for T with corresponding eigenvalue λ , then $f \circ S$ is an eigenfunction corresponding to the eigenvalue $\bar{\lambda}$.*

(ii) *Let H_0 be the closed linear span of the eigenfunctions of T . If for any $S \in \mathcal{B}(T)$, $H_S = \{f \in L^2(X, \mu) : S^2(f) = f\}$, then H_0 and H_S are both T and S invariant subspaces of $L^2(X, \mu)$ which satisfy $H_0 \subseteq H_S$.*

Proof. (i) follows directly from the last theorem.

(ii) can be proved directly, or alternatively we can argue as follows:

Denote by $S_0 = S|_{H_0}$ and $T_0 = T|_{H_0}$. Then $T_0 : H_0 \rightarrow H_0$ has discrete spectrum and hence has simple spectrum. Furthermore $S_0T_0 = T_0^{-1}S_0$ and the conditions of Theorem 1 of [2] are satisfied. (In particular, H_0 may be represented as $L^2(Y)$ for suitable Y corresponding to the maximal Kronecker factor of T .) This implies that $S_0^2 = I$ and the result follows. \square

Corollary 2 gives another proof that if S^2 is ergodic, then T is weak mixing (and hence S is weak mixing). This is because the ergodicity of S^2 implies that H_S is the one-dimensional subspace of $L^2(X, \mu)$ consisting only of the constant functions. The same therefore applies to H_0 and the result follows.

The following corollary is now an immediate consequence of Theorem 2 and Proposition 1. The quasi-discrete spectrum transformations of [2] give rise to examples for which (iii) below holds.

Corollary 3. *For each $S \in \mathcal{B}(T)$, one of the following must hold:*

- (i) $S^{2n} = I$ for some $n \geq 1$,
- (ii) S is weakly mixing,
- (iii) S^2 is aperiodic and non-ergodic.

3. CONJUGATION BY A UNITARY OPERATOR

In this section we attempt to explain why all known ergodic automorphisms having a finite Lebesgue component in their spectrum have Lebesgue components of even multiplicity. (See the examples of Mathew-Nadkarni and Lemańczyk, where the Lebesgue component in the spectrum arises in the ortho-complement of the eigenfunctions.) In particular, we show that for a unitary operator $T : L^2(X, \mu) \rightarrow$

$L^2(X, \mu)$, preserving real-valued functions, and $S \in \mathcal{B}(T)$ (also preserving real-valued functions), T has a multiplicity function which only takes even values (a.e.) in the ortho-complement of the subspace $\{f \in L^2(X, \mu) : S^2(f) = f\}$. It follows that if $T : X \rightarrow X$ is an automorphism having simple spectrum and $S \in \mathcal{B}(T)$, then $S^2 = I$, giving a new proof of Theorem 1 of [2].

The following is the main theorem of the paper, and gives Theorem 1 as an immediate consequence. In this case $\mathcal{B}(T)$ denotes the unitary conjugations between T and T^{-1} .

Theorem 3. *Suppose that $T : L^2(X, \mu) \rightarrow L^2(X, \mu)$ is a unitary operator which preserves real-valued functions, and suppose there exists $S \in \mathcal{B}(T)$ which also preserves real-valued functions. Then in the ortho-complement of the subspace*

$$\{f \in L^2(X, \mu) : S^2(f) = f\},$$

the essential values of the multiplicity function of T are even (∞ is considered as an even number).

Proof. Let $H_1 = \{f \in L^2(X, \mu) : S^2(f) = f\}$, $H_{-1} = \{f \in L^2(X, \mu) : S^2(f) = -f\}$ and $H = H_1 \oplus H_{-1}$; then since $ST = T^{-1}S$, H_1 and H_{-1} are both T and S invariant, so the same is true for H^\perp .

We shall see that T preserves \mathcal{P}_1 and \mathcal{P}_2 where

$$(1) \quad H^\perp = \mathcal{P}_1 \oplus \mathcal{P}_2,$$

and where $\mathcal{P}_1 =$ those functions whose spectral type is concentrated on the ‘‘upper’’ half, S^+ , of the unit circle in the complex plane (excluding ± 1), and $\mathcal{P}_2 =$ those whose spectral type is concentrated on the ‘‘lower’’ half S^- (again excluding ± 1) of the unit circle (spectral type here means with respect to S^2). More precisely, let

$$\mathcal{P}_1 = \{f \in H^\perp : \text{supp } \sigma_f \subseteq S^+\},$$

$$\mathcal{P}_2 = \{f \in H^\perp : \text{supp } \sigma_f \subseteq S^-\}$$

(where $\text{supp } \sigma_f$ denotes the support of the measure σ_f , the spectral type of f with respect to S^2). Since S^2 preserves real-valued functions, it also preserves complex conjugation (i.e. $S^2(\bar{f}) = \overline{S^2(f)}$ for all $f \in L^2(X, \mu)$). It follows that on H^\perp , the maximal spectral type of S^2 is invariant under complex conjugation, so if h realizes its maximal spectral type, we can define measures ν_1 and ν_2 on S^1 by

$$\nu_1(A) = \sigma_h(A \cap S^+), \quad \nu_2(A) = \sigma_h(A \cap S^-),$$

and since σ_h is symmetric on S^1 ,

$$\nu_1(\bar{A}) = \sigma_h(\bar{A} \cap S^+) = \sigma_h(\overline{A \cap S^-}) = \sigma_h(A \cap S^-) = \nu_2(A).$$

Now $\nu_i \ll \sigma_h$, $i = 1, 2$, implies that there exists h_1 and h_2 with $\nu_1 = \sigma_{h_1}$, $\nu_2 = \sigma_{h_2}$. It follows that $\nu_2 = \sigma_{\bar{h}_1}$, where

$$h = h_1 + \bar{h}_1,$$

and where $h_1 \in \mathcal{P}_1$, $\bar{h}_1 \in \mathcal{P}_2$. Therefore

$$\mathcal{P}_1 = \{f \in H^\perp : \sigma_f \ll \sigma_{h_1}\},$$

$$\mathcal{P}_2 = \{f \in H^\perp : \sigma_f \ll \sigma_{\bar{h}_1}\}.$$

Here the spectral type is with respect to S^2 , and it is clear that these subspaces are S^2 and T invariant and also that equation (1) holds.

Now we claim that

$$T : \mathcal{P}_1 \rightarrow \mathcal{P}_1 \quad \text{and} \quad T : \mathcal{P}_2 \rightarrow \mathcal{P}_2$$

are unitarily equivalent.

First note that if $f \in \mathcal{P}_1$, then $S(\bar{f}) \in \mathcal{P}_2$ and

$$\begin{aligned} (T^n f, f) &= (ST^n f, S f) = (T^{-n} S f, S f) \\ &= (S f, T^n S f) = \overline{(T^n S f, S f)} = (T^n S \bar{f}, S \bar{f}). \end{aligned}$$

Consequently, if $Z(f)$ and $Z(S\bar{f})$ are the corresponding cyclic subspaces with respect to T , they are also subspaces of \mathcal{P}_1 and \mathcal{P}_2 respectively, so that $T|Z(f)$ and $T|Z(S\bar{f})$ are unitarily equivalent (this already shows that T has non-simple spectrum when H^\perp is non-trivial).

Now let

$$\mathcal{P}_1 = \bigoplus_{i=1}^{\infty} Z(k_i)$$

be a spectral decomposition (with respect to T) for $T|\mathcal{P}_1$. Then

$$\sigma_{k_1} \gg \sigma_{k_2} \gg \dots$$

Now $S\bar{k}_i \in \mathcal{P}_2$ and $SZ(\bar{k}_i) = Z(S\bar{k}_i)$, for $i = 1, 2, \dots$. Clearly

$$\bigoplus_{i=1}^{\infty} Z(S\bar{k}_i) = \mathcal{P}_2,$$

with $\sigma_{S\bar{k}_i} = \sigma_{k_i}$, $i = 1, 2, \dots$, and $\sigma_{S\bar{k}_1} \gg \sigma_{S\bar{k}_2} \gg \dots$

It follows that

$$\bigoplus_{i=1}^{\infty} Z(S\bar{k}_i) \quad \text{is a spectral decomposition for} \quad T|\mathcal{P}_2.$$

The unitary equivalence between $Z(k_i)$ and $Z(S\bar{k}_i)$ for $i = 1, 2, \dots$, now gives the required result.

We now consider the case of $T|H_{-1}$. As before, if $f \in H_{-1}$, then f and $S\bar{f}$ have the same spectral type, but in this case we also have $S\bar{f} \in H_{-1}$. Here we proceed as follows:

$$(T^n f, S\bar{f}) = (ST^n f, S^2\bar{f}) = (T^{-n} S f, -\bar{f}) = -(S f, T^n \bar{f}) = -(T^n f, S\bar{f}).$$

It follows that $(T^n f, S\bar{f}) = 0$ for all $n \in \mathbb{Z}$, so that as before, $T|Z(f)$ and $T|Z(S\bar{f})$ are unitarily equivalent, and similar arguments to those earlier complete the proof. \square

4. EXAMPLES

We now give two examples of automorphisms which have conjugations to their inverses which are weak mixing. Suppose that R is a weak mixing transformation having two distinct square roots S_1 and S_2 , i.e. $R = S_1^2 = S_2^2$; then it follows from Proposition 2 that $\hat{T} = S_1 \circ S_2^{-1}$ is isomorphic to its inverse and S_1, S_2 are weak mixing conjugations. It may be that \hat{T} is not ergodic, for example, suppose T is weakly mixing and $R = T^2 \times T^2$, $S_1 = T \times T$ and $S_2(x, y) = (y, T^2x)$. If

$\widehat{T}(x, y) = S_1 \circ S_2^{-1}(x, y) = (T^{-1}y, Tx)$, then $\widehat{T}^2 = I$. Below we give two examples giving rise to ergodic (and hence weak mixing) \widehat{T} .

(i) Let $T : X \rightarrow X$ be a weak mixing automorphism; then $T \times T$ is also weak mixing. Define $S : X \times X \rightarrow X \times X$ by $S(x, y) = (y, Tx)$. Clearly S is a square root of $T \times T$, so is weak mixing. Furthermore, S conjugates $T \times T^{-1}$ to its inverse $T^{-1} \times T$, even if T and T^{-1} are non-isomorphic. This is the situation of the above remark where we have chosen $R = T \times T$, having the two distinct square roots, $S(x, y) = (y, Tx)$ and $S'(x, y) = (Ty, x)$, so that $T \times T^{-1} = S' \circ S^{-1}$.

In [3], del Junco and Lemańczyk have shown that there is a dense G_δ subset \mathcal{K} of $\mathcal{U}(X)$ such that for each $T \in \mathcal{K}$, $C(T \times T^{-1}) = C(T) \times C(T)$. It follows from their results that for $T \in \mathcal{K}$

$$\mathcal{B}(T \times T^{-1}) = \{(P \times Q) \circ F : P, Q \in C(T)\},$$

where F is the flip map, $F(x, y) = (y, x)$. Note that $\mathcal{B}(T \times T^{-1})$ contains many involutions, in fact any map of the form $U(x, y) = (Py, P^{-1}x)$ is such an involution.

Remark. Theorem 3 implies that the essential values of $T \times T^{-1}$ are even for every $T \in \mathcal{U}(X)$, and since it is not hard to see that $T \times T$ is spectrally isomorphic to $T \times T^{-1}$, it follows that the essential values of $T \times T$ are always even. Katok [4] has shown that generically the essential values of $T \times T$ are either $\{2\}$ or $\{2, 4\}$.

(ii) This next example was communicated to us by J. Feldman, and although somewhat similar to example (i), it is of interest because $\mathcal{B}(\widehat{T})$ contains maps of a different nature, namely types of shift maps.

We start with an ergodic automorphism T of (X, \mathcal{F}, μ) , a non-atomic standard Borel space. Let $(Y, \mathcal{B}, m) = \prod_{i=-\infty}^{\infty} (X, \mathcal{F}, \mu)_i$ be the doubly infinite direct product of the space (X, \mathcal{F}, μ) . Define $\widehat{T} : Y \rightarrow Y$ by

$$\widehat{T}(\dots, x_{-1}, x_0^*, x_1, x_2, \dots) = (\dots, T^{-1}x_{-1}, T^*x_0, T^{-1}x_1, Tx_2, \dots),$$

where the $*$ indicates the zeroth coordinate.

Suppose now that $S : Y \rightarrow Y$ is the shift map; then S is an ergodic automorphism and it is easy to check that $S\widehat{T} = \widehat{T}^{-1}S$. In fact S is a mixing automorphism.

Note that there are elements of $\mathcal{B}(\widehat{T})$ which are involutions. For example, define $S_1 : Y \rightarrow Y$ by

$$S_1(\dots, x_{-1}, x_0^*, x_1, x_2, \dots) = (\dots, x_{-1}, x_{-2}, x_1^*, x_0, x_3, x_2, \dots);$$

then $S_1 \in \mathcal{B}(\widehat{T})$ and $S_1^2 = I$.

(iii) Mathew and Nadkarni [6] defined a continuum of examples, each having a spectrum consisting of a discrete component and a Lebesgue component of multiplicity equal to 2. It was shown in [2] that each of these transformations T_ϕ is isomorphic to its inverse, and that every $S \in \mathcal{B}(T_\phi)$ satisfies $S^4 = I$ and $S^2 \neq I$. Consequently the results of sections 2 and 3 are applicable. In fact, using the notation of Theorem 3 we have here the situation where $\mathcal{P}_1 = \{0\} = \mathcal{P}_2$ and $L^2(X, \mu) = H_1 \oplus H_{-1}$, with $H_{-1} \neq \{0\}$.

$T_\phi : [0, 1) \times \mathbb{Z}_2 \rightarrow [0, 1) \times \mathbb{Z}_2$ is defined by $T_\phi(x, g) = (Tx, \phi(x) + g)$ where T is the von Neumann Kakutani adding machine, and ϕ is defined using a “random” Toeplitz sequence with “2 holes”, i.e. at the n th stage in the construction of the 2^n levels of T , ϕ is left undefined on the 2^{n-1} th and top levels of the stack. See [2] for details.

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