

DERIVATIONS WITH ENGEL CONDITIONS ON MULTILINEAR POLYNOMIALS

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ABSTRACT. Let R be a prime algebra over a commutative ring K with unity and let $f(X_1, \dots, X_n)$ be a multilinear polynomial over K . Suppose that d is a nonzero derivation on R such that for all r_1, \dots, r_n in some nonzero ideal I of R , $[d(f(r_1, \dots, r_n)), f(r_1, \dots, r_n)]_k = 0$ with k fixed. Then $f(X_1, \dots, X_n)$ is central-valued on R except when $\text{char } R = 2$ and R satisfies the standard identity s_4 in 4 variables.

Throughout this note K will denote a commutative ring with unity and R will denote a prime K -algebra with center \mathcal{Z} . By d we always mean a nonzero derivation on R . For $x, y \in R$, set $[x, y]_0 = x$, $[x, y]_1 = [x, y] = xy - yx$ and, for $k > 1$, $[x, y]_k = [[x, y]_{k-1}, y]$.

A well-known result proved by Posner [10] states that R must be commutative if $[d(x), x] \in \mathcal{Z}$ for all $x \in R$. In [7], the authors generalized Posner's theorem by showing that a Lie ideal L of R must be contained in \mathcal{Z} if $\text{char } R \neq 2$ and $[d(x), x] \in \mathcal{Z}$ for all $x \in L$. As to the case when $\text{char } R = 2$, Lanski [5] obtained the same conclusion except when R satisfies the standard identity s_4 in 4 variables. On the other hand, Vukman [11] showed that R is commutative if $\text{char } R \neq 2$ and $[d(x), x]_2 = 0$ for all $x \in R$, or if $\text{char } R \neq 2, 3$ and $[d(x), x]_2 \in \mathcal{Z}$ for all $x \in R$. In a recent paper [6], a full generalization of these results was proved by Lanski. He showed that a Lie ideal L of R is in \mathcal{Z} if for some fixed $k > 0$, $[d(x), x]_k = 0$ for all $x \in L$, unless $\text{char } R = 2$ and R satisfies s_4 . Note that a noncentral Lie ideal of R contains all the commutators $[x, y]$ for x, y in some nonzero ideal of R except when $\text{char } R = 2$ and R satisfies s_4 . It is natural to consider the situation when $[d(x), x]_k = 0$ for all commutators $x = [x_1, x_2]$, or more generally, when $[d(x), x]_k = 0$ for all $x = f(x_1, \dots, x_n)$ where $f(X_1, \dots, X_n)$ is a multilinear polynomial over K . In the present paper, we shall extend Lanski's theorem by imposing the condition $[d(f(x_1, \dots, x_n)), f(x_1, \dots, x_n)]_k = 0$ on some nonzero ideal of R .

First we dispose of the simplest case when R is the matrix ring $M_m(F)$ over a field F and d is an inner derivation on R .

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Proposition. *Let F be a field and $R = M_m(F)$, the $m \times m$ matrix algebra over F . Suppose that $a \in R$ and that $f(X_1, \dots, X_n)$ is a multilinear polynomial over F such that $[a, f(x_1, \dots, x_n)]_k = 0$ for all $x_i \in R$, where $k > 0$ is a fixed integer. Then either $a \in \mathcal{Z}$ or $f(X_1, \dots, X_n)$ is central-valued on R unless $\text{char } F = 2$ and $m = 2$.*

Proof. If $m = 1$, there is nothing to prove, so we assume that $m \geq 2$. We assume further that either $m \neq 2$ or $\text{char } F \neq 2$ and proceed to show that $a \in \mathcal{Z}$ if $f(X_1, \dots, X_n)$ is not central-valued on R . By [8, Lemmas 2 and 9], there exists a sequence of matrices $r = (r_1, \dots, r_n)$ in R such that $f(r) = f(r_1, \dots, r_n) = \sum_{i=1}^m \alpha_i e_{ii}$ is a noncentral diagonal matrix, say, $\alpha_1 \neq \alpha_2$. Let φ be the inner automorphism on R defined by $x^\varphi = (1 + e_{21})x(1 - e_{21})$. Write $D = f(r)$; then $E = f(r^\varphi) = f(r_1^\varphi, \dots, r_n^\varphi) = D^\varphi = D + (\alpha_1 - \alpha_2)e_{21}$. By assumption, we have $[a, E]_k = 0$, that is, $\sum_{i=0}^k (-1)^i \binom{k}{i} E^i a E^{k-i} = 0$. Multiplying e_{22} from both sides, we have $(\alpha_2 - \alpha_1)^k e_{21} a e_{22} = 0$ by using $E^i = D^i + (\alpha_1^i - \alpha_2^i)e_{21}$, $e_{22}E^i = \alpha_2^i(e_{22} - e_{21}) + \alpha_1^i e_{21}$ and $E^i e_{22} = \alpha_2^i e_{22}$. Thus, $e_{21} a e_{22} = 0$ and so the $(1, 2)$ -entry a_{12} of a is 0. For $s \neq t$, let σ be a permutation in the symmetric group S_m such that $\sigma(1) = s$ and $\sigma(2) = t$. Let ψ be the automorphism on R defined by $x^\psi = \left(\sum_{i,j} \xi_{ij} e_{ij} \right)^\psi = \sum_{i,j} \xi_{ij} e_{\sigma(i), \sigma(j)}$. Then $f(r^\psi) = D^\psi$ is a diagonal matrix with (s, s) - and (t, t) -entries distinct. Replacing D by D^ψ and proceeding as before, we have $a_{st} = 0$ for $s \neq t$. In other words, a is a diagonal matrix. For any F -automorphism θ of R , a^θ enjoys the same property as a does, namely, $[a^\theta, f(x_1, \dots, x_n)]_k = 0$ for all $x_i \in R$. Hence, a^θ must be diagonal. Write $a = \sum_{i=1}^m a_{ii} e_{ii}$; then for $s \neq t$, we have $(1 + e_{ts})a(1 - e_{ts}) = \sum_{i=1}^m a_{ii} e_{ii} + (a_{ss} - a_{tt})e_{ts}$ diagonal. Hence, $a_{ss} = a_{tt}$ and so a is a scalar matrix.

We are now ready to prove the main

Theorem. *Let R be a prime K -algebra and $f(X_1, \dots, X_n)$ a multilinear polynomial over K . Suppose that d is a nonzero derivation on R such that*

$$\left[d(f(x_1, \dots, x_n)), f(x_1, \dots, x_n) \right]_k = 0$$

for all x_i in some nonzero ideal I of R , where $k > 0$ is fixed. Then $f(X_1, \dots, X_n)$ is central-valued on R except when $\text{char } R = 2$ and R satisfies s_4 .

Proof. Assume first that d is Q -inner, that is, $d(x) = [a, x]$ for all $x \in R$, where a is a noncentral element in the symmetric quotient ring Q of R [4]. Then

$$[a, f(x_1, \dots, x_n)]_{k+1} = 0 \quad \text{for all } x_i \in I.$$

By a theorem due to Chuang [1, Theorem 2], this generalized polynomial identity $[a, f(X_1, \dots, X_n)]_{k+1}$ is also satisfied by Q . In case the center C of Q is infinite, we

have $[a, f(x_1, \dots, x_n)]_{k+1} = 0$ for all $x_i \in Q \otimes_C \bar{C}$ where \bar{C} is the algebraic closure of C . Since both Q and $Q \otimes_C \bar{C}$ are prime and *centrally closed* [2, Theorems 2.5 and 3.5], we may replace R by Q or $Q \otimes_C \bar{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed and $[a, f(x_1, \dots, x_n)]_{k+1} = 0$ for all $x_i \in R$.

By Martindale's theorem [9], R is then a primitive ring having a nonzero socle H with C as the associated division ring. In light of Jacobson's theorem [3, p. 75], R is isomorphic to a dense ring of linear transformations on some vector space V over C , and H consists of the linear transformations in R of finite rank. Assume first that V is finite-dimensional over C . Then the density of R on ${}_C V$ implies that $R \cong M_m(C)$ where $m = \dim_C V$. By the preceding proposition, $f(X_1, \dots, X_n)$ is central-valued on R unless $\text{char } R = 2$ and $m = 2$. Assume next that V is infinite-dimensional over C . For any $e = e^2 \in H$, we have $eRe \cong M_m(C)$ with $m = \dim_C Ve$. Since R satisfies $e[a, f(ex_1e, \dots, ex_ne)]_{k+1}e = 0$, eRe satisfies $[eae, f(x_1, \dots, x_n)]_{k+1} = 0$. As we have seen above, eae must be central in eRe if $m \geq 3$ and if $f(X_1, \dots, X_n)$ is not central-valued on eRe . Since $a \notin C$, a does not centralize the nonzero ideal H of R , so $ah_0 \neq h_0a$ for some $h_0 \in H$. Also, $f(X_1, \dots, X_n)$ is not central-valued on H , for otherwise R would satisfy the polynomial identity $[f(X_1, \dots, X_n), X_{n+1}]$ contrary to the infinite-dimensionality of ${}_C V$. So, $[f(h_1, \dots, h_n), h_{n+1}] \neq 0$ for some $h_1, \dots, h_n, h_{n+1} \in H$. By Litoff's theorem [5, p. 280], there is an idempotent $e \in H$ so that $h_0, h_0a, ah_0, h_1, \dots, h_n, h_{n+1}$ are all in eRe . Since $\dim_C V$ is infinite, we may choose e so that $m = \dim_C Ve \geq 3$. Then eae is central in eRe because $f(X_1, \dots, X_n)$ is not central-valued on eRe . Thus

$$ah_0 = eah_0 = eae h_0 = h_0 eae = h_0 a e = h_0 a,$$

a contradiction. Hence $f(X_1, \dots, X_n)$ must be central-valued on R except when $\text{char } R = 2$ and R satisfies s_4 .

Now assume that d is not Q -inner. Recall that d can be extended uniquely to a derivation \bar{d} on Q [4]. We denote by $f^d(X_1, \dots, X_n)$ the polynomial obtained from $f(X_1, \dots, X_n)$ by replacing each coefficient α with $\bar{d}(\alpha \cdot 1)$. Since

$$\begin{aligned} & \left[d(f(x_1, \dots, x_n)), f(x_1, \dots, x_n) \right]_k \\ &= \left[f^d(x_1, \dots, x_n) + f(d(x_1), x_2, \dots, x_n) \right. \\ & \quad \left. + \dots + f(x_1, \dots, x_{n-1}, d(x_n)), f(x_1, \dots, x_n) \right]_k = 0, \end{aligned}$$

we have

$$\begin{aligned} & \left[f^d(x_1, \dots, x_n) + f(y_1, x_2, \dots, x_n) \right. \\ & \quad \left. + \dots + f(x_1, \dots, x_{n-1}, y_n), f(x_1, \dots, x_n) \right]_k = 0 \end{aligned}$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in R$ by Kharchenko's theorem [4]. In particular, $[f^d(x_1, \dots, x_n), f(x_1, \dots, x_n)]_k = 0$ and $[f(y_1, x_2, \dots, x_n), f(x_1, \dots, x_n)]_k = 0$ for all $x_1, \dots, x_n, y_1 \in R$. Applying d to $[f(y_1, x_2, \dots, x_n), f(x_1, \dots, x_n)]_{2k} = 0$,

we have

$$\begin{aligned} & \left[f^d(y_1, x_2, \dots, x_n) + f(d(y_1), x_2, \dots, x_n) \right. \\ & \quad \left. + \sum_{i=2}^n f(y_1, x_2, \dots, d(x_i), \dots, x_n), f(x_1, \dots, x_n) \right]_{2k} \\ & \quad + \sum_{j=0}^{2k-1} \left[\left[[f(y_1, x_2, \dots, x_n), f(x_1, \dots, x_n)]_j, d(f(x_1, \dots, x_n)) \right] \right. \\ & \qquad \qquad \qquad \left. f(x_1, \dots, x_n) \right]_{2k-j-1} = 0. \end{aligned}$$

The second sum vanishes since, for each j ,

$$\begin{aligned} & \left[\left[[f(y_1, x_2, \dots, x_n), f(x_1, \dots, x_n)]_j, d(f(x_1, \dots, x_n)) \right], f(x_1, \dots, x_n) \right]_{2k-j-1} \\ & = \sum_{i=0}^{2k-j-1} \binom{2k-j-1}{i} \left[[f(y_1, x_2, \dots, x_n), f(x_1, \dots, x_n)]_{j+i}, \right. \\ & \qquad \qquad \qquad \left. [d(f(x_1, \dots, x_n)), f(x_1, \dots, x_n)]_{2k-j-i-1} \right] = 0 \end{aligned}$$

where either $j + i \geq k$ or $2k - j - i - 1 \geq k$. Thus we have

$$\begin{aligned} & \left[f^d(y_1, x_2, \dots, x_n) + f(d(y_1), x_2, \dots, x_n) \right. \\ & \quad \left. + \sum_{i=2}^n f(y_1, x_2, \dots, d(x_i), \dots, x_n), f(x_1, \dots, x_n) \right]_{2k} = 0. \end{aligned}$$

Again by Kharchenko’s theorem, we have

$$\begin{aligned} & \left[f^d(y_1, x_2, \dots, x_n) + f(z, x_2, \dots, x_n) \right. \\ & \quad \left. + f(y_1, y_2, x_3, \dots, x_n) + \dots + f(y_1, x_2, \dots, y_n), f(x_1, \dots, x_n) \right]_{2k} = 0 \end{aligned}$$

for all $x_1, \dots, x_n, y_1, \dots, y_n, z \in R$. In particular,

$$\left[f^d(y_1, x_2, \dots, x_n), f(x_1, \dots, x_n) \right]_{2k} = 0$$

and

$$\left[f(y_1, y_2, x_3, \dots, x_n), f(x_1, \dots, x_n) \right]_{2k} = 0$$

for all $x_1, \dots, x_n, y_1, y_2 \in R$. Continuing this process, we will finally get

$$\left[f(y_1, \dots, y_n), f(x_1, \dots, x_n) \right]_{nk} = 0$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in R$. In light of the inner case, we have $f(y_1, \dots, y_n) \in \mathcal{Z}$ for all $y_1, \dots, y_n \in R$, that is, $f(X_1, \dots, X_n)$ is central-valued on R , except when $\text{char } R = 2$ and R satisfies s_4 . \square

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