

**THE LOCAL ZETA FUNCTION FOR THE NON-TRIVIAL
CHARACTERS ASSOCIATED WITH THE SINGULAR
JORDAN ALGEBRAS**

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ABSTRACT. This paper investigates the local integrals

$$Z_m(t, \chi) = \int_{H_m(O_C)} \chi(\det(x)) |\det(x)|^s dx$$

where O_C represents the integers of a composition algebra over a non-archimedean local field K and χ is a non-trivial character on the units in the ring of integers of K extended to K^* by setting $\chi(\pi) = 1$. The local zeta function for the trivial character is known for all composition algebras C . In this paper, we show in the quaternion case that $Z(t, \chi) = 0$ for all non-trivial characters and then compute the local zeta function in the ramified quadratic extension case for χ equal to the quadratic character. In this latter case, $Z(t, \chi) = 0$ for any character of order greater than 2.

1. INTRODUCTION

Let K be a finite, algebraic extension of \mathbf{Q}_p , O_K the ring of integers in K , π the uniformizing element in O_K , πO_K the maximal ideal of O_K , $U_K = O_K - \pi O_K$ the group of units in O_K , and $O_K/\pi O_K$ the residue field of K with cardinality q .

With $f(x) \in K[x_1, x_2, \dots, x_n] - \{0\}$ a non-constant polynomial over K and $\chi : U_K \rightarrow \mathbf{C}^\times$ a character of U_K which has been extended to K^* by setting $\chi(\pi) = 1$, one associates the Igusa local zeta function

$$Z(s, \chi) = \int_{O_K^n} \chi(f(x)) |f(x)|^s dx,$$

for $\text{Re}(s) > 0$, where $|\cdot|$ represents the p -adic absolute value on K , $\chi(0)$ is set equal to 0, and dx denotes the Haar measure on K^n normalized so that the measure of O_K^n is 1. Igusa [6] showed that this local zeta function has a meromorphic continuation to the whole complex plane and is, in fact, a rational function of $t = q^{-s}$. We write $Z(s, \chi) = Z(t, \chi)$.

Let $K' = K(\sqrt{\epsilon})$ be a quadratic unramified extension of K , where ϵ is a non-square unit in O_K , let $L = K(\sqrt{-\pi})$ be a ramified quadratic extension of K and let $D = K' \oplus \sqrt{\pi}K'$ be a quaternion division algebra. These vector spaces over K have natural involutions and form composition algebras C over K [2] and are called the singular composition algebras, as they arise for only finitely many primes.

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In addition, we can use the composition algebras and their involutions to form the vector space of $m \times m$ hermitian matrices over K , $H_m(C)$. These hermitian matrices are Jordan algebras with norm form equal to the formal determinant of the matrix [10, 5, 9].

This paper investigates the following integrals:

$$Z_m(t, \chi) = \int_{H_m(O_C)} \chi(\det(x)) |\det(x)|^s dx$$

where O_C represents the integers of the composition algebra. In [8, 11] the local zeta functions for the trivial character are given for all composition algebras. In this paper, we quickly show in the quaternion case that $Z(t, \chi) = 0$ for all non-trivial characters and then compute the local zeta function in the ramified quadratic extension case for χ equal to the quadratic character. In this case, $Z(t, \chi) = 0$ for any character of order greater than 2.

2. QUATERNION CASE

Before we consider the quaternion case, we need the following simple lemma.

Lemma 1 ([7, page 82]). *Suppose that U_K acts on K^n as a group of measure-preserving homomorphisms in such a way that $f(u \cdot x) = u^\lambda f(x)$ for a fixed λ and for all $u \in U_K$ and $x \in K^n$. Then $Z(t, \chi) = 0$ for all χ of order not dividing λ .*

Proof. Since $x \rightarrow u \cdot x$ is a measure-preserving transformation and χ is multiplicative, we see immediately that $Z(t, \chi) = \chi(u)^\lambda Z(t, \chi)$ for all $u \in U_K$. Hence, if $\chi^\lambda \neq 1$ then there exists a $u \in U_K$ such that $\chi(u)^\lambda \neq 1$, which implies that $Z(t, \chi) = 0$. □

The division algebra D is not isomorphic to a full matrix algebra. For this reason, it is called the twisted case. The quaternion division algebra is isomorphic to a matrix subalgebra with coefficients in K' where the isomorphism ϕ is defined as follows for $\alpha \in D$ (and $a_i \in K$):

$$\phi : \alpha = a_1 + \sqrt{\epsilon}a_2 + (a_3 + \sqrt{\epsilon}a_4)\sqrt{\pi} \rightarrow \begin{pmatrix} a_1 + \sqrt{\epsilon}a_2 & \pi(a_3 + \sqrt{\epsilon}a_4) \\ a_3 - \sqrt{\epsilon}a_4 & a_1 - \sqrt{\epsilon}a_2 \end{pmatrix}.$$

Using this isomorphism, the $m \times m$ hermitian matrices with coefficients in D are isomorphic to a subalgebra of the $2m \times 2m$ alternating matrices with coefficients in L by the following prescription. For each entry $x_{i,j}$ in the hermitian matrix $H_m(D)$ substitute the 2×2 matrix $\phi(x_{i,j}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In this way, we get a linear isomorphism ψ of $H_m(D)$ to a subalgebra of $\text{Alt}_{2m}(K')$.

For example, the isomorphism ψ is defined as follows for $m = 2$:

$$\begin{pmatrix} a_1 & b_1 - \sqrt{\epsilon}b_2 - (b_3 + \sqrt{\epsilon}b_4)\sqrt{\pi} \\ b_1 + \sqrt{\epsilon}b_2 + (b_3 + \sqrt{\epsilon}b_4)\sqrt{\pi} & c_1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & a_1 & b_3 - \sqrt{\epsilon}b_4 & b_1 - \sqrt{\epsilon}b_2 \\ -a_1 & 0 & -b_1 - \sqrt{\epsilon}b_2 & -\pi(b_3 + \sqrt{\epsilon}b_4) \\ -b_3 + \sqrt{\epsilon}b_4 & b_1 + \sqrt{\epsilon}b_2 & 0 & c_1 \\ -b_1 + \sqrt{\epsilon}b_2 & \pi(b_3 + \sqrt{\epsilon}b_4) & -c_1 & 0 \end{pmatrix}.$$

The Pfaffian of an alternating matrix is the square root of its determinant. If we denote the alternating image of $x \in H_m(D)$ by $\psi(x) = A$ then $\det(x) = \text{Pf}(A)$.

In this case, we have a measure-preserving homomorphism for all $u' \in U_{K'}$ on the vector space of Hermitian matrices over D . For any $x \in H_m(D)$ and $u' \in U_{K'}$,

we define the action $u' \cdot x = \begin{pmatrix} u' & 0 \\ 0 & 1_{m-1} \end{pmatrix} x \begin{pmatrix} \bar{u}' & 0 \\ 0 & 1_{m-1} \end{pmatrix}$, where \bar{u}' is the conjugate of u' in $U_{K'}$.

Now, $f(u' \cdot x) = \det(u' \cdot x) = N_{K'/K}(u') \det(x) = u f(x)$ for any $u \in U_K$ since the norm map is surjective. Finally, we can apply the lemma above with $\lambda = 1$ and $f(x) = \det(x)$ to show that $Z_m(t, \chi) = 0$ for all non-trivial characters χ and for all $m \geq 1$.

3. RAMIFIED QUADRATIC CASE

In this case, we would like to compute the following integral:

$$Z_m(t, \chi) = \int_{H_m(O_L)} \chi(\det(x)) |\det(x)|^s dx$$

for χ a non-trivial character on U_K and for $(2, q) = 1$. For $x \in H_m(L)$, we have a measure-preserving homomorphism for all u by defining $u \cdot x = \begin{pmatrix} u & 0 \\ 0 & 1_{m-1} \end{pmatrix} x \begin{pmatrix} u & 0 \\ 0 & 1_{m-1} \end{pmatrix}$. This action implies that $\det(u \cdot x) = u^2 \det(x)$. We can now apply Lemma 1 above with $\lambda = 2$ and $f(x) = \det(x)$ to show that $Z_m(t, \chi) = 0$ for all χ of order greater than 2. As mentioned above, the local zeta function for the trivial character χ_0 is known [11]. Adopting the notation $(a) = (1 - q^{-a})$, $(a)_+ = (1 + q^{-a})$, and $(a, b) = (1 - q^{-atb})$, we prove the following theorem.

Theorem 1. For χ the unique quadratic character on U_K ,

$$Z_m(t, \chi) = \begin{cases} \chi(-1)^{m/2} q^{-m/2} \prod_{i=1}^{m/2} \frac{(2i-1)}{(2i,1)}, & m \text{ even,} \\ 0, & m \text{ odd.} \end{cases}$$

Proof. We will outline the proof of this theorem but note that it follows closely the procedure used in [11] to compute the integral in the trivial character case. We need to find the orbital decomposition of $H_m(O_L)$, to compute two partial integrals $I_{m,k}(t, \chi)$ and $J_{m,2r}(t, \chi)$, to use them and the orbital decompositions of $H_m(\mathbf{F}_q)$ and $\text{Alt}_m(\mathbf{F}_q)$ to get a recursion relation for $Z_m(t, \chi)$, and finally to show that the expression above is the correct closed form solution.

Consider each entry in $H_m(O_L)$ modulo $\sqrt{-\pi}$ and then modulo π and see that

$$H_m(O_L) = H_m(\mathbf{F}_q) \oplus \sqrt{-\pi} \text{Alt}_m(\mathbf{F}_q) \text{ mod } \pi$$

where $H_m(\mathbf{F}_q)$ is the vector space of symmetric matrices over \mathbf{F}_q and $\text{Alt}_m(\mathbf{F}_q)$ are the skew-symmetric matrices over \mathbf{F}_q . The orbital structure of $H_m(\mathbf{F}_q)$ for $(q, 2) = 1$ under the action of $G = GL_m(\mathbf{F}_q)$ defined by $g \cdot a = g a {}^t g$ for $a \in H_m(\mathbf{F}_q)$ is as follows:

$$H_m(\mathbf{F}_q) = \{0\} \cup \left[\bigcup_{k=1}^m G \cdot \xi_k^1 \cup G \cdot \xi_k^2 \right]$$

where $\xi_k^1 = \begin{pmatrix} 1_k & 0 \\ 0 & 0 \end{pmatrix}$, $\xi_k^2 = \begin{pmatrix} 1_{k-1} & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $\varepsilon \in \mathbf{F}_q - (\mathbf{F}_q)^2$. This follows from the fact that $\alpha_1 x_1^2 + \alpha_2 x_2^2 = 1$ has solutions in \mathbf{F}_q for all $\alpha_i \in \mathbf{F}_q^\times$ and that we can diagonalize any quadratic form by an \mathbf{F}_q -linear change of variables such that $\alpha_1 x_1^2 + \alpha_2 x_2^2 + \dots + \alpha_k x_k^2 = y_1^2 + y_2^2 + \dots + \alpha_1 \alpha_2 \dots \alpha_k y_k^2$ for $\alpha_i \in \mathbf{F}_q^\times$. Note that any square α_i can be absorbed into y_k and that the decomposition above is disjoint.

The cardinality of these orbits can be computed using Dickson's formulae for the following groups [1, pages 78,160,94]:

$$|GL_m(\mathbf{F}_q)| = q^{m^2} \prod_{i=1}^m (q^i - 1), \quad |Sp_{2r}(\mathbf{F}_q)| = q^{r(2r+1)} \prod_{i=1}^r (q^{2i} - 1)$$

$|SO_m(\xi_m^i)(\mathbf{F}_q)| = q^{m(m-1)/2} \begin{cases} \prod_{j=1}^{(m-1)/2} (2i), & m \text{ odd,} \\ (1 - \chi(d)q^{-m/2}) \prod_{j=1}^{m/2-1} (2i), & m \text{ even,} \end{cases}$
 $|O_m(\xi_m^i)(\mathbf{F}_q)| = 2|SO_m(\xi_m^i)(\mathbf{F}_q)|$
 where $d = (-1)^{m(m-1)/2} \det(\xi_m^i)$ for $i = 1, 2$ and χ is the quadratic character on \mathbf{F}_q . Thus, $|SO_{2n}(\xi_{2n}^i)(\mathbf{F}_q)|$ depends on

$$\chi((-1)^n \varepsilon) = \begin{cases} 1, & n \text{ even, } \varepsilon \in (\mathbf{F}_q^\times)^2, \\ -1, & n \text{ even, } \varepsilon \in (\mathbf{F}_q^\times) - (\mathbf{F}_q^\times)^2, \\ \chi(-1), & n \text{ odd, } \varepsilon \in (\mathbf{F}_q^\times)^2, \\ -\chi(-1), & n \text{ odd, } \varepsilon \in (\mathbf{F}_q^\times) - (\mathbf{F}_q^\times)^2. \end{cases}$$

Note that the stabilizer of ξ_k^i is the set of all $\begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix} \in GL_m(\mathbf{F}_q)$ such that $g_{11} \in O_k(\xi_m^i)(\mathbf{F}_q)$, $g_{12} \in \text{Mat}_{k,m-k}(\mathbf{F}_q)$, and $g_{22} \in GL_{m-k}(\mathbf{F}_q)$. Thus, we see that

$$|G \cdot \xi_k^i| = \frac{|GL_m(\mathbf{F}_q)|}{|GL_{m-k}(\mathbf{F}_q)| q^{k(m-k)} 2|SO_k(\xi_m^i)(\mathbf{F}_q)|}.$$

In our calculation, we will need the cardinality of $|G \cdot \xi_k^1| - |G \cdot \xi_k^2|$. We make the convention that if $k = 0$ we simply mean the orbit of the 0 matrix, $|G \cdot 0| = 1$, and if a product is taken from a larger index to a smaller, we set it equal to 1. Using the formulae of Dickson with special attention to the case when k is even, we see that

$$|G \cdot \xi_k^1| - |G \cdot \xi_k^2| = \begin{cases} 1, & k = 0, \\ 0, & k \text{ odd,} \\ \chi(-1)^{k/2} (A - B), & k \text{ even,} \end{cases}$$

where

$$A - B$$

$$\begin{aligned} &= \frac{q^{m^2} \prod_{i=1}^m (i)}{2q^{(m-k)^2} \prod_{j=1}^{m-k} (j) q^{k(m-k)} q^{k(k-1)/2} \prod_{i=0}^{k/2-1} (2i)} \left(\frac{1}{1 - q^{-k/2}} - \frac{1}{1 + q^{-k/2}} \right) \\ &= q^{k(2m-k)/2} \frac{\prod_{i=m-k+1}^m (i)}{\prod_{i=1}^{k/2} (2i)}. \end{aligned}$$

The orbital decomposition of $\text{Alt}(\mathbf{F}_q)$ into disjoint orbits is known [4] to be

$$\text{Alt}(\mathbf{F}_q) = \{0\} \cup \left\{ \bigcup_{k=1}^{\lfloor \frac{m}{2} \rfloor} G \cdot \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} \right\},$$

where $\lfloor \cdot \rfloor$ is the greatest integer function and E_r is the $2r \times 2r$ block matrix with r copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ down the main diagonal and zeros elsewhere. The stabilizer of $\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}$ is the set of all $\begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix} \in GL_m(\mathbf{F}_q)$ such that $g_{11} \in Sp_{2r}(\mathbf{F}_q)$, $g_{12} \in \text{Mat}_{2r,m-2r}(\mathbf{F}_q)$, and $g_{22} \in GL_{m-2r}(\mathbf{F}_q)$. Thus, we see that

$$\left| G \cdot \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} \right| = q^{r(2m-2r-1)} \frac{\prod_{i=1}^{2r} (m - 2r + i)}{\prod_{l=1}^r (2l)}.$$

Following the computation of the partial integrals in [8], we see that

$$\begin{aligned} I_{m,k}(t, \chi) &= \int_{\text{diag}(\alpha_1, \dots, \alpha_k, 0) + \sqrt{-\pi} \text{Alt}_m(\mathbf{F}_q) + \pi H_m(O_L)} \chi(\det(x)) |\det(x)|^s dx \\ &= \chi(\alpha_1) \dots \chi(\alpha_k) q^{-k(2m-k+1)/2} \\ &\quad \times \int_{\sqrt{-\pi} \text{Alt}_{m-k}(\mathbf{F}_q) + \pi H_{m-k}(O_L)} \chi(\det(x)) |\det(x)|^s dx \end{aligned}$$

and that

$$\begin{aligned} J_{m,2r}(t, \chi) &= \int_{\sqrt{-\pi} \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} + \pi H_m(O_L)} \chi(\det(x)) |\det(x)|^s dx \\ &= \chi(-1)^r q^{-m^2} t^{m-r} Z_{m-2r}(t, \chi) \end{aligned}$$

for $\alpha_1, \alpha_2, \dots, \alpha_k \in U_K$, $0 \leq k \leq m$, and $0 \leq r \leq [m/2]$.

Applying the key lemma for orbital decomposition [8] and using the formulae for the cardinality of the orbits of $H_m(\mathbf{F}_q)$ and the first partial integral, we find that

$$\begin{aligned} Z_m(t, \chi) &= \sum_{k=0}^m (|G \cdot \xi_k^1| - |G \cdot \xi_k^2|) q^{-k(2m-k+1)/2} \\ &\quad \times \int_{\sqrt{-\pi} \text{Alt}_{m-k}(\mathbf{F}_q) + \pi H_{m-k}(O_L)} \chi(\det(x)) |\det(x)|^s dx. \end{aligned}$$

By our formula for the difference between these orbits, we know that the difference is non-zero only for even values of k , and our formula becomes

$$\begin{aligned} Z_m(t, \chi) &= \sum_{k=0}^{[m/2]} \chi(-1)^k q^{-k} \frac{\prod_{i=m-2k+1}^m (i)}{\prod_{i=1}^k (2i)} \\ &\quad \times \int_{\sqrt{-\pi} \text{Alt}_{m-2k}(\mathbf{F}_q) + \pi H_{m-2k}(O_L)} \chi(\det(x)) |\det(x)|^s dx. \end{aligned}$$

Applying the key lemma again and using the formulae for the cardinality of the orbits of $\text{Alt}_{m-k}(\mathbf{F}_q)$ and the second partial integral, we have that

$$\begin{aligned} Z_m(t, \chi) &= \sum_{k=0}^{[m/2]} \chi(-1)^k q^{-k} \frac{\prod_{i=m-2k+1}^m (i)}{\prod_{i=1}^k (2i)} \sum_{r=0}^{[(m-2k)/2]} \left| G \cdot \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} \right| \\ &\quad \times \int_{\sqrt{-\pi} \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} + \pi H_{m-2k}(O_L)} \chi(\det(x)) |\det(x)|^s dx \\ &= \sum_{k=0}^{[m/2]} \chi(-1)^k q^{-k} \frac{\prod_{i=m-2k+1}^m (i)}{\prod_{i=1}^k (2i)} \sum_{r=0}^{[(m-2k)/2]} q^{r(2m-4k-2r-1)} \\ &\quad \times \frac{\prod_{i=1}^{2r} (m-2k-2r+i)}{\prod_{i=1}^r (2i)} \chi(-1)^r q^{-(m-2k)^2} t^{m-2k-r} \\ &\quad \times Z_{m-2k-2r}(t, \chi). \end{aligned}$$

We see immediately from this recursion formula that $Z_1(t, \chi) = 0$. This then implies, by induction, that $Z_m(t, \chi) = 0$ for all odd m . To complete the proof of our theorem, we let $m = 2n$, substitute our closed form expression for $Z_m(t, \chi)$ into the recursion formula above, divide both sides by $Z_m(t, \chi)$, and change the order of

summation by letting $k \rightarrow n - k$. After these steps, the following identity remains to be proven:

$$1 = \sum_{k=0}^n \frac{\prod_{i=1}^{2(n-k)} (2k+i)}{\prod_{i=1}^{n-k} (2i)} q^{-4k^2} \frac{\prod_{i=k+1}^n (2i, 1)}{\prod_{i=k+1}^n (2i-1)} \\ \times \sum_{r=0}^k q^{2r(2k-r)} t^{2k-r} \frac{\prod_{i=1}^{2r} (2k-2r+i) \prod_{i=k-r+1}^k (2i, 1)}{\prod_{i=1}^r (2i) \prod_{i=k-r+1}^k (2i-1)}.$$

Using the Gauss identity [3] and Lemma 3 in [11] with $x = q^{-2}$, we can show that the inner sum above is precisely $q^{2k^2} t^k$. For completeness, we state Lemma 3 in [11] without proof.

Lemma 3 ([11]). *For any non-negative integer k , the following identity holds:*

$$1 = \sum_{j=0}^k x^{j^2} t^j \prod_{i=1}^{k-j} \frac{(1-x^{j+i})}{(1-x^i)} \prod_{i=1}^{k-j} (1-x^{j+i}t).$$

Simplifying, our identity becomes:

$$1 = \sum_{k=0}^n q^{-2k^2} t^k \prod_{i=1}^{n-k} \frac{(2(k+i))}{(2i)} \prod_{i=k+1}^n (2i, 1)$$

and another application of Lemma 3 in [11] proves the theorem. \square

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