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COUNTABLE NETWORK WEIGHT AND MULTIPLICATION OF BOREL SETS

D. H. FREMLIN, R. A. JOHNSON, AND E. WAJCH

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Abstract. A space X Borel multiplies with a space Y if each Borel set of $X \times Y$ is a member of the σ-algebra in $X \times Y$ generated by Borel rectangles. We show that a regular space X Borel multiplies with every regular space if and only if X has a countable network. We give an example of a Hausdorff space with a countable network which fails to Borel multiply with any non-separable metric space. In passing, we obtain a characterization of those spaces which Borel multiply with the space of countable ordinals, and an internal necessary and sufficient condition for X to Borel multiply with every metric space.

Introduction

For a topological space X, denote by $\mathcal{B}(X)$ the collection of Borel sets of X, i.e. the smallest σ-algebra containing the open sets of X. If X and Y are topological spaces, then $\mathcal{B}(X) \otimes \mathcal{B}(Y)$ denotes the smallest σ-algebra in $X \times Y$ containing all sets of the form $E \times F$ where E and F are Borel sets of X and Y, respectively. The inclusion $\mathcal{B}(X) \otimes \mathcal{B}(Y) \subseteq \mathcal{B}(X \times Y)$ always holds; if $\mathcal{B}(X) \otimes \mathcal{B}(Y) = \mathcal{B}(X \times Y)$, we say that X Borel multiplies with Y. A space X will be called a universal Borel multiplier if X Borel multiplies with every space. Considerable attention has been devoted to the Borel multiplication of discrete spaces (cf. e.g. [1], [8], [9], [12], [13] & [15]); however, the following natural problem does not seem to have been posed explicitly:

Problem. *Give an internal characterization of universal Borel multipliers.*

It is well known that any second countable space X Borel multiplies with every space (cf. [7, Thm. 8]). It is easy to observe that if X is regular, the assumption of second countability can be weakened to the requirement that X be of countable network weight. We shall prove that countable network weight is also a necessary condition for a regular space to be a universal Borel multiplier. To this end, we shall show first that if a space X Borel multiplies with the space ω_1 of countable ordinals equipped with the order topology, then X is hereditarily Lindelöf. Further, we shall observe that if a regular Hausdorff hereditarily Lindelöf space X Borel multiplies with any one of its Hausdorff compactifications, then X has a countable network. The above-mentioned results imply that a regular (not necessarily Hausdorff) space

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X is a universal Borel multiplier if and only if X is of countable network weight. Since a useful characterization of spaces Borel multiplying with ω_1 seems to be interesting in itself, we shall take the opportunity to prove that X Borel multiplies with ω_1 if and only if X is a hereditarily separable hereditarily Lindelöf space which Borel multiplies with the discrete space of size ω_1. We shall find an internal necessary and sufficient condition for X to Borel multiply with every metric space. Finally, we shall give an example of a Hausdorff space of countable network weight which fails to Borel multiply with the discrete space of size ω_1.

We shall use $hd(X)$ and $hl(X)$ to denote, respectively, the hereditary density and the hereditary Lindelöf degree of X. Basic facts concerning the cardinal functions considered here can be found, for instance, in [2] and [10]. The real line with the usual topology induced by the metric $|x-y|$ will be denoted by $\mathbb{R}$. The symbol $\mathcal{P}(X)$ will stand for the collection of all subsets of X. For a cardinal κ, $D(\kappa)$ will denote the discrete space of size κ.

The results

For completeness, let us begin with the following

1. **Proposition.** *Let X and Y be topological spaces. If X has a countable network consisting of Borel sets, then $\mathcal{B}(X)\otimes\mathcal{B}(Y)=\mathcal{B}(X\times Y)$.*

Proof. Suppose that $\langle E_n\rangle_{n<\omega}$ is a countable collection of Borel sets of X which serves as a network for X. To check that $\mathcal{B}(X\times Y)\subseteq\mathcal{B}(X)\otimes\mathcal{B}(Y)$, it suffices to observe that if $W\subseteq X\times Y$ is open, then $W=\bigcup_{n<\omega}(E_n\times V_n)$ where $V_n=\bigcup\{V : V$ is open in Y and $E_n\times V\subseteq W\}$ for $n<\omega$.

2. **Corollary.** *A regular space of countable network weight is a universal Borel multiplier.*

If $M\subseteq X\times Y$, the horizontal section of M at $y\in Y$ is defined by $M^{-1}[y]=\{x\in X : (x,y)\in M\}$, and the vertical section of M at $x\in X$ is given by $M[x]=\{y\in Y : (x,y)\in M\}$.

3. **Lemma.** *Let $\mathcal{A}_X$ be a σ-algebra of subsets of X, let $\mathcal{I}$ be a σ-ideal of subsets of Y, and let $\mathcal{A}_Y$ be the smallest σ-algebra of subsets of Y which contains $\mathcal{I}$. That is, $\mathcal{A}_Y=\{A\subseteq Y : A\in\mathcal{I}$ or $Y\setminus A\in\mathcal{I}\}$. Then, for each $M\in\mathcal{A}_X\otimes\mathcal{A}_Y$, we have*

$$\bigcup\{M[x] : M[x]\in\mathcal{I}\}\in\mathcal{I}\quad\text{and}\quad\bigcup\{Y\setminus M[x] : M[x]\notin\mathcal{I}\}\in\mathcal{I}.$$

Proof. Let $\mathcal{M}$ be the family of all the sets $M\subseteq X\times Y$ which have the property that there exists $L\in\mathcal{I}$ such that, for each $x\in X$, $M[x]\subseteq L$ if $M[x]\in\mathcal{I}$, and $Y\setminus M[x]\subseteq L$ if $M[x]\notin\mathcal{I}$. Then $\mathcal{M}$ is stable under complements; indeed, the same L will work for a set M and its complement. To show that $\mathcal{M}$ is closed under countable unions, suppose that $\langle M_n\rangle_{n<\omega}$ is a sequence of members of $\mathcal{M}$. For any $n<\omega$, choose $L_n\in\mathcal{I}$ such that, for each $x\in X$, $M_n[x]\subseteq L_n$ if $M_n[x]\in\mathcal{I}$, and $Y\setminus M_n[x]\subseteq L_n$ if $M_n[x]\notin\mathcal{I}$. Put $L=\bigcup_{n<\omega}L_n$. Obviously, $L\in\mathcal{I}$. If $(\bigcup_{n<\omega}M_n)[x]\in\mathcal{I}$, then $M_n[x]\in\mathcal{I}$ for any $n<\omega$, so that $\bigcup_{n<\omega}M_n[x]\subseteq L$. On the other hand, if $(\bigcup_{n<\omega}M_n)[x]\notin\mathcal{I}$, then there exists $m<\omega$ such that $M_m[x]\notin\mathcal{I}$; for this m we have $Y\setminus M_m[x]\subseteq L$, which implies that $Y\setminus(\bigcup_{n<\omega}M_n)[x]\subseteq L$. We have therefore shown that $\mathcal{M}$ is a σ-algebra. Clearly, $\mathcal{M}$ contains each rectangle of the form $E\times F$ where $E\in\mathcal{A}_X$ and $F\in\mathcal{A}_Y$ (simply, let $L=F$ if $F\in\mathcal{I}$, and $L=Y\setminus F$ otherwise). Thus $\mathcal{M}$ contains $\mathcal{A}_X\otimes\mathcal{A}_Y$ and the result follows.

4. **Theorem.** *A topological space X Borel multiplies with ω_1 if and only if X is a hereditarily separable hereditarily Lindelöf space which Borel multiplies with $D(\omega_1)$.*

Proof. Necessity. Assume that $\mathcal{B}(X) \otimes \mathcal{B}(\omega_1) = \mathcal{B}(X \times \omega_1)$. By [8, Lemma 1.1], it is obvious that X Borel multiplies with $D(\omega_1)$. Let μ be the Dieudonné measure on ω_1 (cf. [3, 5.5, p. 974]). Then $\mathcal{B}(\omega_1)$ is contained in the σ-algebra generated by the σ-ideal $\mathcal{I} = \{A \subseteq \omega_1 : \text{there is } B \in \mathcal{B}(\omega_1) \text{ with } A \subseteq B \text{ and } \mu(B) = 0\}$. Suppose, if possible, that $hd(X) > \omega$. There exists a transfinite sequence $\langle x_\xi \rangle_{\xi<\omega_1}$ of points of X such that $x_\xi \notin \operatorname{cl}\{x_\eta : \eta < \xi\}$ for any $\xi < \omega_1$. Put

$$W = \bigcup_{\xi<\omega_1} \big((X \setminus \operatorname{cl}\{x_\eta : \eta < \xi\}) \times \xi\big).$$

The set W being open in $X \times \omega_1$, it follows from Lemma 3 that $\mu(\bigcup\{W[x] : \mu(W[x]) = 0\}) = 0$. On the other hand, since $\mu(\xi) = 0$ and $W[x_\xi] = \xi$ for any $\xi < \omega_1$, we have $\bigcup\{W[x] : \mu(W[x]) = 0\} = \omega_1$; hence $\mu(\bigcup\{W[x] : \mu(W[x]) = 0\}) = 1$. The contradiction obtained shows that $hd(X) \le \omega$.

Now, suppose that $hl(X) > \omega$. There exists a collection $\langle V_\xi \rangle_{\xi<\omega_1}$ of open subsets of X such that $V_\xi \subseteq V_\eta$ and $V_\xi \neq V = \bigcup_{\tau<\omega_1} V_\tau$ for any $\xi < \eta < \omega_1$. Let

$$U = \bigcup_{\xi<\omega_1} \big(V_\xi \times [\xi + 1, \omega_1)\big).$$

For any $x \in V$, let ξ_x be the first ordinal such that $x \in V_{\xi_x}$. Then $U[x] = [\xi_x + 1, \omega_1)$ for $x \in V$, hence $\bigcup\{\omega_1 \setminus U[x] : \mu(U[x]) = 1\} = \omega_1$. Since U is open in $X \times \omega_1$, it follows from Lemma 3 that $\mu(\bigcup\{\omega_1 \setminus U[x] : \mu(U[x]) = 1\}) = 0$, which is absurd. Therefore $hl(X) \le \omega$.

Sufficiency. Given a closed set $F \subseteq X \times \omega_1$, put

$$A = \{x \in X : F[x] \quad \text{is unbounded}\}.$$

Consider any $x \in X \setminus A$. We shall show that there exist an open neighbourhood U_x of x and an ordinal $\gamma_x < \omega_1$, such that

$$\big(U_x \times [\gamma_x, \omega_1)\big) \cap F = \emptyset.$$

To this end, choose $\eta_x < \omega_1$ with $F[x] \subseteq \eta_x$. If $\eta_x < \alpha < \omega_1$, then $(\{x\} \times [\eta_x, \alpha]) \cap F = \emptyset$; hence, by the compactness of $[\eta_x, \alpha]$, there exists an open neighbourhood G_α of x such that $(G_\alpha \times [\eta_x, \alpha]) \cap F = \emptyset$. Let $U_x = \bigcap_{\alpha>\eta_x} \bigcup_{\beta>\alpha} G_\beta$. Since $hd(X) \le \omega$, there exists a countable $B \subseteq X \setminus U_x$ such that $\operatorname{cl} B = X \setminus U_x$. For any $y \in B$, we can find $\alpha_y > \eta_x$ with $y \in \bigcap_{\beta>\alpha_y}(X \setminus G_\beta)$. Let $\gamma_x = \sup_{y\in B} \alpha_y$. Then $B \subseteq \bigcap_{\beta>\gamma_x}(X \setminus G_\beta)$, which implies that $X \setminus U_x = \bigcap_{\beta>\gamma_x}(X \setminus G_\beta)$. In consequence, U_x is an open neighbourhood of x. It is easily seen that $U_x \times [\gamma_x, \omega_1)$ does not meet F. Obviously, $U_x \subseteq X \setminus A$, which means that A is closed. Since $hl(X) \le \omega$, there exists a countable $C \subseteq X \setminus A$ such that $\bigcup_{x\in X\setminus A} U_x = \bigcup_{x\in C} U_x$. For $\zeta_0 = \sup_{x\in C} \gamma_x$, we have

$$\big((X \setminus A) \times [\zeta_0, \omega_1)\big) \cap F = \emptyset.$$

As $hd(X) \le \omega$, there exists a countable $D \subseteq A$ such that $\operatorname{cl} D = A$. Put

$$H = \bigcap_{x\in D} \big(F[x] \cap [\zeta_0, \omega_1)\big).$$

Let us observe that if $\xi \in H$, then $D \subseteq F^{-1}[\xi] \subseteq A$, so that $F^{-1}[\xi] = A$ for any $\xi \in H$. This yields that $F \cap (X \times H) = A \times H \in \mathcal{B}(X) \otimes \mathcal{B}(\omega_1)$. Further, since the

collection of closed unbounded subsets of ω_1 is stable under countable intersections, the set H is closed and unbounded. Accordingly $\omega_1 \setminus H$ is a metrizable open subspace of ω_1. By Theorem 2.5 of [8], X Borel multiplies with $\omega_1 \setminus H$. Therefore $F \cap \big(X \times (\omega_1 \setminus H)\big) \in \mathcal{B}(X) \otimes \mathcal{B}(\omega_1)$, which completes the proof.

A satisfactory solution of the problem of when a space can Borel multiply with $D(\omega_1)$ is unknown. Even the answer to the question whether $\mathcal{B}(D(\kappa)) \otimes \mathcal{B}(D(\omega_1)) = \mathcal{B}(D(\kappa) \times D(\omega_1))$ for $\kappa \leq 2^\omega$ depends on one's set theory (cf. e.g. [1] & [9]). However, with Theorem 4 in hand it is easy to check that, for instance, the Sorgenfrey line Borel multiplies with ω_1. To see this, let us give an internal characterization of those spaces which Borel multiply with every metric space.

Given a collection $\mathcal{G}$ of subsets of X, denote by $[\mathcal{G}]_\sigma$ and $[\mathcal{G}]_c$ the collections of, respectively, countable unions and complements of members of $\mathcal{G}$. Put $\mathcal{B}_0(\mathcal{G}) = [\mathcal{G}]_\sigma \cup \{\emptyset\}$ and, for any non-zero ordinal $\alpha < \omega_1$, define

$$\mathcal{B}_\alpha(\mathcal{G}) = \begin{cases} \big[\bigcup_{\eta<\alpha} \mathcal{B}_\eta(\mathcal{G})\big]_\sigma & \text{when } \alpha \text{ is even,} \\ \big[\bigcup_{\eta<\alpha} \mathcal{B}_\eta(\mathcal{G})\big]_c & \text{when } \alpha \text{ is odd.} \end{cases}$$

Then $\bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha(\mathcal{G})$ equals the smallest σ-algebra containing $\mathcal{G}$.

5. **Theorem.** *A topological space X Borel multiplies with every metric space if and only if there exist a countable ordinal α and a countable collection $\mathcal{G}$ of open subsets of X, such that any open set in X lies in $\mathcal{B}_\alpha(\mathcal{G})$.*

Proof. Necessity. Let $\{W_y : y \in Y\}$ be the collection of all open subsets of X. Give Y the discrete topology and put $W = \bigcup_{y\in Y}(W_y \times \{y\})$. If $W \in \mathcal{B}(X) \otimes \mathcal{B}(Y)$, then, in view of [5, Exercise 59.2, p. 261], there exist countable collections $\mathcal{G}$ of open subsets of X and $\mathcal{A}$ of subsets of Y, such that W lies in the smallest σ-algebra which contains all rectangles of the form $G \times A$ where $G \in \mathcal{G}$ and $A \in \mathcal{A}$. We can find a countable ordinal α such that $W \in \mathcal{B}_\alpha(\{G \times A : G \in \mathcal{G} \quad \& \quad A \in \mathcal{A}\})$. It is readily seen that $\{W_y : y \in Y\} \subseteq \mathcal{B}_\alpha(\mathcal{G})$.

Sufficiency. Suppose now that we are given a countable ordinal α and a countable collection $\mathcal{G}$ of open subsets of X, such that any open set in X lies in $\mathcal{B}_\alpha(\mathcal{G})$. Let Y be any discrete space. One easily checks that if $W \subseteq X \times Y$ is such that $W^{-1}[y] \in \mathcal{B}_0(\mathcal{G})$ for any $y \in Y$, then $W \in \mathcal{B}(X) \otimes \mathcal{B}(Y)$. Suppose that we have already shown that if $\xi < \alpha$, and $W \subseteq X \times Y$ is such that $W^{-1}[y] \in \mathcal{B}_\xi(\mathcal{G})$ for any $y \in Y$, then $W \in \mathcal{B}(X) \otimes \mathcal{B}(Y)$. Consider any set $U \subseteq X \times Y$ such that $U^{-1}[y] \in \mathcal{B}_\alpha(\mathcal{G})$ for any $y \in Y$. Let α be even. For any $y \in Y$, choose a sequence $\langle \xi_n(y)\rangle_{n<\omega}$ of ordinals $< \alpha$ and sets $A_n(y) \in \mathcal{B}_{\xi_n(y)}(\mathcal{G})$, such that $U^{-1}[y] = \bigcup_{n<\omega} A_n(y)$. Let $C_{n,\xi} = \{y \in Y : \xi_n(y) = \xi\}$ and $U_{n,\xi} = \bigcup\{A_n(y) \times \{y\} : y \in C_{n,\xi}\}$ for $\xi < \alpha$ and $n < \omega$. Under the inductive assumption, $U_{n,\xi} \in \mathcal{B}(X) \otimes \mathcal{B}(Y)$ for any $\xi < \alpha$ and $n < \omega$. Since $U = \bigcup_{\xi<\alpha} \bigcup_{n<\omega} U_{n,\xi}$, we have that $U \in \mathcal{B}(X) \otimes \mathcal{B}(Y)$. If α is odd, we can use similar arguments to show that $(X \times Y) \setminus U \in \mathcal{B}(X) \otimes \mathcal{B}(Y)$. In consequence, we obtain that X Borel multiplies with Y. By Theorem 2.5 of [8], X Borel multiplies with every metric space.

6. **Corollary.** *The Sorgenfrey line Borel multiplies with every metric space and with ω_1.*

Obviously, the Sorgenfrey line does not Borel multiply with itself for it contains only 2^ω Borel sets, while its square contains 2^{2^ω} Borel sets.

Let us come back to the question whether a universal Borel multiplier is of countable network weight.

7. **Lemma.** *If $\mathcal{E}$ is a collection of subsets of a set Y and if F lies in the smallest σ-algebra containing $\mathcal{E}$, then*

$$\big(F \times (Y \setminus F)\big) \cup \big((Y \setminus F) \times F\big) \subseteq \bigcup_{E \in \mathcal{E}} \big(\big(E \times (Y \setminus E)\big) \cup \big((Y \setminus E) \times E\big)\big).$$

Proof. It is enough to observe that $\mathcal{M} = \big\{M \subseteq Y : \big(M \times (Y \setminus M)\big) \cup \big((Y \setminus M) \times M\big) \subseteq \bigcup_{E \in \mathcal{E}} \big(\big(E \times (Y \setminus E)\big) \cup \big((Y \setminus E) \times E\big)\big)\big\}$ is a σ-algebra which contains $\mathcal{E}$.

8. **Proposition.** *Let X be a hereditarily Lindelöf regular Hausdorff space. If there exists a Hausdorff compactification αX of X such that $\mathcal{B}(X \times \alpha X) = \mathcal{B}(X) \otimes \mathcal{B}(\alpha X)$, then X has a countable network.*

Proof. The diagonal Δ of X is a closed subset of $X \times \alpha X$; hence, since $\Delta \in \mathcal{B}(X) \otimes \mathcal{B}(\alpha X)$, there exists a countable collection $\mathcal{G}$ of open subsets of αX such that Δ lies in the smallest σ-algebra containing all sets of the form $(G \cap X) \times H$ where $G, H \in \mathcal{G}$ (cf. [5, Exercise 59.2, p. 261]). It follows from Lemma 7 that, for any $x \in X$ and $y \in \alpha X$ with $x \neq y$, there exists $G \in \mathcal{G}$ such that exactly one of the points x, y lies in G. Since X is regular and hereditarily Lindelöf, for any $G \in \mathcal{G}$, there exists a countable collection $\mathcal{F}_G$ of closed subsets of αX such that $G \cap X \subseteq \bigcup_{F \in \mathcal{F}_G} F \subseteq G$. Put $\mathcal{F} = \{\alpha X \setminus G : G \in \mathcal{G}\} \cup \bigcup_{G \in \mathcal{G}} \mathcal{F}_G$. The collection $\mathcal{F}$ is countable and has the property that, for any $x \in X$ and $y \in \alpha X$ with $x \neq y$, there exists $F \in \mathcal{F}$ such that $x \in F$ and $y \notin F$. This implies that if U is an open subset of αX and if $x \in X \cap U$, then the collection $\{\alpha X \setminus F : F \in \mathcal{F} \quad \& \quad x \in F\}$ is an open cover of $\alpha X \setminus U$. There exists a finite subcollection $\mathcal{F}(x)$ of $\mathcal{F}$ such that $x \in \bigcap_{F \in \mathcal{F}(x)} F$ and $\alpha X \setminus U \subseteq \bigcup_{F \in \mathcal{F}(x)} (\alpha X \setminus F)$. This shows that the collection of all finite intersections of members of $\{F \cap X : F \in \mathcal{F}\}$ is a countable network for X.

Since the discrete space $D(\omega_1)$ Borel multiplies with itself (cf. [9, Thm. 12.5(ii)] or [12, Thm. 2]), it can serve as an example of a space of uncountable network weight which Borel multiplies with its one-point compactification (cf. [8, Lemma 1.2]).

Now, we are in a position to state our main theorem.

9. **Theorem.** *A regular space X is a universal Borel multiplier if and only if X has a countable network.*

Proof. For $x, y \in X$, write $x \backsim y$ if and only if x, y belong to just the same open sets in X. It is easy to check that the quotient space $X/_{\backsim}$ Borel multiplies with Y if and only if X does; $X/_{\backsim}$ has a countable network if and only if X has; further, $X/_{\backsim}$ is regular if and only if X is. Obviously, if X is regular, the space $X/_{\backsim}$ is Hausdorff. Hence, in view of Corollary 2, it suffices to prove that a regular Hausdorff universal Borel multiplier is of countable network weight. But this follows directly from Theorem 4, Proposition 8 and from the fact that every regular Hausdorff Lindelöf space is Tychonoff and, in consequence, it has a Hausdorff compactification.

Remarks. Let us observe that the collection $\mathcal{F}$ obtained in the proof of Proposition 8 witnesses that X is a Σ-space; therefore, in view of Theorem 4 and [8, Lemma 1.1], the "only if" part of Theorem 9 can be deduced from Corollary 4.13 of [6]

or from Theorems 4.4 and 4.5 of [4]. This was the original route to Theorem 9, discovered with the help of A.V. Arhangel'skii.

The problem whether every Hausdorff universal Borel multiplier is of countable network weight is still open. It is easily seen that

$$\mathcal{B}((X_1 \times X_2) \times Y) = \mathcal{B}(X_1 \times X_2) \otimes \mathcal{B}(Y)$$

whenever $\mathcal{B}(X_1 \times (X_2 \times Y)) = \mathcal{B}(X_1) \otimes \mathcal{B}(X_2 \times Y)$ and $\mathcal{B}(X_2 \times Y) = \mathcal{B}(X_2) \otimes \mathcal{B}(Y)$. This, along with Lemma 1.5 of [8], yields that if $\langle X_n \rangle_{n<\omega}$ is a sequence of topological spaces each one of which is a universal Borel multiplier, then $\prod_{n<\omega} X_n$ is a universal Borel multiplier. Of course, this remark is of no interest unless there are universal Borel multipliers without countable networks.

Let us finish by giving the promised example of a Hausdorff space of countable network weight which does not Borel multiply with the discrete space of size ω_1.

10. **Example.** Give the set $X = \omega_1$ the cofinite topology. Let $\langle D_\xi \rangle_{\xi<\omega_1}$ be a partition of $\mathbb{R}$ into Bernstein sets (cf. [11, Thm. 27, p. 152]) and let $T = \{(\xi, r) \in X \times \mathbb{R} : r \in D_\xi\}$ be regarded as a subspace of $X \times \mathbb{R}$. If $(\xi_1, r_1), (\xi_2, r_2)$ are distinct points of T, then $r_1 \neq r_2$; hence T is Hausdorff. Since both X and $\mathbb{R}$ are of countable network weight, the space T is of countable network weight.

Put $\mathcal{E} = \big\{E \subseteq T : \text{there exist } \xi < \omega_1 \text{ and } H \in \mathcal{B}(\mathbb{R}) \text{ with } E \cap \big((\omega_1 \setminus \xi) \times \mathbb{R}\big) = T \cap \big((\omega_1 \setminus \xi) \times H\big)\big\}$. Observe that if $\xi < \omega_1$ and $H \in \mathcal{B}(\mathbb{R})$ witness that $E \in \mathcal{E}$, then ξ and $\mathbb{R} \setminus H$ yield that $T \setminus E \in \mathcal{E}$; if, for $n < \omega$, $\xi_n < \omega_1$ and $H_n \in \mathcal{B}(\mathbb{R})$ witness that $E_n \in \mathcal{E}$, then $\sup_{n<\omega} \xi_n$ and $\bigcup_{n<\omega} H_n$ show that $\bigcup_{n<\omega} E_n \in \mathcal{E}$. Hence $\mathcal{E}$ is closed under complements and countable unions.

Let $A \subseteq X$ be a finite set and let G be an open subset of $\mathbb{R}$. Choose $\xi < \omega_1$ with $A \subseteq \xi$. Then $\big((X \setminus A) \times G\big) \cap T \cap \big((\omega_1 \setminus \xi) \times \mathbb{R}\big) = T \cap \big((\omega_1 \setminus \xi) \times G\big)$; hence $\mathcal{E}$ contains all members of the natural base of the topology of T. Since T is hereditarily Lindelöf and $\mathcal{E}$ is closed under countable unions, $\mathcal{E}$ contains all open subsets of T. In consequence, $\mathcal{B}(T) \subseteq \mathcal{E}$ because $\mathcal{E}$ is a σ-algebra.

Let Y be any set. Let $\mathcal{A}$ be the collection of all sets $A \subseteq T \times Y$ such that there is a $\xi < \omega_1$ such that, for each $y \in Y$, there exists a Borel set $H \subseteq \mathbb{R}$ with $\{t \in T : (t, y) \in A\} \cap \big((\omega_1 \setminus \xi) \times \mathbb{R}\big) = T \cap \big((\omega_1 \setminus \xi) \times H\big)$. It is easily seen that $\mathcal{A}$ is a σ-algebra. Since $\mathcal{B}(T) \subseteq \mathcal{E}$, $\mathcal{A}$ contains all the sets $E \times F$ with $E \in \mathcal{B}(T)$ and $F \subseteq Y$. Hence $\mathcal{B}(T) \otimes \mathcal{P}(Y) \subseteq \mathcal{A}$.

Now, let $Y = \omega_1$ be considered with the discrete topology. Put

$$W = \bigcup_{\xi<\omega_1} \big(\big(((X \setminus \{\xi\}) \times \mathbb{R}) \cap T\big) \times \{\xi\}\big).$$

Then W is open in $T \times Y$. Suppose that $W \in \mathcal{B}(T) \otimes \mathcal{B}(Y)$. Then $W \in \mathcal{A}$, thus there exist $\xi < \omega_1$ and a Borel set $H \subseteq \mathbb{R}$, such that $E = \{t \in T : (t, \xi) \in W\} \cap \big((\omega_1 \setminus \xi) \times \mathbb{R}\big) = T \cap \big((\omega_1 \setminus \xi) \times H\big)$. Then $\emptyset = \{r \in \mathbb{R} : (\xi, r) \in E\} = H \cap D_\xi$, so, H is countable. On the other hand, $D_{\xi+1} = \{r \in \mathbb{R} : (\xi+1, r) \in E\} = H \cap D_{\xi+1}$, so that $D_{\xi+1} \subseteq H$ and H is cocountable, which is absurd. Hence $W \notin \mathcal{B}(T) \otimes \mathcal{B}(Y)$.

In view of Proposition 1, the space T cannot have a countable network consisting of Borel sets.

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Department of Mathematics, Essex University, Colchester C04 3SQ, England
E-mail address: `fremdh@essex.ac.uk`

Department of Mathematics, Washington State University, Pullman, Washington 99164
E-mail address: `johnson@beta.math.wsu.edu`

Institute of Mathematics, University of Łódź, S. Banacha 22, 90-238 Łódź, Poland
E-mail address: `ewajch@krysia.uni.lodz.pl`