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HOMOTOPY PERIODICITY AND COHERENCE

ROSS GEOGHEGAN AND ANDREW NICAS

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Abstract. If $f\colon Z \to Z$ is a periodic homotopy equivalence ($\mathrm{id}_Z \simeq f^p$) or a homotopy idempotent ($f \simeq f^2$), the question arises whether this periodicity property can be achieved by a homotopy "compatible with" f. These coherence questions are answered.

0. Introduction

Let Z be a path connected space. A map $f\colon Z \to Z$ is a *periodic homotopy idempotent* if, for some $r \geq 0$ and some $p > 0$, f^r is homotopic to f^{r+p}. This notion unifies two familiar special cases: *homotopy idempotents* ($f \simeq f^2$) and *periodic homotopy equivalences* ($\mathrm{id}_Z \simeq f^p$). In the course of our study of mapping tori in our paper [GN], we encountered a requirement for a "coherence condition" in achieving $f^r \simeq f^{r+p}$. To be precise, we say $f\colon Z \to Z$ is an *eventually coherent periodic homotopy idempotent* if there exist integers $r \geq 0$, $q > 0$ and $m \geq 0$, as well as a homotopy $N\colon f^r \simeq f^{r+q}$, and a map $J\colon Z \times I \times I \to Z$ making the following diagram commute up to homotopy rel $Z \times \{0,1\} \times I$:

$$
\begin{array}{ccccc}
Z \times I & \xrightarrow{f \times \mathrm{id}} & Z \times I & \xrightarrow{N} & Z \\
\downarrow{\scriptstyle N} & & & & \downarrow{\scriptstyle f^m} \\
Z & \xrightarrow{f^{m+1}} & Z & = & Z
\end{array}
\tag{0.1}
$$

The *period* of such a map f is the least $q > 0$ for which there exist r, N, J and m as above. If such r, N, J and m do not exist, the period is undefined (even in the case f is a periodic homotopy idempotent; see Proposition 3.2).

The questions arising in [GN] were: if f is a periodic homotopy idempotent, is f necessarily an eventually coherent periodic homotopy idempotent? and, if it is, how is the period q related to the least p for which $f^r \simeq f^{r+p}$ for some r? In this note we resolve these questions for the special cases of periodic homotopy equivalences (§2) and homotopy idempotents (§3). The main results are Theorems 2.4 and 3.4.

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1. The rotation index

We need some properties of mapping tori.

Suppose Z is a path connected space and has a basepoint $v \in Z$. Given a continuous map $f\colon Z \to Z$, its *mapping torus*, denoted by $T(Z,f)$, is the space obtained from $Z \times [0,1]$ by identifying $(z,1)$ with $(f(z),0)$ for each $z \in Z$. The image of $(z,u) \in Z \times [0,1]$ in $T(Z,f)$ will be denoted by $[z,u]$. Choose a basepath σ from v to $f(v)$.

Let $X = T(Z,f)$. Choose $w = [v,0]$ as a basepoint for X. There is a canonical map of X to the standard circle S^1 (realized as complex numbers of unit modulus) given by $p_f\colon X \to S^1$, $p_f([z,s]) = e^{2\pi\imath s}$. Let $i\colon Z \hookrightarrow X$ be the inclusion $z \mapsto [z,0]$.

For any space Y, Y^Y denotes the space of all maps $Y \to Y$ with the compact-open topology. We define $\Gamma_Y \equiv \pi_1(Y^Y, \mathrm{id})$. The group Γ_Y is abelian because Y^Y is an H-space with unit id_Y.

Let $\bar\gamma\colon I \to X$ be the path $\bar\gamma(u) = [v,u]$, and let $\gamma_0\colon I \to X$ be the path $\gamma_0 = \bar\gamma(i \circ \sigma)^{-1}$. Define a map $P_X\colon X^X \to (S^1)^{S^1}$ by $P_X(g)(e^{2\pi\imath u}) = p_f(g(\gamma_0(u)))$. (For the continuity of P_X and other function space maps, see [D, XII §2, 3].) Then P_X induces a homomorphism $(P_X)_\#\colon \Gamma_X \to \Gamma_{S^1}$. We define an identification $\Gamma_{S^1} \xrightarrow{\cong} \mathbb{Z}$ by sending the generator $[s \mapsto (e^{2\pi\imath u} \mapsto e^{2\pi\imath(u+s)})] \in \Gamma_{S^1}$ to $1 \in \mathbb{Z}$.

Definition 1.1. The *rotation degree* of $\gamma \in \Gamma_X$ is the integer $(P_X)_\#(\gamma)$.

Proposition 1.2. *If Z is path connected, then the rotation degree of $\gamma \in \Gamma_X$ is independent of the choice of basepoint and basepath.*

Proof. We first show that $(P_X)_\#(\gamma)$ does not depend on the choice of basepath. There is a commutative diagram

$$\begin{array}{ccc} X^X & \xrightarrow{\mathrm{ev}_w} & X \\ \big\downarrow{\scriptstyle P_X} & & \big\downarrow{\scriptstyle p_f} \\ (S^1)^{S^1} & \xrightarrow{\mathrm{ev}_1} & S^1 \end{array} \tag{1.3}$$

where ev_w is evaluation at the basepoint $w \equiv [v,0] \in X$ and ev_1 is evaluation at $1 \in S^1$. The map ev_1 is a fibration whose fiber over $1 \in S^1$ is a $K(\mathbb{Z},0)$; from the long exact homotopy sequence of this fibration we have that $(\mathrm{ev}_1)_\#\colon \Gamma_{S^1} \to \pi_1(S^1,1)$ is an isomorphism. Clearly, the composite $p_f \circ \mathrm{ev}_w$ is independent of the basepath and so by Diagram (1.3), $(P_X)_\#$ is also independent of the basepath.

Suppose Z is path connected and $v, v' \in Z$ are basepoints. Let $\mu\colon I \to Z$ be a path from v to v'. There is a basepoint-preserving homotopy $H\colon (X^X, \mathrm{id}) \times I \to (S^1, 1)$ given by $H(g,t) = p_f(g([\mu(t),0]))$. We have $H_0 = p_f \circ \mathrm{ev}_w$ and $H_1 = p_f \circ \mathrm{ev}_{w'}$, where $w' \equiv [v',0] \in X$, and so by Diagram (1.3), $(P_X)_\#$ is independent of the basepoint. □

Here is a more geometric interpretation of the rotation degree. Let $F\colon X \times I \to X$ be a homotopy from id_X to itself representing $\gamma \in \Gamma_X$. Consider the covering space $\bar X$ of X which is the infinite "mapping telescope" of f formed by gluing together end-to-end countably many copies of the mapping cylinder of f indexed by the integers. We may lift F to a homotopy $\bar F\colon \bar X \times I \to \bar X$ from $\mathrm{id}_{\bar X}$ to an integer covering translation; this integer is the rotation degree of γ.

Definition 1.4. The *rotation index* of a self-map $f\colon Z \to Z$ is the non-negative integer, $\mathrm{RI}(f)$, given by:

$$\mathrm{RI}(f) = \begin{cases} 0 & \text{if } (P_X)_\# = 0, \\ \text{index of image}(P_X)_\# \text{ in } \mathbb{Z} & \text{if } (P_X)_\# \neq 0 \end{cases}$$

where $X = T(Z, f)$ and $(P_X)_\#$ is the rotation degree homomorphism.

The connection between the rotation index and "period" as defined in the introduction is:

Proposition 1.5. *If $f\colon Z \to Z$ is a map for which $\mathrm{RI}(f) > 0$, then f is an eventually coherent periodic homotopy idempotent of period $\mathrm{RI}(f)$. Conversely, if f is an eventually coherent periodic homotopy idempotent, then its period is $\mathrm{RI}(f)$ (which is therefore positive).*

Proof. See Theorem 6.3 of [GN] and the paragraph preceding it. □

Proposition 1.6 (Homotopy Invariance). *Let $h\colon Z \to Z'$ be a homotopy equivalence. Suppose $f\colon Z \to Z$ and $f'\colon Z' \to Z'$ are such that the diagram*

$$\begin{array}{ccc} Z & \xrightarrow{f} & Z \\ \downarrow h & & \downarrow h \\ Z' & \xrightarrow{f'} & Z' \end{array}$$

is homotopy commutative. Then $\mathrm{RI}(f) = \mathrm{RI}(f')$.

For the proof of Proposition 1.6 we need:

Lemma 1.7. *Let $X = T(Z, f)$ and $X' = T(Z', f')$. There is a homotopy equivalence $\alpha\colon X \to X'$ such that the following diagram is homotopy commutative:*

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X' \\ \downarrow p_f & & \downarrow p_{f'} \\ S^1 & = & S^1. \end{array}$$

Proof. Write $\bar{X}$ and $\bar{X}'$ for the infinite mapping telescope covering spaces of X and X' respectively (see the discussion preceding Definition 1.4). The existence of an equivariant, end-preserving homotopy equivalence $\bar{\alpha}\colon \bar{X} \to \bar{X}'$ follows easily from Corollaries 2.4 and 2.5 of [F]. The details are most easily conveyed by means of a picture. (See Figure 1, where k is a homotopy inverse for h.) □

Proof of Proposition 1.6. Let $\alpha\colon X \to X'$ be as in Lemma 1.7, and let $\beta\colon X' \to X$ be a homotopy inverse for α which preserves basepoints (i.e. $\beta(w') = w$). Define $C\colon X^X \to (X')^{X'}$ by $C(g) = \alpha \circ g \circ \beta$. Consider the diagram:

$$\begin{array}{ccc} X^X & \xrightarrow{C} & (X')^{X'} \\ \downarrow \mathrm{ev}_w & & \downarrow \mathrm{ev}_{w'} \\ X & \xrightarrow{\alpha} & X' \\ \downarrow p_f & & \downarrow p_{f'} \\ S^1 & = & S^1. \end{array}$$

The top square is strictly commutative and the bottom square is homotopy commutative. Since C is a homotopy equivalence, $\text{image}((p_f \circ \text{ev}_w)_\#) = \text{image}((p_{f'} \circ \text{ev}_{w'})_\#)$ in $\pi_1(S^1, 1)$ and so by Diagram (1.3) $\text{image}(P_X)_\# = \text{image}(P_{X'})_\#$. Hence $\text{RI}(f) = \text{RI}(f')$. □

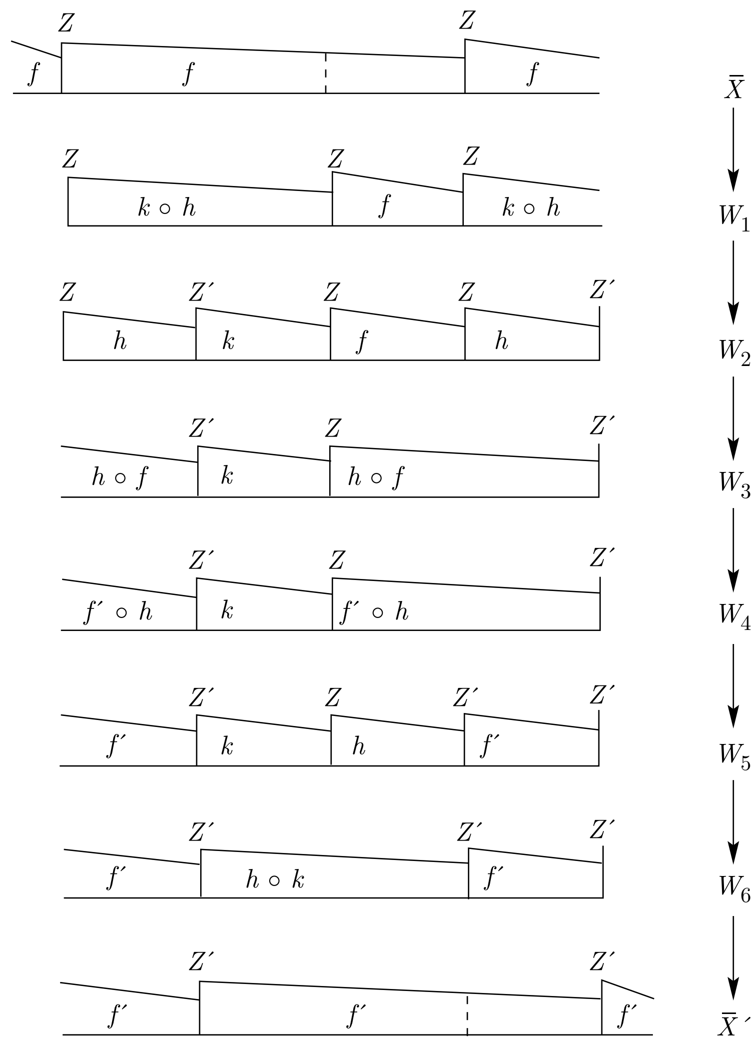

FIGURE 1. Construction of $\bar{\alpha}\colon \bar{X} \to \bar{X}'$

Remark. In [D] and [F] there is a standing hypothesis that all spaces are Hausdorff, but that restriction is not needed in the parts cited here.

2. Coherence of periodic homotopy equivalences

Let T be the infinite cyclic group with multiplication written additively. For a positive integer r, let $C_r \equiv T/rT$ be the cyclic group of order r. For positive integers r, s, let $\phi\colon C_{rs} \to C_r$ be the quotient homomorphism $T/rsT \to T/rT$.

The central extension $0 \to T \xrightarrow{\times r} T \to C_r \to 0$ defines a generator of $H^2(C_r; \mathbb{Z})$ which we denote by α_r. It is straightforward to show:

Lemma 2.1. $\phi^*(\alpha_r) = s\alpha_{rs}$. □

Let M be any left C_r–module. There is a cup product pairing

$$H^i(C_r; M) \otimes H^j(C_r; \mathbb{Z}) \xrightarrow{\cup} H^{i+j}(C_r; M).$$

Lemma 2.2. *For $\ell > 0$ and $k \geq 0$, the homomorphism $H^\ell(C_r; M) \to H^{\ell+2k}(C_r; M)$ given by $y \mapsto y \cup \alpha_r^k$ is an isomorphism.*

Proof. See [Br, Exercise 3, p.114]. □

Proposition 2.3. *For $k \geq 3$, $\phi^*\colon H^k(C_r; M) \to H^k(C_{r^2}; M)$ is zero.*

Proof. Let $x \in H^k(C_r; M)$. Since $k \geq 3$, by Lemma 2.2 there is $y \in H^{k-2}(C_r; M)$ such that $x = y \cup \alpha_r$. Then $\phi^*(x) = \phi^*(y \cup \alpha_r) = \phi^*(y) \cup \phi^*(\alpha_r)$. By Lemma 2.1, $\phi^*(\alpha_r) = r\alpha_{r^2}$ and so $\phi^*(x) = \phi^*(y) \cup r\alpha_{r^2} = \phi^*(ry) \cup \alpha_{r^2}$. The group $H^{k-2}(C_r; M)$ has exponent r ([Br, Corollary (10.2), p.84]) and thus $ry = 0$. It follows that $\phi^*(x) = 0$. □

Remark. When combined with the remark preceding [GN, Theorem 7.4], Proposition 2.3 gives a new proof of [GN, Proposition 7.3].

Theorem 2.4. *Let Z have the homotopy type of a CW complex, and let $f\colon Z \to Z$ be a periodic homotopy equivalence such that f^p is homotopic to the identity map of Z for some positive integer p. Then f is an eventually coherent periodic homotopy idempotent of period q where $q > 0$ divides p^2.*

Proof. The hypotheses define a homomorphism $\psi\colon C_p \to \pi_0(\mathcal{E}(Z))$, where $\mathcal{E}(Z)$ is the H-group of selfhomotopy equivalences of Z. By Proposition 2.3, the Cooke obstructions ([C, Theorem 1.1]) for the homotopy action given by $\psi \circ \phi\colon C_{p^2} \to \pi_0(\mathcal{E}(Z))$ all vanish and so there is a CW complex Y, a homotopy equivalence $h\colon Z \to Y$ and a homeomorphism $\bar{f}\colon Y \to Y$ such that $h \circ f \simeq \bar{f} \circ h$ and $\bar{f}^{p^2} = \mathrm{id}$. By Proposition 1.6, $\mathrm{RI}(f) = \mathrm{RI}(\bar{f})$. Clearly, $\mathrm{RI}(\bar{f}) > 0$ and divides p^2 and so, applying Proposition 1.5, the conclusion follows. □

In case Z is aspherical, the group $\pi_0(\mathcal{E}(Z))$ is isomorphic to $\mathrm{Out}(\pi_1(Z))$, the group of outer automorphisms of $\pi_1(Z)$, and Theorem 2.4 yields a result of group-theoretic interest:

Corollary 2.5. *Let $\theta\colon G \to G$ be an automorphism of a group G whose image in the group $\mathrm{Out}(G)$ of outer automorphisms of G has finite order r, and let $g \in G$ be such that $\theta^r(\cdot) = g(\cdot)g^{-1}$. Then $g^r \in Z(G)\,\mathrm{Fix}(\theta)$ where $Z(G)$ denotes the center of G and $\mathrm{Fix}(\theta)$ denotes the fixed subgroup of θ.* □

For a purely group-theoretic proof of this corollary (supplied to us by Peter Neumann) see [GN, Proposition 7.3].

3. Coherence of split homotopy idempotents

A homotopy idempotent $f\colon Z \to Z$ *splits* if there is a space Y and maps $d\colon Z \to Y$, $u\colon Y \to Z$ such that $d \circ u \simeq \mathrm{id}_Y$ and $u \circ d \simeq f$. A theorem of Hastings and Heller ([HH], see [BG, Corollary 8.2] for another proof) says that f splits whenever Z has the homotopy type of a finite-dimensional CW complex, so a split homotopy idempotent arises from any domination of a space Y by a finite-dimensional CW complex.

Proposition 3.1. *Let $f\colon Z \to Z$ be a homotopy idempotent which splits. Then f is an eventually coherent periodic homotopy idempotent of period* 1.

Proof. With splitting data as above, replace d by a fibration, $\bar{d}\colon \bar{Z} \to Y$, and $Z \subset \bar{Z}$. Then we have $\bar{u}\colon Y \to \bar{Z}$ with $\bar{d} \circ \bar{u} = \mathrm{id}_Y$ and $\bar{f} \equiv \bar{u} \circ \bar{d}\colon \bar{Z} \to \bar{Z}$ a strict idempotent (i.e. $\bar{f} = \bar{f}^2$). Thus $\bar{f}$ is an eventually coherent homotopy idempotent of period 1. So $\mathrm{RI}(\bar{f}) = 1$. The conclusion now follows by Propositions 1.6 and 1.5. □

There is a well-known example of a homotopy idempotent which does not split. Let F be the "Thompson group" with presentation

$$\langle x_0, x_1, \dots \mid x_i^{-1} x_n x_i = x_{n+1}, \quad n \geq 1,\ 0 \leq i < n \rangle.$$

Let $\phi\colon F \to F$ be the endomorphism $x_i \mapsto x_{i+1}$, $i \geq 0$. Let Z' be a $K(F,1)$ complex, and let $f'\colon Z' \to Z'$ a basepoint-preserving map inducing ϕ. Then (see, for example, [BG, §1] for this and all other claims about F and ϕ) f' is a homotopy idempotent which does not split. Moreover, we have:

Proposition 3.2. *The homotopy idempotent $f'\colon Z' \to Z'$ is not an eventually coherent periodic homotopy idempotent.*

Proof. Suppose f' is an eventually coherent periodic homotopy idempotent with coherence data (r, N', J', m) as in the Introduction (where Z' and f' play the roles of Z and f). Let τ be the loop $N'(v', \cdot)$ in Z'. Diagram (0.1) asserts that two homotopies $Z' \times I \to Z'$ are homotopic rel $Z' \times \{0,1\}$. Under these two homotopies, the basepoint v' traces out the loops $(f')^m \circ \tau$ and $(f')^{m+1} \circ \tau$ respectively, which must therefore represent the same element of F. Thus $\phi^m(x) = \phi^{m+1}(x)$ where $x \in F$ is represented by τ. But ϕ maps normal forms to normal forms, and the normal forms of y and $\phi(y)$ are different when $y \neq 1$. Hence $\mathrm{Fix}(\phi)$ is the trivial group; in particular $\phi^m(x) = 1$, and since ϕ is monic, this implies that $x = 1$. Thus τ is homotopically trivial, N' can be replaced by a basepoint preserving homotopy, and $\phi^r = \phi^{r+q}$ with $q > 0$. But different iterates of ϕ are not equal and so the supposition that f' is an eventually coherent periodic homotopy idempotent cannot be valid. □

Proposition 3.3. *Let $f\colon Z \to Z$ be a homotopy idempotent which does not split. Then f is not an eventually coherent periodic homotopy idempotent.*

Proof. Suppose f is a (basepoint-preserving) eventually coherent periodic homotopy idempotent with coherence data (r, N, J, m). Let τ be the loop $N(v, \cdot)$ where v is the basepoint of Z. By [BG, Proposition 2.1 and Theorem 8.1], there is a basepoint-preserving map $e\colon Z' \to Z$ inducing a monomorphism $e_\#\colon F \to \pi_1(Z, v)$ such that $ef' = fe$ and $eN' = N(e \times \mathrm{id})$. The proof of Proposition 3.2 is readily adapted to derive a contradiction. □

In summary:

Theorem 3.4. *Let $f\colon Z \to Z$ be a homotopy idempotent; f is an eventually coherent periodic homotopy idempotent if and only if f splits, and in that case the period is* 1.

Acknowledgments

In our paper [GN], Peter Neumann provided us with a key group-theoretic proof of [GN, Proposition 7.3] which is applicable here to aspherical spaces Z (see Corollary 2.5). The search for a geometric explanation of this result led to the present note. We thank him, and also Steve Ferry who suggested the relevance of [C].

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Department of Mathematics, State University of New York at Binghamton, Binghamton, New York 13902–6000

E-mail address: `ross@math.binghamton.edu`

Department of Mathematics, McMaster University, Hamilton, Ontario, Canada L8S 4K1

E-mail address: `nicas@mcmaster.ca`