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ON THE REALIZABILITY OF LEWY STRUCTURES

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Abstract. We prove that a nondegenerate CR structure with signature $(p, n-p)$ at $0 \in \mathbb{R}^{2n+1}$ and with n first integrals $z_1, \cdots, z_n$ satisfying

$$dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \neq 0$$

is realizable if and only if an action of the group $O(2p, 2n-2p)$ leaves invariant a one-dimensional subbundle of the structure bundle.

1

The local realizability (embeddability) of CR structures has been the focus of several papers, among others we mention [Ak], [AH], [BR], [Ha], [HJ], [Ja1], [JT1, JT2, JT3, Ku, Ni]. It has been proved ([AH]) that abstract real analytic CR structures are realizable. It is also proved ([Ak], [Ku]) that C^∞ strictly pseudoconvex CR structures in $\mathbb{R}^{2n+1}$ with $n \geq 3$ are embeddable. On the other hand, there exist nonrealizable CR structures: of strictly pseudoconvex nature in $\mathbb{R}^3$ ([Ni, JT1, JT2, JT3]); of Lewy type with signature $(1, n-1)$ in $\mathbb{R}^{2n+1}$ ([JT1, JT2, JT3]); and of high codimension ([M1]).

In this note, we consider the realizability problem for a class of Lewy structures of hypersurface type at $0 \in \mathbb{R}^{2n+1}$ with signature $(p, n-p)$. More precisely, we consider those structures with one first integral not listed. We prove that the "missing" first integral exists if and only if an action of the group $O(2p, 2n-2p)$ leaves invariant a 1-dimensional subbundle of the Lewy bundle.

2

A Lewy structure at $0 \in \mathbb{R}^{2n+1}$ with signature $(p, n-p)$ is the datum of a subbundle $\mathcal{C}$ of $\mathbb{C}T^*U$, where U is a neighborhood of $0 \in \mathbb{R}^{2n+1}$ and $\mathbb{C}T^*U$ is the complexified cotangent bundle T^*U, such that $\mathcal{C}$ is generated by $n+1$ smooth C^∞ one-forms $\omega_1, \cdots, \omega_{n+1}$ satisfying

$$\begin{aligned} &\omega_1 \wedge \cdots \wedge \omega_n \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_n \wedge \omega_{n+1} \neq 0, \\ &\omega_1 \wedge \cdots \wedge \omega_{n+1} \wedge d\omega_k = 0 \quad \text{for} \quad k = 1, \cdots, n+1, \end{aligned} \tag{1}$$

and such that the Levi form has p positive and $n-p$ negative eigenvalues. It follows from the formal integrability and the nondegeneracy of the Levi form that in suitable coordinates $(x_1, y_1, \cdots, x_n, y_n, u)$ of $\mathbb{R}^{2n+1}$ the second jets of $\omega_1, \cdots, \omega_{n+1}$

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are given by

$$(2)\qquad \begin{aligned} j^2\omega_k &= dz_k, \quad \text{for} \quad k = 1, \cdots, n, \\ j^2\omega_{n+1} &= du + i\left(\sum_{l=1}^{p} z_l d\bar{z}_l - \sum_{l=p+1}^{n} z_l d\bar{z}_l\right), \end{aligned}$$

where $z_k = x_k + iy_k$. The Lewy structure is said to be realizable if the bundle $\mathcal{C}$ is generated by exact forms. (For a comprehensive treatment of CR structures we refer to [Ja2] and [Tr].)

In this note, we are concerned with those Lewy structures $\mathcal{L}$ with signature $(p, n-p)$ generated, in suitable coordinates, by the forms

$$(3)\qquad \begin{aligned} \omega_k &= dz_k \quad \text{for} \quad k = 1, \cdots, n, \\ \omega_{n+1} &= du + \sum_{l=1}^{n} \lambda_l d\bar{z}_l. \end{aligned}$$

Hence ω_{n+1} satisfies

$$d\omega_{n+1} \wedge dz_1 \wedge \cdots \wedge dz_n \wedge \omega_{n+1} = 0$$

and its coefficients λ_l satisfy

$$(4)\qquad \lambda_l = i\epsilon_l z_l + o(|(z,u)|^2),$$

with $\epsilon_l = 1$ if $l \le p$ and $\epsilon_l = -1$ if $l \ge p+1$. Notice that in order for the Lewy structure $\mathcal{L}$ to be realizable, there must be C^∞ functions $\mu_1, \cdots, \mu_n$ such that

$$\omega_{n+1} + \mu_1 dz_1 + \cdots + \mu_n dz_n = gdf$$

for some C^∞ functions f and g. The function f is called ([Ha], [HJ]) the missing first integral of $\mathcal{L}$.

3

When the missing first integral f exists, the form df defines a *Mizohata* structure with signature $|2n-4p|$ at $0 \in \mathbb{R}^{2n+1}$ (see [Tr] and [M2] for details about Mizohata structures). We know from [Tr] that there exists a germ of a diffeomorphism Φ at $0 \in \mathbb{R}^{2n+1}$, tangent to the identity ($D\Phi(0) = I$), such that if $\Phi = (\phi_1, \cdots, \phi_{2n+1})$ and

$$\begin{aligned} t_k &= \phi_k(x,u) \quad \text{for} \quad k = 1, \cdots, 2n, \\ s &= \phi_{2n+1}(x,u), \end{aligned}$$

then

$$(5)\qquad f \circ \Phi^{-1}(t,s) = s + i\left(\sum_{k=1}^{2p} t_k^2 - \sum_{k=2p+1}^{2n} t_k^2\right).$$

Hence the invariance group $O(2p, 2n-2p)$ of the quadratic form

$$(6)\qquad q(t) = \sum_{k=1}^{2p} t_k^2 - \sum_{k=2p+1}^{2n} t_k^2$$

acting on $\mathbb{R}^{2n+1}$ by

$$A(t,s) = (At, s)$$

leaves invariant the bundle generated by $d(f \circ \Phi^{-1})$. Therefore the action of the group

$$G = \Phi^{-1} O(2p, 2n-2p)\Phi \tag{7}$$

leaves invariant the one-dimensional subbundle of $\mathcal{L}$ generated by df.

4

Conversely, we prove that the existence of a group G, conjugate to $O(2p, 2n-2p)$ under a diffeomorphism Φ with $D\Phi(0) = I$, and the existence of a one-dimensional subbundle $\mathcal{T}$ of the Lewy bundle $\mathcal{L}$ such that

$$G^*\mathcal{T} = \mathcal{T} \tag{8}$$

is a sufficient condition for the realizability of the Lewy structure $\mathcal{L}$.

Suppose that $\mathcal{T}$ satisfying (8) is spanned by a form

$$\omega = a\omega_{n+1} + \sum_{k=1}^{n} b_k dz_k. \tag{9}$$

We claim that $a(0) \neq 0$ and $b_k(0) = 0$ for $k = 1, \cdots, n$. Indeed, let $A \in O(2p, 2n-2p)$ be defined by

$$A(t_1, \cdots, t_{2n}, s) = (t_1, -t_2, \cdots, t_{2n-1}, -t_{2n}, s).$$

Since $D\Phi(0) = I$, then $\Psi = \Phi^{-1} \circ A \circ \Phi$ satisfies

$$\Psi(x_1, y_1, \cdots, x_n, y_n, u) = (x_1, -y_1, \cdots, x_n, -y_n, u) + o(|(z, u)|). \tag{10}$$

Thus,

$$\Psi^*\omega(0) = a(0)du + \sum_{k=1}^{n} b_k(0) d\bar{z}_k. \tag{11}$$

This shows that for $\Psi^*\omega$ to be a section of $\mathcal{T}$, it is necessary to have $a(0) \neq 0$ and $b_k(0) = 0$ as claimed.

After dividing ω by a, we see that $\mathcal{T}$ is spanned by a form

$$\begin{aligned} \Omega &= \omega_{n+1} + \sum_{k=1}^{n} \mu_k dz_k \\ &= du + \sum_{k=1}^{n} \lambda_k d\bar{z}_k + \sum_{k=1}^{n} \mu_k dz_k. \end{aligned} \tag{12}$$

Now, observe that since

$$\Psi^*\Omega \wedge \Omega = 0, \tag{13}$$

(Ψ is defined in (10)) and since the λ_k's satisfy (4), it follows that necessarily

$$\mu_k = i\epsilon_k \bar{z}_k + o(|(z, u)|^2). \tag{14}$$

This means that the first jet of Ω is

$$j^1\Omega = du + i\sum_{k=1}^{n} \epsilon_k d(z_k \bar{z}_k). \tag{15}$$

With this, our aim would be acheived if we could show that

$$\Omega \wedge d\Omega = 0. \tag{16}$$

For then Ω would generate a Mizohata structure with signature $|2n-4p|$ at $0 \in \mathbb{R}^{2n+1}$ and we know from [M2] that such a structure is locally integrable, i.e.,

$$\Omega = g df. \tag{17}$$

The function f would then be the missing first integral of the Lewy structure $\mathcal{L}$.

To prove the formal integrability (16), we use again the change of coordinates Φ and write

$$(\Phi^{-1})^*\Omega = \alpha(t,s)\left[ds + \sum_{k=1}^{2n} a_k(t,s)dt_k\right] = \alpha(t,s)\Theta. \tag{18}$$

Since the bundle generated by the form Θ is invariant under the action of the group $O(2p, 2n-2p)$, then it is not difficult to see that

$$A^*\Theta = \Theta \qquad \forall A \in O(2p, 2n-2p). \tag{19}$$

To prove that Θ is formally integrable, it is enough to prove that it descends to the orbit space $\mathbb{R}^{2n+1}/O(2p, 2n-2p)$ as a form $\bar{\Theta}$. The latter form would, necessarily, be formally integrable because of the dimension two of the orbit space. More precisely, we prove the following lemma.

Lemma. *Let $q(t)$ be the quadratic form defined in (6) and Θ the 1-form defined in (18) and satisfying (19). Then, in each connected component of $\mathbb{R}^{2n+1}\backslash\{q(t)=0\}$ we have*

$$\Theta = ds + g(q(t), s)d(q(t)) \tag{20}$$

for some function g of two variables.

Proof. Consider the spherical coordinates

$$\Pi_1 : \mathbb{R} \times S^{2p-1} \longrightarrow \mathbb{R}^{2p}, \quad \Pi_1(r_1, \theta^1) = r_1\theta^1,$$
$$\Pi_2 : \mathbb{R} \times S^{2n-2p-1} \longrightarrow \mathbb{R}^{2n-2p}, \quad \Pi_2(r_2, \theta^2) = r_2\theta^2.$$

We have then

$$\Theta = ds + U_1 dr_1 + U_2 dr_2 + \Lambda \tag{21}$$

in $s, r_1, r_2, \theta^1, \theta^2$ coordinates, where U_1, U_2 are functions and Λ is a 1-form on $S^{2p-1} \times S^{2n-2p-1}$ with s, r_1, r_2 as parameters.

Let R^1 and R^2 be arbitrary rotations of S^{2p-1} and $S^{2n-2p-1}$, respectively. It follows from (19) that

$$U_k(s, r_1, r_2, R^1\theta^1, R^2\theta^2) = U_k(s, r_1, r_2, \theta^1, \theta^2), \qquad k = 1, 2. \tag{22}$$

Since the group of rotations is transitive on the sphere, then U_1 and U_2 are independent of θ^1, θ^2. Now we are going to prove that in a connected component of $\mathbb{R}^{2n+1}\backslash\{q=0\}$ we have

$$U_1(s, r_1, r_2)dr_1 + U_2(s, r_1, r_2)dr_2 = g(r_1^2 - r_2^2, s)d(r_1^2 - r_2^2) \tag{23}$$

for some function g of two variables. For this, we consider the hyperbolic radius $\rho^2 = r_1^2 - r_2^2$ and the change of variables

$$r_1 = \rho\cosh t, \qquad r_2 = \rho\sinh t. \tag{24}$$

In the coordinates (ρ, t) the form $\Gamma = U_1 dr_1 + U_2 dr_2$ has the expression

$$\begin{aligned} \Gamma &= (U_1\cosh t + U_2\sinh t)d\rho + \rho(U_1\sinh t + U_2\cosh t)dt \\ &= V d\rho + \rho W dt. \end{aligned} \tag{25}$$

From (19), it follows that Γ is invariant under the translations $t \mapsto t+\tau$ and under the reflection $t \mapsto -t$. The invariance under translations implies that V and W are independent of t, and then the invariance under the reflection gives $W = 0$. This proves (23).

To complete the proof of the lemma, we go back to (21) and prove that $\Lambda = 0$. Again from (19) it follows that $A^*\Lambda = \Lambda$ for every $A \in O(2p, 2n-2p)$. Let $\Lambda = \Lambda_1 + \Lambda_2$, where Λ_1 is a 1-form on S^{2p-1} with parameters s, r_1, r_2, θ^2 and Λ_2 is a form on $S^{2n-2p-1}$ with parameters s, r_1, r_2, θ^1. We deduce that

$$A_1^*\Lambda_1 = \Lambda_1 \qquad \text{and} \qquad A_2^*\Lambda_2 = \Lambda_2 \tag{26}$$

for every $A_1 \in O(2p)$ and every $A_2 \in O(2n-2p)$.

Now suppose that $\alpha = a(x,y)dx + b(x,y)dy$ is a 1-form in a two-dimensional plane satisfying $B^*\alpha = \alpha$ for every $B \in O(2)$. We write α in polar coordinates (r, ϕ) as

$$\begin{aligned} \alpha &= (a\cos\phi + b\sin\phi)dr + r(-a\sin\phi + b\cos\phi)d\phi \\ &= p(r,\phi)dr + rq(r,\phi)d\phi. \end{aligned} \tag{27}$$

Since α is invariant under the translations $\phi \mapsto \phi + \tau$, it follows at once that p and q are independent of ϕ. The invariance of α under the reflection $\phi \mapsto -\phi$ shows that $q = 0$. That is,

$$\alpha = p(r)dr. \tag{28}$$

With this at hand, we prove that $\Lambda_1 = 0$ as follows. Take any point $p_0 \in S^{2p-1}$ and consider the stereographic projection with center p_0. Consider the action of the group $O(2)$ on the chart $\mathbb{R}^{2p-1} = \mathbb{R}^2 \times \mathbb{R}^{2p-3}$, with p_0 as the origin, as $(x, x') \mapsto (Bx, x')$. The form Λ_1 is invariant under this action and so it follows from (28) that Λ_1 is independent of the polar angle ϕ in $\mathbb{R}^2$. Since p_0 is arbitrary in S^{2p-1}, the nullity of Λ_1 follows. Similarly we prove that $\Lambda_2 = 0$. The lemma is proved.

In conclusion, we have proved the following

Theorem. *A Lewy structure $\mathcal{L}$ with signature $(p, n-p)$ and with only one "missing" first integral is realizable if and only if there exist a germ of a diffeomorphism Φ at $0 \in \mathbb{R}^{2n+1}$, tangent to the identity, and a 1-dimensional subbundle $\mathcal{T}$ of the Lewy bundle such that*

$$\Psi^*\mathcal{T} = \mathcal{T} \qquad \forall\Psi \;\in\; \Phi^{-1}O(2p, 2n-2p)\Phi. \tag{22}$$

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