THE DISTANCE FROM THE APOSTOL SPECTRUM

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Abstract. If $T$ is an $s$-regular operator in a Banach space (i.e. $T$ has closed range and $N(T) \subset R^\infty(T)$) and $\gamma(T)$ is the Kato reduced minimum modulus, then

$$\lim_{n \to \infty} \gamma(T^n)^{1/n} = \sup \{ r : T - \lambda \text{ is } s\text{-regular for } |\lambda| < r \}.$$  

Let $x$ be an element of a Banach algebra $A$. The spectral radius of $x$ is given by the well-known spectral radius formula:

$$r(x) = \lim_{n \to \infty} \|x^n\|^{1/n}.$$  

There are a number of generalizations of this formula. If we set $d(x) = \inf \{ \|xy\| : y \in A, \|y\| = 1 \}$ and denote by $\tau(x) = \{ \lambda \in \mathbb{C} : d(x - \lambda) = 0 \}$ the left approximate point spectrum of $x$, then $\text{dist}(0, \tau(x)) = \lim_{n \to \infty} d(x^n)^{1/n}$; see [13], [9]. In particular, in the algebra $B(X)$ of all bounded linear operators in a Banach space $X$ this gives formulas for radii of boundedness below or surjectivity:

$$\sup \{ r : T - \lambda \text{ is bounded below for } |\lambda| < r \} = \lim_{n \to \infty} j(T^n)^{1/n}$$

and

$$\sup \{ r : T - \lambda \text{ is onto for } |\lambda| < r \} = \lim_{n \to \infty} k(T^n)^{1/n},$$

where $j(T)$ and $k(T)$ are the moduli of injectivity and surjectivity of $T$:

$$j(T) = \inf \{ \|Tx\| : x \in X, \|x\| = 1 \} \text{ and } k(T) = \sup \{ r : TU_X \supset rU_X \},$$

where $U_X$ is the closed unit ball in $X$.

For a bounded linear operator $T$ in a Banach space $X$ denote by $N(T)$ and $R(T)$ its kernel and range, respectively. Denote further $R^\infty(T) = \bigcap_{n=1}^{\infty} R(T^n)$ and $N^\infty(T) = \bigcup_{n=1}^{\infty} N(T^n)$.

The injectivity and surjectivity moduli of an operator which is bounded below (onto) are special cases of the Kato reduced minimum modulus [7]

$$\gamma(T) = \inf \left\{ \frac{\|Tx\|}{\text{dist}(x, N(T))} : x \in X \setminus N(T) \right\}$$

(for $T = 0$ we define formally $\gamma(T) = \infty$).

The existence and the meaning of the limit $\lim_{n \to \infty} \gamma(T^n)^{1/n}$ in a more general setting were studied by Apostol [1] and Mbekhta [10].

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Definition. Let $T \in B(X)$. We say that $T$ is s-regular (= semi-regular) if $R(T)$ is closed and $N(T) \subset R^\infty(T)$.

The s-regular operators and closely related classes of operators were studied (under various names) by many authors; see [3, 4, 5, 6, 8, 16]. We list some of the most important equivalent conditions for s-regular operators; see [11, 12].

Theorem. Let $T \in B(X)$ be an operator with a close range. The following conditions are equivalent:

(1) $T$ is s-regular,

(2) the function $\lambda \mapsto R(T - \lambda)$ is continuous at 0 in the gap topology,

(3) the function $\lambda \mapsto N(T - \lambda)$ is continuous at 0 in the gap topology,

(4) the function $\lambda \mapsto \gamma(T - \lambda)$ is continuous at 0,

(5) $\liminf_{\lambda \to 0} \gamma(T - \lambda) > 0$,

(6) $N^\infty(T) \subset R(T)$,

(7) $N^\infty(T) \subset R^\infty(T)$.

Denote further $\sigma_\gamma(T) = \{ \lambda \in C : T - \lambda \text{ is not s-regular} \}$. The set $\sigma_\gamma(T)$ was studied by Apostol [1], Rakocević [15], Mbekhta and Ouahab [11, 12] and Mbekhta [10]. The terminology is not unified; we suggest to call $\sigma_\gamma(T)$ the Apostol spectrum of $T$.

The Apostol spectrum $\sigma_\gamma(T)$ is always a non-empty compact subset of the complex plane, $\partial \sigma(T) \subset \sigma_\gamma(T) \subset \sigma(T)$ and $\sigma_\gamma(T) = f\sigma_\gamma(T)$ for any function $f$ analytic in a neighbourhood of $\sigma(T)$.

If $T$ is an s-regular operator in a Hilbert space, then the limit $\lim_{n \to \infty} \gamma(T^n)^{1/n}$ exists and

$$\lim_{n \to \infty} \gamma(T^n)^{1/n} = \text{dist}(0, \sigma_\gamma(T)) = \sup\{ r : T - \lambda \text{ is s-regular for } |\lambda| < r \};$$

see [1, Theorem 3.2] or [10, Theorem 3.1].

The aim of this paper is to prove equality (1) for operators in Banach spaces. This gives a positive answer to the conjecture of Rakocević [15] and Mbekhta and generalizes the above-mentioned results for radii of injectivity and surjectivity.

Further, we study the essential version of this result.

If $T$ is a semi-Fredholm operator, then the limit $\lim_{n \to \infty} \gamma(T^n)^{1/n}$ exists by [2] and it is equal to the semi-Fredholm radius of $T$:

$$\lim_{n \to \infty} \gamma(T^n)^{1/n} = \sup\{ r : T - \lambda \text{ is semi-Fredholm for } |\lambda| < r \};$$

see [17] and [2].

We prove a similar formula for essentially s-regular operators which generalizes the semi-Fredholm case.

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Lemma 1. $T \in B(X)$ is s-regular if and only if there exists a closed subspace $M \subset X$ such that $TM = M$ and the operator $\tilde{T} : X/M \to X/M$ induced by $T$ is bounded below.

Proof. If $T$ is s-regular, then set $M = R^\infty(T)$. It is well known that $M$ is closed and (see e.g. [4, Theorem 3.4]) that $TM = M$ and $\tilde{T} : X/M \to X/M$ is bounded from below.
Conversely, let $M$ be the subspace of $X$ with the required properties. Then $TM = M$ implies $M \subset R^\infty(T)$. If $Tx = 0$, then $\tilde{T}(x + M) = 0$ and the injectivity of $\tilde{T}$ implies $x \in M$. Thus $N(T) \subset M \subset R^\infty(T)$.

It remains to prove that $T$ has closed range. Let $\pi : X \to X/M$ be the canonical projection. We show $R(T) = \pi^{-1}R(\tilde{T})$. If $y \in R(T), y = Tx$ for some $x \in X$, then $\pi y = Tx + M = \tilde{T}(x + M) \in R(\tilde{T})$ so that $R(T) \subset \pi^{-1}R(\tilde{T})$. If $y \in X$ and $\pi y \in R(\tilde{T})$, i.e. $y + M = Tx + M$ for some $x \in X$, then $y \in R(T)$ since $M \subset R(T)$. Thus $R(T) = \pi^{-1}R(\tilde{T})$, which is closed since $R(\tilde{T})$ is closed and $\pi$ continuous.

**Lemma 2.** Let $T \in B(X)$, and let $M$ be a closed subspace of $X$ such that $TM = M$ and the operator $\tilde{T} : X/M \to X/M$ induced by $T$ is bounded below. Denote by $T_1 : M \to M$ the restriction of $T$ to $M$. Then

$$\lim_{n \to \infty} \gamma(T^n)^{1/n} = \min \{ \lim_{n \to \infty} \gamma(T^n_1)^{1/n}, \lim_{n \to \infty} \gamma(\tilde{T}^n)^{1/n} \}.$$ 

**Proof.** The limits on the right-hand side exist by [17]. If $T^nx = 0$, then $\tilde{T}^n(x+M) = 0$, i.e. $x \in M$. Thus $N(T^n) \subset M$ and $N(T^n_1) = N(T^n)$. We have

$$\gamma(T^n) = \inf \left\{ \frac{\|T^n x\|}{\text{dist}\{ x, N(T^n_1) \}} : x \in M \setminus N(T^n) \right\}$$

$$= \inf \left\{ \frac{\|T^n x\|}{\text{dist}\{ x, N(T^n) \}} : x \in M \setminus N(T^n) \right\} \geq \gamma(T^n).$$

Further, since $TM = M$, 

$$\gamma(\tilde{T}^n) = \inf \left\{ \frac{\|\tilde{T}^n(x + M)\|}{\|x + M\|} : x \not\in M \right\} = \inf \left\{ \frac{\|T^n x + M\|}{\text{dist}\{ x, M \}} : x \not\in M \right\}$$

$$\geq \inf \left\{ \frac{\|T^n x\|}{\text{dist}\{ x, M \}} : x \not\in M \right\} \geq \inf \left\{ \frac{\|T^n x\|}{\text{dist}\{ x, N(T^n) \}} : x \not\in M \right\} \geq \gamma(T^n).$$

Thus $\gamma(T^n) \leq \min \{ \gamma(T^n_1), \gamma(\tilde{T}^n) \}$ and

$$\lim_{n \to \infty} \gamma(T^n)^{1/n} \leq \min \{ \lim_{n \to \infty} \gamma(T^n_1)^{1/n}, \lim_{n \to \infty} \gamma(\tilde{T}^n)^{1/n} \}.$$ 

Denote

$$s = \min \{ \lim_{n \to \infty} \gamma(T^n_1)^{1/n}, \lim_{n \to \infty} \gamma(\tilde{T}^n)^{1/n} \}.$$ 

We prove $\lim \inf_{n \to \infty} \gamma(T^n)^{1/n} \geq s$.

Let $n \geq 1$, $x = x_0 \in R(T^n)$, $\|x\| = 1$, and let $s > \varepsilon > 0$. Then $x + M \in R(\tilde{T}^n)$ and

$$\|\tilde{T}^{-i}(x + M)\| \leq \gamma(\tilde{T}^{-1}) \|x + M\| \leq \gamma(\tilde{T}^{-1})^{-1} (i = 1, \ldots, n).$$

Thus there exist vectors $x_i \in \tilde{T}^{-i}(x + M)$ such that

$$\|x_i\| \leq \gamma(\tilde{T}^{-i})^{-1} (1 + \varepsilon) \quad (i = 1, \ldots, n).$$

Denote $m_i = Tx_{i+1} - x_i$ $(i = 0, \ldots, n - 1)$. Then

$$\|m_i\| \leq \|T\| \|x_{i+1}\| + \|x_i\| \leq (1 + \varepsilon) [\|T\| \gamma(\tilde{T}^{i+1})^{-1} + \gamma(\tilde{T}^{-1})^{-1}] \quad (i = 0, \ldots, n - 1).$$
Further, \( \tilde{T}^i(m_i + M) = T^{i+1}x_{i+1} - T^ix_i + M = M \) so that \( m_i \in M \) for each \( i \). We have
\[
\sum_{i=0}^{n-1} T^i m_i = (T^n x_n - T^{n-1}x_{n-1}) + (T^{n-1} x_{n-1} - T^{n-2}x_{n-2}) + \cdots + (T_1 x - x)
\]
\[
= T^n x_n - x.
\]
Since \( T_1 M \to M \) is onto, there exist vectors \( m_i' \in M \) such that \( T^{n-i} m_i' = m_i \) and \( \|m_i'\| \leq (1 + \varepsilon)\gamma(T_1^{n-i})^{-1}\|m_i\| \). Thus
\[
T^n \left( x_n - \sum_{i=0}^{n-1} m_i' \right) = T^n x_n - \sum_{i=0}^{n-1} T^i m_i = x
\]
and
\[
\left\| x_n - \sum_{i=0}^{n-1} m_i' \right\| \leq (1 + \varepsilon)\gamma(T^n)^{-1} + \sum_{i=0}^{n-1} (1 + \varepsilon)^2 \gamma(T_1^{n-i})^{-1} \left( \|T\|\gamma(T_1^{i+1})^{-1} + \gamma(T_i)^{-1} \right).
\]
Thus
\[
\gamma(T^n)^{-1} \leq (1 + \varepsilon)\gamma(T^n)^{-1} + \sum_{i=0}^{n-1} (1 + \varepsilon)^2 \gamma(T_1^{n-i})^{-1} \left( \|T\|\gamma(T_1^{i+1})^{-1} + \gamma(T_i)^{-1} \right).
\]
Find \( n_0 \) such that
\[
\gamma(T_1^i) \geq (s - \varepsilon)^i, \quad \gamma(T_i^i) \geq (s - \varepsilon)^i \quad (i \geq n_0).
\]
Denote
\[
K = \max_{1 \leq i \leq n_0+1} \max \{ \gamma(T_1^i)^{-1}, \gamma(T_i^i)^{-1}, (s - \varepsilon)^{-i} \}.
\]
For \( n \) large enough we have
\[
\gamma(T^n)^{-1} \leq (1 + \varepsilon)^2 \left( (s - \varepsilon)^{-n} + \sum_{i=n_0}^{n-n_0} (s - \varepsilon)^{1-n} (\|T\|(s - \varepsilon)^{-i-1} + (s - \varepsilon)^{-i}) 
\]
\[
+ \sum_{i=n_0}^{n-1} (s - \varepsilon)^{1-n} (\|T\| \cdot K + K) + \sum_{i=n-n_0}^{n-1} K((\|T\|(s - \varepsilon)^{-i-1} + (s - \varepsilon)^{-i})) \right]
\]
\[
\leq (1 + \varepsilon)^2 (s - \varepsilon)^{n_0-n} \left[ K + (n - 2n_0)(K \cdot \|T\| + K) + 2n_0K(\|T\| \cdot K + K) \right]
\]
\[
\leq (1 + \varepsilon)^2 (s - \varepsilon)^{n_0-n} \cdot K',
\]
where \( K' \) is a constant independent of \( n \). Hence
\[
\liminf_{n \to \infty} \gamma(T^n)^{1/n} \geq \liminf_{n \to \infty} (s - \varepsilon)^{\frac{n-n_0}{n}} = s - \varepsilon.
\]
Since \( \varepsilon > 0 \) was arbitrary, we conclude that \( \liminf_{n \to \infty} \gamma(T^n)^{1/n} \geq s \), so that
\[
\lim_{n \to \infty} \gamma(T^n)^{1/n} = s.
\]
Theorem 3. Let $T \in B(X)$ be s-regular. Then
\[
\text{dist}\{0, \sigma(T)\} = \lim_{n \to \infty} \gamma(T^n)^{1/n}.
\]

Proof. Denote $r = \text{dist}\{0, \sigma(T)\}$. Let $M = R^\infty(T), T_1 = T|_M$, and let $\tilde{T} : X/M \to X/M$ be the operator induced by $T$. If $\lambda$ is a complex number satisfying

\[
|\lambda| < \lim_{n \to \infty} \gamma(T^n)^{1/n} = \min\{\lim_{n \to \infty} \gamma(T_1^n)^{1/n}, \lim_{n \to \infty} \gamma(\tilde{T}^n)^{1/n}\},
\]

then $T_1 - \lambda$ is onto and $\tilde{T} - \lambda$ is bounded below. Thus $T - \lambda$ is s-regular by Lemma 1 and $\lim_{n \to \infty} \gamma(T^n)^{1/n} \leq r$.

Conversely, it is well known (see e.g. [15, Theorem 5.2]) that $R^\infty(T - \lambda)$ is constant on the component of $C\setminus\sigma(T)$ containing 0, in particular $R^\infty(T - \lambda) = M$ for $|\lambda| < r$. If $|\lambda| < r$, then $(T - \lambda)M = M$ and $\tilde{T} - \lambda = \tilde{T} - \lambda : X/M \to X/M$ is bounded below. Thus $\lim_{n \to \infty} \gamma(T_1^n)^{1/n} \geq r$ and $\lim_{n \to \infty} \gamma(\tilde{T}^n)^{1/n} \geq r$. Hence $\lim_{n \to \infty} \gamma(T^n)^{1/n} \geq r$ by Lemma 2.

Remark. It is possible to deduce the inequality $\text{dist}\{0, \sigma(T)\} \geq \lim_{n \to \infty} \gamma(T^n)^{1/n}$ from [11, Theorem 2.10]. We have obtained a new direct proof of this result.

Definition. $T \in B(X)$ is called essentially s-regular if $R(T)$ is closed and there exists a finite-dimensional subspace $F \subset X$ such that $N(T) \subset R^\infty(T) + F$.

Define further $\sigma_{e\gamma}(T) = \{\lambda \in C : T - \lambda$ is not essentially s-regular$\}$.

For properties of essentially s-regular operators and the set $\sigma_{e\gamma}(T)$ see [14, 15].

Theorem 4. Let $T \in B(X)$ be essentially s-regular. Then $\lim_{n \to \infty} \gamma(T^n)^{1/n}$ exists and
\[
\lim_{n \to \infty} \gamma(T^n)^{1/n} = \max\{r : T - \lambda$ is s-regular for $0 < |\lambda| < r\} = \text{dist}\{0, \sigma(T)\}\{0\}.
\]

Proof. By [14, Theorem 3.1] or [15, Theorem 2.1] there exist subspaces $X_1, X_2 \subset X$ such that $X = X_1 \oplus X_2, \dim X_1 < \infty, TX_1 \subset X_1, TX_2 \subset X_2, T_1 = T|_{X_1}$ is nilpotent and $T_2 = T|_{X_2}$ is s-regular (the Kato decomposition). By the previous theorem $\text{dist}\{0, \sigma(T_2)\} = \lim_{n \to \infty} \gamma(T_2^n)^{1/n}$. For $n \geq \dim X_1$ we have $T_1^n = 0$ so that $N(T^n) = X_1 \oplus N(T_2^n)$. Let $P$ be the projection with $R(P) = X_2$ and $N(P) = X_1$. Let $x_2 \in X_2$. We have
\[
\text{dist}\{x_2, N(T_2^n)\} = \inf\{\|x_2 - y_2\| : y_2 \in X_2, T_2^n y_2 = 0\}
\leq \|P\| \inf\{\|y_1 \oplus (x_2 - y_2)\| : y_1 \in X_1, y_2 \in X_2, T_2^n y_2 = 0\}
\leq \|P\| \text{dist}\{x_2, N(T_2^n)\} \leq \|P\| \text{dist}\{x_2, N(T^n)\} = \|P\| \text{dist}\{x_2, N(T_2^n)\}.
\]

Then
\[
\gamma(T_2^n) = \inf\left\{\frac{\|T_2^n x_2\|}{\text{dist}\{x_2, N(T_2^n)\}} : x_2 \in X_2 \setminus N(T_2^n)\right\}
\leq \inf\left\{\frac{\|T^n x_2\|}{\text{dist}\{x_2, N(T)\}} : x_2 \in X_2 \setminus N(T)\right\}
\leq \inf\left\{\frac{\|T^n (x_1 \oplus x_2)\|}{\text{dist}\{x_1 \oplus x_2, N(T)\}} : x_1 \oplus x_2 \in X \setminus N(T)\right\} = \gamma(T^n).
\]
and
\[ \gamma(T^n) \leq \inf \left\{ \frac{\|T^n x\|}{\text{dist}\{x, N(T^n)\}} : x \in X \setminus N(T^n) \right\} \]
\[ \leq \|P\| \inf \left\{ \frac{\|T^n x\|}{\text{dist}\{x, N(T^n)\}} : x \in X \setminus N(T^n) \right\} = \|P\| \gamma(T^n). \]

Hence \( \lim_{n \to \infty} \gamma(T^n)^{1/n} = \lim_{n \to \infty} \gamma(T^n)_{2}^{1/n} \).

If \( \lambda \neq 0 \), then \( T - \lambda \) is s-regular if and only if \( T_2 - \lambda \) is s-regular. Then
\[ \max\{r : T - \lambda \text{ is s-regular for } 0 < |\lambda| < r \} = \text{dist}\{0, \sigma_x(T_2)\} = \lim_{n \to \infty} \gamma(T^n)^{1/n}. \]

The following lemma is an analog of Lemma 1 for essentially s-regular operators:

**Lemma 5.** \( T \in B(X) \) is essentially s-regular if and only if there exists a closed subspace \( M \subset X \) such that \( TM = M \) and the operator \( \tilde{T} : X/M \to X/M \) induced by \( T \) is upper semi-Fredholm.

**Proof.** If \( T \) is essentially s-regular, then set \( M = R^\infty(T) \). If \( X = X_1 \oplus X_2 \) is the Kato decomposition (\( \dim X_1 < \infty, TX_1 \subset X_1, TX_2 \subset X_2, T_1 = T|X_1 \) nilpotent and \( T_2 = T|X_2 \) s-regular), then \( M = R^\infty(T_2) \subset X_2 \) and \( TM = T_2M = M \). If \( x = x_1 \oplus x_2 \) satisfies \( Tx \in M \), then \( T_2x_2 \in M \) so that \( x \in X_1 + M \). Thus \( x \in X_1 + M \) and \( N(\tilde{T}) \subset X_1 + M \). Hence \( \dim N(\tilde{T}) < \infty \).

Let \( \pi : X \to X/M \) be the canonical projection. Since \( M \subset R(T) \) and \( R(\tilde{T}) = \{Tx + M : x \in X\} = \pi R(T) \), the range of \( \tilde{T} \) is closed. Thus \( \tilde{T} \) is upper semi-Fredholm.

Conversely, let \( M \) be a subspace of \( X \) with the required properties. We can prove that \( R(T) \) is closed in exactly the same way as in Lemma 1.

Further, \( M \subset R^\infty(T) \). If \( Tx = 0 \), then \( \tilde{T}(x + M) = 0 \), i.e. \( \pi x \in N(\tilde{T}) \). Thus \( N(\tilde{T}) \subset \pi^{-1}N(T) \subset M + F \subset R^\infty(T) + F \) for a finite-dimensional subspace \( F \subset X \).

**Theorem 6.** Let \( T, A \in B(X), TA = AT \), and let \( A \) be a quasinilpotent. Then

1. \( \sigma_c(T + A) = \sigma_c(T) \),
2. \( \sigma_{\gamma}(T + A) = \sigma_{\gamma}(T) \).

**Proof.** Let \( T \) be an essentially s-regular operator, and let \( A \) be a quasinilpotent commuting with \( T \). Denote \( M = R^\infty(T), T_1 = T|M \), and let \( \tilde{T} : X/M \to X/M \) be the operator induced by \( T \). Clearly \( AM \subset M \) so that we can define operators \( A_1 = A|M \) and \( A : X/M \to X/M \) induced by \( A \). Clearly \( r(A_1) = \lim_{n \to \infty} \|A_1^n\|^{1/n} \leq \lim_{n \to \infty} \|A^n\|^{1/n} = 0 \) and \( r(A) = \lim_{n \to \infty} \|A^n\|^{1/n} \leq \lim_{n \to \infty} \|A^n\|^{1/n} = 0 \) so that \( \sigma(A_1) = \{0\} \) and \( \sigma(A) = \{0\} \). Further \( T_1A_1 = A_1T_1 \) and \( TA = AT \). Denote by
\[ \sigma_{\delta}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not onto}\}, \]
\[ \sigma_{*}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not bounded below}\}, \]
\[ \sigma_{\pi e}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-Fredholm}\} \]
the defect spectrum, the approximate point spectrum and the essential approximate point spectrum, respectively.
By the spectral mapping property for these spectra we have
\[ \sigma_\delta(T_1 + A_1) = \sigma_\delta(T), \]
\[ \sigma_\pi(\tilde{T} + \tilde{A}) = \sigma_\pi(\tilde{T}), \]
\[ \sigma_{\pi e}(\tilde{T} + \tilde{A}) = \sigma_{\pi e}(\tilde{T}). \]
Thus \( 0 \notin \sigma_\delta(T + A) \), i.e. \((T + A)M = M\). Similarly \( 0 \notin \sigma_{\pi e}(\tilde{T} + \tilde{A}) \), i.e. \( \tilde{T} + \tilde{A}\) is upper semi-Fredholm. By the previous lemma \( T + A \) is essentially s-regular. This proves (2).

If \( T \) is s-regular and \( A \) a quasinilpotent commuting with \( T \), then in the same way \((T + A)M = M\) and \( \tilde{T} + \tilde{A}\) is bounded below. Hence \( T + A \) is s-regular by Lemma 1.

**Remark.** Statement (1) for Hilbert space operators was proved in [10, Theorem 4.8]. The second statement gives a positive answer to Question 3 of [15].

**References**