

CONTINUOUS SINGULAR MEASURES WITH ABSOLUTELY CONTINUOUS CONVOLUTION SQUARES

ANTHONY H. DOOLEY AND SANJIV KUMAR GUPTA

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ABSTRACT. We prove for every non-abelian compact connected group G there is a continuous, singular, central measure μ with $\mu * \mu$ in L^p for all p , $1 \leq p < \infty$. We also construct such measures on some families of non-abelian compact totally disconnected groups. These results settle an open question of Ragozin.

1. INTRODUCTION

In 1966, Hewitt and Zuckerman [6] proved that if G is a non-discrete locally compact abelian group with Haar measure λ , then there exists a non-negative, continuous, regular measure μ on G that is singular to λ ($\mu \perp \lambda$) such that $\mu(G) = 1$, $\mu * \mu$ is absolutely continuous with respect to λ ($\mu * \mu \ll \lambda$) and the Lebesgue-Radon-Nikodym derivative of $\mu * \mu$ with respect to λ is in $L^p(G, \lambda)$ for all p , $1 \leq p < \infty$. This result was partially generalised to non-abelian connected compact simple groups by Ragozin [10] and to any locally compact group (nondiscrete) by Karanikas and Koumandos [7]. They showed that there exist continuous singular measures μ with $\mu * \mu \in L^1(G, \lambda)$, λ being the left Haar measure on G . Ragozin [10] mentioned that even for connected compact simple Lie groups it is an open question whether there exists a singular measure μ with $\mu * \mu \in L^p(G, \lambda)$ for all p , $1 \leq p < \infty$.

In this paper, we show that on any non-abelian compact, connected group or on any compact Lie group G , there exist central, continuous, singular measures μ with $\mu * \mu \in L^p(G, \lambda)$ for all p , $1 \leq p < \infty$ and also prove this result on some specific disconnected non-abelian compact groups. We also note two remarkable but difficult works of Saeki [11], [12] and a paper of Stempak [14] on this subject.

In section 2, we establish our notation. We state and prove our main theorem (Theorem 3.1) in section 3. Section 4 contains the results on some specific totally disconnected compact groups.

2. NOTATION

G always denotes a non-abelian compact group unless stated otherwise. Let G be a compact connected Lie group and $T = \mathbb{T}^\ell$ be a maximal torus for G . Let R denote the set of roots of G with respect to T , Δ a base for R , and R_+ the set of positive roots with respect to Δ .

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The Weyl group W is the finite group $N_G(T)/T$, where $N_G(T)$ is the normaliser of T in G and one has an action on T given by

$$w = gT : t \rightarrow wt = gtg^{-1}.$$

For each $w \in W$, $H \rightarrow dw(e)H$ defines an action of w on the Lie algebra \underline{t} of T which we denote also by $H \rightarrow wH$. Let $\text{sgn}(w)$ denote the determinant of $dw(e)$.

Let \underline{g}^* and \underline{t}^* denote the duals of the real Lie algebras \underline{g} and \underline{t} respectively. The derivatives of the roots of G on T are purely imaginary linear forms on \underline{t} . For $\alpha \in R$, let $d\alpha(e) = i\tilde{\alpha}$ where $\tilde{\alpha} \in \underline{t}^*$. Let $(,)$ denote the Killing form on \underline{t} , and by duality transfer it to a form on \underline{t}^* . For $H_1, H_2 \in \underline{t}$, we denote $\frac{2(H_1, H_2)}{(H_2, H_2)}$ by $\langle H_1, H_2 \rangle$.

For $p \in \underline{t}^*$, we denote by $H_p \in \underline{t}$ the element satisfying

$$p(H) = (H_p, H) \quad \forall H \in \underline{t}.$$

Let $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. We denote by $\check{\alpha}_i = \frac{2\tilde{\alpha}_i}{(\tilde{\alpha}_i, \tilde{\alpha}_i)}$, $1 \leq i \leq \ell$.

Let $\delta = 1/2 \sum_{\alpha \in R_+} \tilde{\alpha}$ and $q = \prod_{\alpha \in R_+} (1 - \tilde{\alpha})$. Define j , an Ad-invariant function on \underline{g} as follows: for $H \in \underline{t}$, set

$$j(H) = \frac{e^{i\delta(H)} q(\exp H)}{\prod_{\alpha \in R_+} i\tilde{\alpha}(H)}.$$

Then $|j|^2$ is exactly the determinant of the exponential map, and in fact j takes real values.

Now we define the wrapping map Φ [2]. Let $\mu \in M(\underline{g})$ be an Ad-invariant measure of compact support. Define

$$\langle \Phi\mu, \varphi \rangle = \langle \mu, j\tilde{\varphi} \rangle \quad \forall \varphi \in C(G)$$

where

$$\tilde{\varphi}(X) = \varphi(\exp X).$$

It follows easily that $\Phi\mu$ is a central measure on G and if μ is continuous then $\Phi\mu$ is continuous too.

Also for $\sigma \in \hat{G}$, $(\Phi\mu)(\hat{\sigma}) = \hat{\mu}(\lambda + \delta)I_{d_\sigma}$ where $e^{i\lambda}$, $\lambda \in \underline{t}^*$ is the highest weight of σ [2].

3. THE MAIN THEOREM

In this section we state the main result (Theorem 3.1) of this paper and prove it.

Theorem 3.1. *Let G be a compact connected group or a compact Lie group. Then there exists a central, singular, continuous measure $\mu \in M(G)$ such that $\mu * \mu$ is absolutely continuous and the Lebesgue-Radon-Nikodym derivative of $\mu * \mu$ is in $L^p(G)$ for all p , $1 \leq p < \infty$.*

To prove Theorem 3.1, we will show that for G satisfying the hypothesis of Theorem 3.1, there exists a central, singular, continuous measure μ defined on G whose Fourier-Stieltjes transform $\hat{\mu}$ is p -summable for every $p > 2$. Then $\mu * \mu$ is absolutely continuous and the Lebesgue-Radon-Nikodym derivative of $\mu * \mu$ is in L^p for all p , $1 \leq p < \infty$ by the Hausdorff-Young theorem ([4], Theorem 31.24).

Proof of Theorem 3.1 for G compact simply connected semisimple Lie group. Let $\{\gamma(n)\}_{n=1}^\infty$ be an even sequence of non-negative numbers increasing to infinity. Let μ be a non-negative continuous, singular measure on the circle group \mathbb{T} such that [13]

$$\hat{\mu}(n) \approx 0 \left(\frac{\gamma(n)}{n^{1/2}} \right) \quad \text{and} \quad \mu(\mathbb{T}) = 1.$$

We define a compactly supported measure ν on \underline{t} as follows: Let $f \in C(\underline{t})$,

$$\langle \nu, f \rangle = \int_0^{2\pi} \dots \int_0^{2\pi} \bar{f}(a_1, \dots, a_\ell) \bar{\pi}(a_1, \dots, a_\ell) d\mu(a_1) \dots d\mu(a_\ell)$$

where

$$\begin{aligned} \bar{f}(a_1, \dots, a_\ell) &= f(H_p), \\ \bar{\pi}(a_1, \dots, a_\ell) &= \prod_{\alpha \in R_+} \tilde{\alpha}(H_p) \quad \text{when } p = 2a_1\tilde{\alpha}_1 + \dots + 2a_\ell\tilde{\alpha}_\ell. \end{aligned}$$

Let $\wedge = \{H_p | p = 2a_1\tilde{\alpha}_1 + \dots + 2a_\ell\tilde{\alpha}_\ell, 0 \leq a_i \leq 2\pi, i = 1, \dots, \ell\}$. Then ν is a singular measure on \underline{t} supported in \wedge .

Next we define an Ad-invariant measure μ_g on \underline{g} . Let $f \in C_0(\underline{g})$,

$$\langle \mu_g, f \rangle = \int_{\underline{t}} \int_{\mathcal{O}_H} f(Y) d\beta_H(Y) d\nu(H),$$

where \mathcal{O}_H denotes the orbit of H in \underline{g} and β_H is the surface measure on \mathcal{O}_H . Then μ_g is a continuous, singular, Ad-invariant measure of compact support on \underline{g} . Let

$$\mu_G = \Phi\mu_g.$$

It is easily seen that μ_G is a continuous, central, singular measure on G . We show that $\|\hat{\mu}_G\|_p < \infty$ for $p > 2$. Let $\sigma \in \hat{G}$; then $\hat{\mu}_G(\sigma) = \hat{\mu}_g(\lambda + \delta)I_{d_\sigma}$ where λ is the highest weight for the representation σ .

Now

$$\hat{\mu}_g(\lambda + \delta) = \int_{\wedge} \left(\int_{\mathcal{O}_H} e^{-i(\lambda + \delta)(X)} d\beta_H(X) \right) d\nu(H).$$

Using Harish-Chandra's formula for the Fourier-Stieltjes transform of a surface measure [1], we have

$$\begin{aligned} \hat{\mu}_g(\lambda + \delta) &= \frac{1}{\pi(H_{\lambda + \delta})} \int_{\wedge} \frac{1}{\pi(H)} \sum_{w \in W} \text{sgn}(w) e^{i(w \cdot (\lambda + \delta))(H)} d\nu(H) \\ &= \frac{1}{\pi(H_{\lambda + \delta})} \sum_{w \in W} \text{sgn}(w) \int_0^{2\pi} \dots \int_0^{2\pi} e^{i(w(\lambda + \delta), 2a_1\tilde{\alpha}_1 + \dots + 2a_\ell\tilde{\alpha}_\ell)} d\mu(a_1) \dots d\mu(a_\ell). \\ &= \frac{1}{\pi(H_{\lambda + \delta})} \sum_{w \in W} \text{sgn}(w) \prod_{i=1}^\ell \hat{\mu}(2\langle \lambda + \delta, w^{-1}\tilde{\alpha}_i \rangle). \end{aligned}$$

Hence

$$\|\hat{\mu}_G\|_p^p = \sum_{\sigma \in \hat{G}} d_\sigma \|\hat{\mu}_G(\sigma)\|_p^p = \sum_{\sigma \in \hat{G}} d_\sigma^2 |\hat{\mu}_g(\lambda + \delta)|^p.$$

By Weyl’s dimension formula, $d_\sigma = \frac{\prod_{\alpha \in R_+} \langle \lambda + \delta, \tilde{\alpha} \rangle}{\prod_{\alpha \in R_+} \langle \delta, \tilde{\alpha} \rangle}$, thus we obtain

$$\|\hat{\mu}_G\|_p^p \leq C \sum_{\sigma \in \hat{G}} \frac{1}{\left(\prod_{\alpha \in R_+} \langle \lambda + \delta, \tilde{\alpha} \rangle\right)^{p-2}} \left| \sum_{w \in W} \operatorname{sgn}(w) a_w \right|^p$$

where
$$a_w = \frac{\prod_{i=1}^\ell \gamma(2\langle \lambda + \delta, w^{-1} \tilde{\alpha}_i \rangle)}{\prod_{i=1}^\ell (2\langle \lambda + \delta, w^{-1} \tilde{\alpha}_i \rangle)^{1/2}}, \quad w \in W.$$

Since G is simply connected, \hat{G} is in one to one correspondence with the dominant integral weights given by

$$\lambda = \sum_{i=1}^\ell m_i \lambda_i, \quad m_i \in \mathbb{Z}^+.$$

Here, $\{\lambda_1, \dots, \lambda_\ell\} \subseteq \underline{t}^*$ is the set of fundamental dominant weights, i.e. $\langle \lambda_i, \tilde{\alpha}_j \rangle = \delta_{ij}$. Fix $\lambda \in \hat{G}$. Suppose $\lambda = \sum_{i=1}^\ell m_i \lambda_i$, $m_i \in \mathbb{Z}^+$ and $m = \max_{\alpha \in R} 2|\langle \lambda + \delta, \tilde{\alpha} \rangle|$.

Then $m \leq C_1(m_1 + \dots + m_\ell)$. Also for $w \in W$, $\prod_{i=1}^\ell 2|\langle \lambda + \delta, w^{-1} \tilde{\alpha}_i \rangle| \geq \prod_{i=1}^\ell (m_i + 1)$. Hence

$$\begin{aligned} \|\hat{\mu}_G\|_p^p &\leq C \sum_{m_1, \dots, m_\ell \in \mathbb{Z}^+} \frac{(\gamma(m))^{p\ell} |W|^p}{\left(\prod_{i=1}^\ell \langle \lambda + \delta, \tilde{\alpha}_i \rangle\right)^{p-2} \prod_{i=1}^\ell (m_i + 1)^{p/2}} \\ &\leq C \sum_{m_1, \dots, m_\ell \in \mathbb{Z}^+} \frac{(\gamma(m))^{p\ell} |W|^p}{\prod_{i=1}^\ell (m_i + 1)^{\frac{3p-4}{2}}}. \end{aligned}$$

Let

$$\begin{aligned} \gamma(n) &= 0 \quad \text{if } |n| \leq 1 \\ &= \log |n| \quad \text{if } |n| > 1. \end{aligned}$$

Then

$$\begin{aligned} \|\hat{\mu}_G\|_p^p &\leq C |W|^p \sum_{m_1, \dots, m_\ell \in \mathbb{Z}^+} \frac{(\log C_1 + \sum_{i=1}^\ell \log(m_i + 1))^{p\ell}}{\prod_{i=1}^\ell (m_i + 1)^{\frac{3p-4}{2}}} \\ &\leq C_2 \sum_{m_1, \dots, m_\ell \in \mathbb{Z}^+} \frac{\left(\prod_{i=1}^\ell \log(m_i + 1)\right)^{p\ell}}{\prod_{i=1}^\ell (m_i + 1)^{\frac{3p-4}{2}}} \\ &= C_2 \prod_{i=1}^\ell \sum_{m=0}^\infty \frac{(\log(m + 1))^{p\ell}}{(m + 1)^{\frac{3p-4}{2}}} < \infty \text{ for } p > 2. \end{aligned}$$

The proof is complete.

To complete the proof of Theorem 3.1 in the general case, we first prove two lemmas. For convenience, we will say that the group G has property \underline{P} if there exists a continuous, singular, central measure on G whose Fourier-Stieltjes transform is p -summable for every $p > 2$.

Lemma 3.2. *Let T be a compact connected abelian group, $\{G_\alpha\}_{\alpha \in I}$ a family of compact, simply connected, simple Lie groups and K a totally disconnected closed subgroup of the centre of $T \times \prod_{\alpha \in I} G_\alpha$. Suppose that there exists a $\beta \in I$ such that G_β has property \underline{P} . Then*

$$G = T \times \prod_{\alpha \in I} G_\alpha / K$$

has property \underline{P} .

Proof. Let μ_β be a measure on G_β corresponding to property \underline{P} of G_β . Define $\nu = \prod_{\alpha \in (I - \{\beta\})} \mu_\alpha \times \mu_\beta$, where μ_α is Haar measure on G_α for $\alpha \in (I - \{\beta\})$.

Then ν is a continuous, central, singular measure on $T \times \prod_{\alpha \in I} G_\alpha$ and the measure μ induced by ν on G shows that G has property \underline{P} . This completes the proof.

Lemma 3.3. *Let G be any compact group. Let G_0 be an open normal subgroup of G satisfying the property \underline{P} . Then G satisfies \underline{P} .*

Proof. Define a measure μ on G as follows: Let E be a Borel measurable subset of G ,

$$\mu(E) = \mu_0(E \cap G_0),$$

when μ_0 is a measure on G_0 satisfying \underline{P} . It is clear that μ is central, singular, continuous measure on G . We show that $\|\hat{\mu}\|_p < \infty$ for every $p > 2$.

Let n be the index of G_0 in G . If $\sigma \in \hat{G}$, then by Clifford's theorem ([8], Theorem 12.1) there exists $\alpha \in \hat{G}_0$ and natural numbers k, n_σ ($k, n_\sigma \leq n$) such that

$$\begin{aligned} \sigma|_{G_0} &= n_\sigma(\alpha^{(g_1)} + \dots + \alpha^{(g_k)}) \quad \text{when} \\ \alpha^{(g)}(h) &= \alpha(g^{-1}hg), \quad g \in G, h \in G_0, \end{aligned}$$

and $\{e = g_1, g_2, \dots, g_k\}$ is a complete set of coset representatives for $H_\alpha = \{g \in G | \alpha^{(g)} \text{ is equivalent to } \alpha\}$.

Let $\hat{\mu}(\sigma) = c_\sigma I_{d_\sigma}$. Now

$$\begin{aligned} c_\sigma &= \frac{1}{d_\sigma} \int_G \chi_\sigma(x^{-1}) d\mu(x) \\ &= \frac{n_\sigma d_\alpha}{d_\sigma} (c_{\alpha^{(g_1)}} + \dots + c_{\alpha^{(g_k)}}) \quad \text{where } \hat{\mu}_0(\alpha^{(g_i)}) = c_{\alpha^{(g_i)}} I_{d_\alpha}. \end{aligned}$$

Let $p > 2$. Then

$$\begin{aligned} \|\hat{\mu}\|_p^p &= \sum_{\sigma \in \hat{G}} d_\sigma \|\hat{\mu}(\sigma)\|_p^p = \sum_{\sigma \in \hat{G}} d_\sigma^2 |c_\sigma|^p \\ &= \sum_{\sigma \in \hat{G}} d_\sigma^2 \left(\frac{n_\sigma d_\alpha}{d_\sigma}\right)^p \left(\sum_{i=1}^k |c_{\alpha^{(g_i)}}|\right)^p. \end{aligned}$$

Since $d_\sigma = d_\alpha n_\sigma k$, we get

$$\|\hat{\mu}\|_p^p = \sum_{\sigma \in \hat{G}} \frac{d_\alpha^2 n_\sigma^2}{k^{p-2}} \left(\sum_{i=1}^k |c_{\alpha^{(g_i)}}|\right)^p.$$

Since ‘ α ’ can occur in at most n irreducible representations of G (as $1 \leq n_\sigma \leq n$), we get

$$\begin{aligned} \|\hat{\mu}\|_p^p &\leq n \sum_{\alpha \in \hat{G}_0} d_\alpha^2 n_\sigma^2 \left(\sum_{i=1}^k |c_{\alpha(g_i)}| \right)^p \\ &\leq n^4 \sum_{\alpha \in \hat{G}_0} d_\alpha^2 \max_{1 \leq i \leq k} |c_{\alpha(g_i)}|^p. \end{aligned}$$

Now $\sum_{\alpha \in \hat{G}_0} d_\alpha^2 \max_{1 \leq i \leq k} |c_{\alpha(g_i)}|^p \leq \sum_{\alpha \in \hat{G}_0} d_\alpha^2 |c_\alpha|^p$. Therefore

$$\|\hat{\mu}\|_p^p \leq n^4 \sum_{\alpha \in \hat{G}_0} d_\alpha^2 |c_\alpha|^p = n^4 \|\hat{\mu}_0\|_p^p < \infty$$

for $p > 2$.

This completes the proof.

End of the proof of Theorem 3.1. The theorem follows now by a standard argument using the structure theorem of compact connected groups (see [9]), Lemma 3.2 and Lemma 3.3.

Remark 3.4. S. Sakai [12] has shown that there exists a singular probability measure μ on the circle group \mathbb{T} such that $\mu * \mu$ is absolutely continuous and the Radon-Nikodym derivative of $\mu * \mu$ has a uniformly convergent Fourier series. We are unable to show that the measure μ constructed in Theorem 3.1 has this stronger property.

4. TOTALLY DISCONNECTED GROUPS

In this section we prove the following analogue of Theorem 3.1 on some totally disconnected groups.

Theorem 4.1. *Let $\{G_n\}_{n=1}^\infty$ be a sequence of finite non-abelian groups such that there exist two constants $C_1, C_2 > 0$ satisfying: For every n , there exists an irreducible character χ_n of G_n such that*

$$(4.2) \quad \operatorname{Re}(\chi_n) \geq -C_1 \quad \text{on } G_n$$

$$(4.3) \quad \text{and} \quad \frac{1}{|G_n|} \sum_{x \in A_n} (\operatorname{Re} \chi_n(x))^2 \geq C_2$$

$$\text{when} \quad A_n = \{x \in G_n \mid \operatorname{Re} \chi_n(x) < 0\}.$$

*Then there exists a continuous, singular, central measure μ on $G = \prod_{n=1}^\infty G_n$ such that $\mu * \mu$ is absolutely continuous with respect to Haar measure on G and the Lebesgue Radon-Nikodym derivative of $\mu * \mu$ belongs to L^p for all p , $1 \leq p < \infty$.*

Proof. Let $\{b_n\}_{n=1}^\infty$ be a sequence of non-negative numbers such that

$$0 < b_n C_1 < 1 \text{ for every } n, \quad \sum_{n=1}^\infty b_n^2 = \infty \text{ and } \sum_{n=1}^\infty b_n^p < \infty, \text{ for every } p > 2.$$

Define $g_n = \operatorname{Re} \chi_n$, and $f_n = 1 + b_n g_n$.

Let $\mu = \prod_{n=1}^{\infty} f_n dx$, where dx denotes the Haar measure on G_n . Then μ is a central continuous measure on G .

μ is **singular**: By Kakutani's criterion ([5], Theorem 22.36) for singularity of product measures, it is sufficient to show that

$$\prod_{n=1}^{\infty} \int_{G_n} f_n^{1/2} dx = 0.$$

Now

$$\int_{G_n} f_n^{1/2} dx = \int_{A_n} f_n^{1/2} dx + \int_{G_n - A_n} f_n^{1/2} dx.$$

Also

$$\begin{aligned} \int_{A_n} f_n^{1/2} dx &= \frac{1}{|G_n|} \sum_{x \in A_n} (1 - (-b_n g_n)(x))^{1/2} \\ (4.4) \qquad \qquad &\leq \frac{1}{|G_n|} \sum_{x \in A_n} \left(1 + \frac{b_n g_n(x)}{2} - \frac{b_n^2 g_n^2(x)}{16} \right), \end{aligned}$$

and

$$\begin{aligned} \int_{G_n - A_n} f_n^{1/2} dx &\leq \left(\int_{G_n - A_n} f_n dx \right)^{1/2} \left(\frac{|G_n| - |A_n|}{|G_n|} \right)^{1/2} \\ (4.5) \qquad \qquad &\leq \frac{(|G_n| - |A_n|)}{|G_n|} \left(1 + \frac{\sum_{x \in A_n} (-b_n g_n(x))}{2(|G_n| - |A_n|)} \right). \end{aligned}$$

By (4.4) and (4.5), we get

$$\int_{G_n} f_n^{1/2} dx \leq 1 - \frac{1}{|G_n|} \sum_{x \in A_n} \frac{b_n^2 g_n^2(x)}{16} \leq 1 - \frac{b_n^2 C_2}{16}.$$

Since $\sum_{n=1}^{\infty} b_n^2 = \infty$, we conclude ([5], Lemma 22.25) that $\prod_{n=1}^{\infty} \int_{G_n} f_n^{1/2} = 0$.

Computation of $\|\hat{\mu}\|_p$: Let $p > 2$.

$$\begin{aligned} \|\hat{\mu}\|_p^p &= \sum_{\sigma \in \hat{G}} d_{\sigma} \|\hat{\mu}(\sigma)\|_p^p = \sum_{\sigma \in \hat{G}} d_{\sigma}^{2-p} \prod_{i=1}^k \left| \left(\frac{b_{n_i}}{2} \left(1 + \int_{G_{n_i}} \overline{\chi_{n_i}(x)}^2 dx \right) \right) \right|^p \\ &\leq \sum_{n_1, \dots, n_k \in \mathbb{N}} \prod_{i=1}^k (b_{n_i}^p) = \prod_{i=1}^{\infty} (1 + b_i^p) < \infty \text{ as } \sum_{i=1}^{\infty} b_i^p < \infty. \end{aligned}$$

This completes the proof of the theorem.

Remark 1. Conditions (4.2) and (4.3) are satisfied by $G = \prod_{n=1}^{\infty} G_n$ if G_n 's are symmetric groups, dihedral groups and generalised quaternion groups (see [4], 27.61).

REFERENCES

1. J.L. Clerc, *Orbites dans le plan tangent d'un espace symétrique, mesures orbitales et leurs transformées de Fourier*, *Topics in Modern Harmonic Analysis, Vol.I*, Proc. of a seminar held in Torino and Milano, May-June 1982, pp. 259-300. MR **86c**:22013
2. A.H. Dooley and N.J. Wildberger, *Global formulae on compact Lie groups*, *Funktsional. Anal. i Prilozhen.* **27** (1) (1993), 25-32; English transl. , *Functional Anal. Appl.* **27** (1993), 21-27. MR **94e**:22032
3. E. Hewitt and K. Ross, *Abstract Harmonic Analysis, Vol.I*, Springer-Verlag, Berlin, 1963. MR **28**:158

4. E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Vol.II, Springer-Verlag, Berlin, 1970. MR **42**:7378
5. E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, 3rd printing, Springer-Verlag, Berlin, 1975. MR **51**:3363
6. E. Hewitt and H. Zuckerman, *Singular measures with absolutely continuous convolution squares*, Proc. Cambridge Phil. Soc. **62** (1966), 399–420; , vol. 63, 1967, pp. 367–368. MR **33**:1655; MR **34**:8497
7. C. Karanikas and S. Koumandos, *Continuous singular measures with absolutely continuous convolution squares and locally compact groups*, Illinois J. of Math. **35** (3) (1991), 490–495. MR **92b**:43004
8. G.W. Mackey, *Induced representations of locally compact groups I*, Ann. of Math. **55** (1952), 101–139. MR **13**:434
9. J.F. Price, *Lie groups and compact groups*, Cambridge University Press, Cambridge, 1977. MR **56**:8743
10. D.L. Ragozin, *Central measures on compact simple Lie groups*, J. of Functional Analysis **10** (1972), 212–229. MR **49**:5715
11. S. Saeki, *Singular measures having absolutely continuous convolution powers*, Illinois J. Math. **21** (1977), 395–412. MR **58**:6719
12. S. Saeki, *On convolution squares of singular measures*, Illinois J. Math. **24** (1980), 225–232. MR **81i**:42005
13. A.C. Schaeffer, *The Fourier-Stieltjes coefficients of a function of bounded variation*, Amer. J. Math. **61** (1939), 934–940. MR **1**:12
14. K. Stempak, *On convolution products of radial measures on the Heisenberg group*, Colloq. Math. **50** (1985), 125–128. MR **87d**:22014

SCHOOL OF MATHEMATICS, UNIVERSITY OF NEW SOUTH WALES, NEW SOUTH WALES 2052, AUSTRALIA

E-mail address: `tony@solution.maths.unsw.edu.au`

E-mail address: `sanjiv@solution.maths.unsw.edu.au`

Current address, S. Gupta: Department of Mathematics, University of South Pacific, Suva, Fiji Islands