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EQUIVALENT CONDITIONS INVOLVING COMMON FIXED POINTS FOR MAPS ON THE UNIT INTERVAL

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ABSTRACT. Let g be a continuous self-map of the unit interval I. Equivalent conditions are given to ensure that g has a common fixed point with every continuous map $f : I \mapsto I$ that commutes with g on a suitable subset of I. This extends a recent result of Gerald Jungck.

1. Introduction

Let f and g be two commuting continuous self-maps of I, the unit interval. It is known that f and g need not have then a common fixed point (for counterexamples, see, e.g., [6]). However, if one of the maps, say g, has appropriate additional properties then f and g possess a common fixed point. In particular, W. Boyce [1, Corollary 5] has shown that it suffices to assume the family $\{g^n : n \in \boldsymbol{N}\}$, iterates of g, is equicontinuous on I. This result has been extended by J. Cano [2, Theorems 1 and 2] who has required that either g has a closed interval for its set of fixed points $F(g)$, or $F(g)$ coincides with $P(g)$, the set of all periodic points of g. It is worth underlining here that Corollary 5 [1] as well as Theorem 1 [2] give only sufficient conditions for the existence of a common fixed point of f and g.

On the other hand, recently, Gerald Jungck [7, Theorem 3.6] established the following interesting equivalence: a continuous self-map g of I has a common fixed point with every continuous map $f : I \mapsto I$ that nontrivially commutes with g on the set of coincidence points of f and g if and only if $P(g) = F(g)$.

Our purpose here is to give other necessary and sufficient criteria of this type (see Theorems 1, 2 and 3). We also obtain a variant of Jungck's Theorem in more abstract settings as compact and convex subsets of a normed linear space (see Proposition 1).

2. Equivalent conditions

Following Boyce [1] and Cano [2] we define the classes of maps:

$$\boldsymbol{B} \doteq \{g : I \mapsto I \mid \{g^n : n \in \boldsymbol{N}\} \text{ is equicontinuous on } I\},$$

$$\boldsymbol{C}_1 \doteq \{g : I \mapsto I \mid g \text{ is continuous and } F(g) \text{ is a closed interval}\},$$

$$\boldsymbol{C}_2 \doteq \{g : I \mapsto I \mid g \text{ is continuous and } F(g) = P(g)\}.$$

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Let f and g be continuous self-maps of I. Then we have :

(1) If f and g commute on I and $g \in \boldsymbol{B}$ then $F(f) \cap F(g) \neq \emptyset$ [1, Corollary 5].

(2) If f and g commute on I and $g \in \boldsymbol{C}_1 \cup \boldsymbol{C}_2$ then $F(f) \cap F(g) \neq \emptyset$; moreover, $\boldsymbol{B} \subseteq \boldsymbol{C}_1$. If $g \in B$ and $F(g)$ is not a singleton then $g \in \boldsymbol{C}_2$ [2, Theorems 1 and 2].

Next, by [7, Theorem 3.6], $g \in \boldsymbol{C}_2$ if and only if $F(f) \cap F(g) \neq \emptyset$ for every continuous map $f : I \mapsto I$ such that the set of coincidence points of f and g is non-empty, and f and g commute on it.

Inspired by the last result we give other characterizations of the classes $\boldsymbol{C}_1$, $\boldsymbol{C}_2$ and $\boldsymbol{B}$.

Theorem 1. *Let g be a continuous self-map of I. Then the following conditions are equivalent:*

(i) *$g \in \boldsymbol{C}_1$;*

(ii) *the family $\{g^n : n \in \boldsymbol{N}\}$ is equicontinuous on $F(g)$, or $F(g)$ is a singleton;*

(iii) *g has a common fixed point with every continuous map $f : I \mapsto I$ that commutes with g on $F(g)$.*

Proof. (i) $\Rightarrow$ (ii). If $F(g)$ is a singleton, we are done. So suppose $F(g) = [a, b]$ and $a < b$. Obviously, it suffices to show that $\{g^n : n \in \boldsymbol{N}\}$ is equicontinuous at the points a and b. And because of symmetry, we consider only the point a. The case when $a = 0$ is trivial. So suppose $a > 0$. Fix an $\epsilon \in (0, b - a)$. By the continuity, there exists a $\delta \in (0, \epsilon)$ such that

$$a - \epsilon < g(x) < a + \epsilon \text{ for all } x \in (a - \delta, a) \cap I. \tag{1}$$

We shall apply induction on n to show that, for all $n \in \boldsymbol{N}$,

$$a - \epsilon < g^n(x) < a + \epsilon \text{ for all } x \in (a - \delta, a) \cap I. \tag{2}$$

By (1), (2) holds if $n = 1$. Assuming (2) holds for $n = 1, 2, ..., k$, we shall prove it for $k + 1$. Fix an $x \in (a - \delta, a) \cap I$. If $a \leq g^k(x) < a + \epsilon$ then $g^k(x) \in F(g)$ since $a + \epsilon < b$ so (2) is fulfilled for $n = k + 1$. Assume now that $g^k(x) < a$. Then $g^i(x) < a$ for $i = 1, 2, ..., k$; for otherwise, by induction hypothesis, $a \leq g^i(x) < a + \epsilon$ for some i, $1 \leq i \leq k$ so $g^i(x) \in F(g)$, which implies $g^k(x) \in F(g)$ and hence $g^k(x) \geq a$, a contradiction. Since $F(g) = [a, b]$, we have $g(y) > y$ for all $y \in [0, a)$. In particular, $g^i(x) > g^{i-1}(x)$ for $i = 1, 2, ..., k$, which implies $g^k(x) > x$. Since $x > a - \delta$ and $g^k(x) < a$, we obtain that $g^k(x) \in (a - \delta, a) \cap I$. By (1), $a - \epsilon < g^{k+1}(x) < a + \epsilon$, which completes the induction.

Since $\delta < \epsilon < b - a$, we have $g^n(x) = x$ for $x \in [a, a + \delta)$ and $n \in \boldsymbol{N}$. So finally, $a - \epsilon < g^n(x) < a + \epsilon$ for all $x \in (a - \delta, a + \delta)$ and $n \in \boldsymbol{N}$. This proves $\{g^n : n \in \boldsymbol{N}\}$ is equicontinuous at the point a.

(ii) $\Rightarrow$ (i). This implication follows from the proof of Cano's Theorem 1 [2].

(i) $\Rightarrow$ (iii). If f commutes with g on $F(g)$ then $F(g)$ is f-invariant so $f|_{F(g)}$ has a fixed point since $F(g)$ is a closed interval.

(iii) $\Rightarrow$ (i). Suppose $F(g)$ is not an interval. There exist $a, b \in F(g)$, $a < b$, such that $(a, b) \cap F(g) = \emptyset$. Define the map f: $f(x) \doteq b$ for $x \in [0, a]$, $f(x) \doteq -x + a + b$ for $x \in (a, b)$, and $f(x) \doteq a$ for $x \in [b, 1]$. Then f is continuous and for $x \in F(g)$, either $x \in [0, a]$ and then $f(g(x)) = g(f(x)) = b$, or $x \in [b, 1]$ and then $f(g(x)) = g(f(x)) = a$. Thus f and g commute on $F(g)$, but $F(f) \cap F(g) = \emptyset$, which contradicts (iii). ∎

The following example shows that we cannot omit the condition "$F(g)$ is a singleton" in (ii) of Theorem 1.

Example 1. Define the map g on I as follows:

$$g(x) \doteq 1 \text{ for } x \in [0, \frac{1}{4}],\ g(x) \doteq -2x + \frac{3}{2} \text{ for } x \in (\frac{1}{4}, \frac{3}{4}),\ g(x) \doteq 0 \text{ for } x \in [\frac{3}{4}, 1].$$

Clearly, $F(g) = \{\frac{1}{2}\}$ so $g \in \boldsymbol{C}_1$. However, it is easy to verify that the family $\{g^n : n \in \boldsymbol{N}\}$ is not equicontinuous at the point $\frac{1}{2}$.

Theorem 2. *Let g be a continuous self-map of I. Then the following conditions are equivalent:*

(i) $g \in \boldsymbol{C}_2$;
(ii) *the sequence $\{g^n\}_{n=1}^{\infty}$ is pointwise convergent on I;*
(iii) *g has a common fixed point with every continuous map $f : I \mapsto I$ that commutes with g on $F(f)$.*

Proof. That (i) implies (ii) was proved by S. C. Chu and R. D. Moyer [3, Theorem 1] and, independently, by E. M. Coven and G. A. Hedlund [4, Theorem 2]. To prove (ii) implies (iii) fix an $x \in F(f)$. Since, by the commutativity, $F(f)$ is g-invariant we have $g^n(x) \in F(f)$ for $n \in \boldsymbol{N}$. By (ii), $\{g^n(x)\}_{n=1}^{\infty}$ converges to some $z \in I$. Then $z \in F(f) \cap F(g)$ since $F(f)$ is closed and g is continuous. To prove (iii) implies (i) it suffices to show that for any non-empty closed g-invariant set $C \subseteq I$, $C \cap F(g) \neq \emptyset$ and then apply [3, Theorem 1]. Fix such a set C. There exists a continuous map $f : I \mapsto I$ such that $F(f) = C$. If $x \in F(f)$ then $g(f(x)) = g(x)$ and $f(g(x)) = g(x)$ since C is g-invariant. Thus f and g commute on $F(f)$ so, by (iii), $F(f) \cap F(g) \neq \emptyset$, i.e., $C \cap F(g) \neq \emptyset$. □

Remark 1. *The sufficiency part of Jungck's Theorem 3.6* [7] *is easily subsumed by Theorem 2: if f and g commute at their coincidence points, $P(g) = F(g)$ and $f(a) = g(a)$ then $f^n(a) = g^n(a)$ for $n \in \boldsymbol{N}$. By Theorem 2 ((i) ⇒ (ii)), $\{g^n(a)\}_{n=1}^{\infty}$ is convergent to some b. Then, by the continuity, b is a common fixed point of f and g.*

Before stating the next theorem let us notice that a common fixed point theorem for a family of commuting maps would be trivial if we assumed one of them had a *unique* fixed point. This justifies a use of the assumption "$F(g)$ is not a singleton" in Theorem 3 below.

Theorem 3. *Let g be a continuous self-map of I such that $F(g)$ is not a singleton. Then the following conditions are equivalent:*

(i) *$g \in \boldsymbol{B}$;*
(ii) *the sequence $\{g^n\}_{n=1}^{\infty}$ is uniformly convergent on I;*
(iii) *g has a common fixed point with every continuous map $f : I \mapsto I$ that commutes with g either on $F(f)$, or on $F(g)$.*

Proof. (i) ⇒ (ii). If $g \in \boldsymbol{B}$ and $F(g)$ is not a singleton then, by [2, Theorem 2], $g \in \boldsymbol{C}_2$. By Theorem 2, $\{g^n\}_{n=1}^{\infty}$ is pointwise convergent on I which implies (ii), since $\{g^n : n \in \boldsymbol{N}\}$ is equicontinuous.

(ii) ⇒ (iii). This implication easily follows from Theorems 1 and 2.

(iii) ⇒ (i). By Theorem 2, $\{g^n\}_{n=1}^{\infty}$ is pointwise convergent, which implies $F(g) = F(g^2)$. On the other hand, by Theorem 1, $F(g)$ is an interval. Therefore, $F(g^2)$ is an interval, so by [1, Lemma 1 and Theorem 5], $\{g^n : n \in \boldsymbol{N}\}$ is equicontinuous, i.e., $g \in \boldsymbol{B}$. □

Remark 2. The condition that $F(g)$ is not a singleton is necessary in Theorem 3 (consider the map $g(x) \doteq 1 - x$ $(x \in I)$, for which (i) holds but (ii) is not fulfilled). Further, Example 1 shows that one cannot modify Theorem 3 similarly to Theorem 1. More precisely, we cannot add in (ii) of Theorem 3 the text "or $F(g)$ is a singleton" and, simultaneously, remove the assumption "$F(g)$ is not a singleton" occurring at the beginning of Theorem 3.

We now give a partial extension of Theorem 2 for maps on subsets of a normed linear space.

Proposition 1. *Let A be a non-empty compact and convex subset of a normed linear space and let g be a continuous self-map of A. Then the following conditions are equivalent:*

(i) *for any non-empty closed and g-invariant set $C \subseteq A$, $C \cap F(g) \neq \emptyset$;*
(ii) *g has a common fixed point with every continuous map $f : A \mapsto A$ that commutes with g on $F(f)$.*

Remark 3. It follows from [3, Theorem 1] that in case when $A = I$, the conditions (i) of Proposition 1 and (i) of Theorem 2 are equivalent.

Proof of Proposition 1. (i) ⇒ (ii). Let a continuous map $f : A \mapsto A$ commute with g on $F(f)$. Then $F(f)$ is g-invariant and closed. Moreover, by Schauder's Fixed Point Theorem, $F(f)$ is non-empty. So by (i), we get $F(f) \cap F(g) \neq \emptyset$.

(ii) ⇒ (i). Let C be a non-empty closed g-invariant subset of A. We show that $C \cap F(g) \neq \emptyset$. The case when $C = A$ is trivial. So assume $C \neq A$ and fix a point $a \in A \setminus C$. There exists a continuous function $\phi : A \mapsto I$ such that $\phi^{-1}(0) = a$ and $\phi^{-1}(1) = C$ (see, e.g., [5, Theorem 1.5.19, p.69]). Assume further that $0 \in C$. Define a map f by $f(x) \doteq \phi(x)x$ for $x \in A$. Then $f(A) \subseteq A$ by convexity, and $F(f) = C$, since $x = f(x)$ iff $x = 0$ or $\phi(x) = 1$. For $x \in C$, $g(f(x)) = g(x)$ since $f(x) = x$, and $f(g(x)) = g(x)$ since C is g-invariant and $F(f) = C$. Thus f and g commute on $F(f)$. By (ii), $F(f) \cap F(g) \neq \emptyset$, i.e., $C \cap F(g) \neq \emptyset$. In case when $0 \notin C$, fix a point $c \in C$, consider the sets $A' \doteq A - c$, $C' \doteq C - c$, and repeat the above argument to deduce there exists a continuous map $f' : A' \mapsto A'$ such that $F(f') = C'$. Next, define a map f as $f(x) \doteq f'(x - c) + c$ for $x \in A$ and apply (ii) for such a map f to obtain that $C \cap F(g) \neq \emptyset$. □

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