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A NOTE ON MILLER'S THEOREM ABOUT MAPS OUT OF CLASSIFYING SPACES

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Abstract. Let X be a connected infinite loop space whose fundamental group is a torsion group and let Y be a finite nilpotent CW-complex. The main result of this paper is that the space of based maps from X to the profinite completion of Y is weakly contractible.

1. Introduction

The theorem to which the title refers is Haynes Miller's celebrated result,

Theorem 1. *Let G be a discrete, locally finite group and let Y be a connected finite dimensional CW-complex. Then the space of based maps from the classifying space of G to Y is weakly contractible.*

Given two spaces X and Y with base points, let $Map_*(X,Y)$ denote the function space of all maps from X to Y which preserve those base points. The iterated loop space $\Omega^k Map_*(X,Y)$ is easily seen to be homeomorphic to $Map_*(X,\Omega^k Y)$. Thus if X is arcwise connected, the weak contractibility of $Map_*(X,Y)$ is equivalent to the condition that every based map from X to $\Omega^k Y$ is null homotopic for every $k \geq 0$.

Miller's theorem is the crucial ingredient in the following result. In it $\widehat{Y}$ denotes the profinite completion of a space Y in the sense of Sullivan [1], [8]. The first case in this theorem is essentially due to Zabrodsky [9], while the second is due to Friedlander and Mislin [3].

Theorem 2. *$Map_*(X,\widehat{Y})$ is weakly contractible if Y is a nilpotent finite CW-complex and if either*
1) X is a connected CW-complex whose fundamental group is locally finite and $\pi_n X = 0$ for n sufficiently large, or
2) $X = BG$ where G is a Lie group with only a finite number of path components.

This result provides one with a large source of *phantom maps* ([5], Section 5). This term will be used in this paper to describe any based map $f : X \to Y$ with the property that for any finite CW-complex K and any map $g : K \to X$, the

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composition fg is null homotopic.[1] When Y is a nilpotent target of finite type, the profinite completion $Y \to \widehat{Y}$ induces a function $Map_*(X,Y) \to Map_*(X,\widehat{Y})$ whose kernel is precisely the set of all phantom maps from X to Y. Thus Theorem 2 asserts that every map from X to $\Omega^k Y$ is a phantom map when X and Y satisfy the stated conditions.

The purpose of this note is to derive another result similar to Theorem 2 from Miller's Theorem.

Theorem 3. *Let X be a connected infinite loop space whose fundamental group is a torsion group and assume that Y is a nilpotent finite CW-complex. Then the space of based maps from X to $\widehat{Y}$ is weakly contractible.*

Remarks. 1) Notice that it is necessary in Theorem 3 that $\pi_1 X$ does not contain an infinite cyclic summand. Otherwise X would dominate S^1 and thus $Map_*(X,\widehat{Y})$ would be weakly contractible if and only if $\Omega\widehat{Y}$ was.

2) The weak contractibility of $Map_*(X,\widehat{Y})$ in Theorems 2 and 3 implies that, for all integers $k \geq 0$,

$$[X,\Omega^k Y] \;\approx\; [X_o,\Omega^k Y]$$

by Theorem 5.4 of [5]. Here X_o denotes the rationalization of X and $[X,Z]$ denotes the set $\pi_0 Map_*(X,Z)$. It then follows from Theorem 5.2, ibid., that

$$[X_o, Z] \;\approx\; \prod_{j\geq 1} H^j(X;\pi_{j+1}(Z)\otimes\mathbb{R}),$$

where $\mathbb{R}$ is a rational vector space whose cardinality equals that of the real numbers. Moreover, when Z is rationally an H-space, there is a natural group structure on $[X_o, Y]$ so that the above bijection is an isomorphism of rational vector spaces. Thus in these special cases one has a simple and complete classification of homotopy classes of maps (all of which are phantoms) from X to the iterated loop spaces of Y.

Proof of Theorem 3. We start with a special case, where X is a path component of QS^0. As usual, $QA = \operatorname{colim}\Omega^n\Sigma^n A$ for a space A. Let $\mathcal{S}_\infty$ denote the infinite symmetric group. There is a map

$$B\mathcal{S}_\infty \longrightarrow X$$

which induces an isomorphism in integral homology and in mod p homology for all primes p [7]. Let C denote the cofiber of this map and note that C is acyclic. The cofiber sequence

$$B\mathcal{S}_\infty \longrightarrow X \longrightarrow C$$

induces a fibration

$$Map_*(B\mathcal{S}_\infty, Y) \longleftarrow Map_*(X,Y) \longleftarrow Map_*(C,Y),$$

for any based target Y. Assume now that Y is finite dimensional. The group $\mathcal{S}_\infty$ is locally finite as it is isomorphic to the direct limit of the finite symmetric groups. Thus the base space of this fibration is weakly contractible by Miller's Theorem.

[1] These are referred to as phantom maps of the second kind in [5]. Phantom maps of the first kind are maps out of a CW-complex whose restrictions to each skeleton are null-homotopic. Of course when the domain is a CW-complex with only a finite number of cells in each dimension, the two kinds of phantom maps coincide.

Assume in addition that Y is nilpotent and let $Y^{(n)}$ denote the nth stage in the principal refinement of its Postnikov decomposition. It follows that $[C, Y^{(n)}] = *$ for each n, by an easy induction argument. Thus the only maps from C to Y are phantom maps. However, if Y has finite type, it follows from Example 4.1 of [4] that there are no essential phantom maps from C to Y. Thus if Y is a nilpotent finite complex, it follows that $Map_*(C, Y)$ is weakly contractible and hence the same is true of $Map_*(X, Y)$. □

Now let $n \geq 2$ and consider the principal fibration sequence

$$F_n \longrightarrow QS^n \longrightarrow K(\mathbb{Z}, n)$$

where the second map is an n-equivalence. To this fibration we will apply the following result from [6]. It is often referred to as the Zabrodsky lemma.

Lemma 4. *Let $E \to B$ be a principal bundle with structure group G. If $Map_*(G, Y)$ is contractible, then $Map_*(B, Y) \to Map_*(E, Y)$ is a homotopy equivalence.*

Choose a simplicial model $G \to E \to B$ for the fibration $F_n \to QS^n \to K(\mathbb{Z}, n)$. Let Y be a nilpotent finite complex and assume for the moment that $Map_*(G, \widehat{Y})$ is contractible. The contractibility of $Map_*(B, \widehat{Y})$ follows at once from the first part of Theorem 2. The weak contractibility of $Map_*(QS^n, \widehat{Y})$ is then a consequence of the Zabrodsky lemma. To show that $Map_*(G, \widehat{Y})$ is contractible note that $F_{n-k} \simeq \Omega^k F_n$ and note that the case of F_0 was established at the beginning of the proof. The weak contractibility of $Map_*(F_n, \widehat{Y})$ then follows by n applications of the Zabrodsky lemma to fibrations of the form

$$F_{k-1} \longrightarrow * \longrightarrow F_k.$$

Suppose that K and L are finite CW-complexes which occur in a cofiber sequence

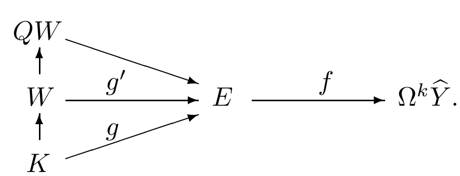
$$S^n \longrightarrow K \longrightarrow L.$$

The functor $Q(\ \ )$ converts this into a principal fibration sequence

$$QS^n \longrightarrow QK \longrightarrow QL.$$

Hence if $n \geq 2$ and $Map_*(QK, \widehat{Y})$ is weakly contractible, the Zabrodsky lemma implies the same is true of $Map_*(QL, \widehat{Y})$. Thus, by induction, Theorem 3 is true whenever the source is an infinite loop space of the form QW where W is a 1-connected finite complex.

Let E be a 1-connected infinite loop space and let $f : E \to \Omega^k \widehat{Y}$ be any map. Let $g : K \to E$ be a map of a finite CW-complex into E. Since E is 1-connected, the map g factors though a 1-connected finite CW-complex, say $W = K/K_1$. The quotient map $g' : W \to E$ then factors through QW as in the following diagram:



Since we already know that Theorem 3 holds for QW, it follows that the composition fg is null-homotopic and thus f is a phantom map. Since there are no essential phantom maps into a complete space, it follows that Theorem 3 now holds for all

1-connected infinite loop spaces. The non-simply-connected case follows using the fibration

$$\tilde{E} \longrightarrow E \longrightarrow K(\pi_1 E, 1),$$

where the second map induces an isomorphism on fundamental groups. The Zabrodsky lemma together with Miller's theorem applied to the base implies that $Map_*(E, \widehat{Y})$ is weakly contractible and completes the proof of Theorem 3.

I would like to acknowledge the referee's valuable help on this paper. Among other things, he pointed out that Zabrodsky's Theorem 2.1 should not require the domain X to be simply connected or to have finite type as it originally did. The referee's proof goes as follows. In the simply connected version one can use Lemma 4 to reduce the theorem to the case $X = K(\pi, n)$ or even to the case $X = K(\pi, 2)$. Choose a free resolution of π

$$0 \longrightarrow F_0 \longrightarrow F_1 \longrightarrow \pi \longrightarrow 0,$$

and consider the associated fibration sequence

$$K(F_0, 2) \longrightarrow K(F_1, 2) \longrightarrow K(\pi, 2).$$

Another application of Lemma 4 shows that it is enough to prove Theorem 2 in the case $X = K(F, 2)$ for a free abelian group F. Now consider the fibration sequence

$$K(\mathbb{Q}/\mathbb{Z} \otimes F, 1) \longrightarrow K(F, 2) \longrightarrow K(\mathbb{Q} \otimes F, 2).$$

The theorem is clearly true in the case $X = K(\mathbb{Q} \otimes F, 2)$, since this space is rational, so it is enough (again by Lemma 4) to prove it for $X = K(\mathbb{Q}/\mathbb{Z} \otimes F, 1)$. This then follows directly from Miller's Theorem and an arithmetic square argument ([6], Theorem 1.5).

To finish the proof of Theorem 2.1 in the non-simply-connected case one has to deal with the possibility that X might not be a simple space. In this case one uses the following unbased enhancement of the Zabrodsky lemma.

Lemma 4′. *Let $F \to E \to B$ be a bundle where B is a connected CW-complex. If $F \to *$ induces weak equivalence $Y \to Map(F, Y)$, then $E \to B$ induces a weak equivalence $Map(B, Y) \to Map(E, Y)$.*

This is easily proved by induction on the cells of B.

The referee also pointed out the following very different proof of Theorem 3. As has been shown already it suffices to prove the result for QW where W is a finite 1-connected complex. After replacing W if necessary by the realization of its singular complex, we can assume that $W = |K|$ for some simplicial set K. Consider the n-fold product W^n as a G-space where G is the symmetric group on n letters. It is clear that W^n can be given the structure of a $G - CW$-complex. ($W^n = |K^n|$ in such a way that the action of G on W^n is the geometric realization of an action of G on K^n; the cells of W^n then correspond to the nondegenerate simplices of K^n. and they are permuted in a $G - CW$ way by the action of G.)

Consider the quotient map

$$(W^n \times EG)/G \to W^n/G = \mathrm{SP}^n\, W$$

and observe that W^n/G inherits a CW-structure from the $G - CW$-structure on W^n. Let e be the interior of a cell in W^n/G. It is clear that the inverse image of e in $(W^n \times EG)/G$ is $e \times BH$, where H is the isotropy subgroup of e. One then sees by induction on the cell decomposition of W^n/G that the above quotient map induces a weak equivalence

$$Map(\mathrm{SP}^n W, Y) \to Map((W^n \times EG)/G, Y)$$

for a finite dimensional target Y. Passing to the limit and identifying QW as a completion of $\mathrm{colim}(W^n \times E\Sigma_n)/\Sigma_n$ (see [2], pp. 50–59), one concludes that the map

$$QW \to \mathrm{SP}^\infty W$$

induces a weak equivalence

$$Map(\mathrm{SP}^\infty W, Y) \to Map(QW, Y).$$

The proof is completed by noting that $\mathrm{SP}^\infty W$ has the homotopy type of a finite product of Eilenberg Mac Lane spaces, and then appealing to Zabrodsky's Theorem 2.1.

Let me conclude by mentioning one aspect of Miller's theorem which fails to generalize. Since Theorem 1 can be regarded as saying that every map from $\Sigma^k BG$ to Y is null-homotopic, for every k, it seemed natural to wonder if the same conclusion would hold if $\Sigma^k BG$ were replaced by the Thom space of a vector bundle over BG. After all, a Thom space is something like a twisted suspension. One does not have to look very far to see that the answer is no. Let P_n denote infinite real projective space with the subspace $\mathbb{R}P^{n-1}$ collapsed to a point. Then P_n is the Thom space of an n-plane bundle over $\mathbb{R}P^\infty$. Moreover, it is easy to check that $Hom_{\mathcal{A}}(\bar{H}_*(\Sigma^t P_n), M) = 0$ for every bounded $\mathcal{A}$-module M, just as in Lemma 1.15 of [6]. Nevertheless there is an obvious map $P_n \to \Sigma\mathbb{R}P^{n-1}$ which is essential.

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