

A CLASS OF COMPLETE SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper is concerned with a class of complete second order linear differential equations in a Banach space. We show the existence and uniqueness of classical solutions of

$$(SE) \quad \begin{cases} u''(t) = A(t)u'(t) + B(t)u(t) + f(t) \text{ for } t \in [0, T] \\ u(0) = x \text{ and } u'(0) = y. \end{cases}$$

1. INTRODUCTION

In this paper we study complete second order linear differential equations

$$(SE) \quad \begin{cases} u''(t) = A(t)u'(t) + B(t)u(t) + f(t) \text{ for } t \in [0, T] \\ u(0) = x \text{ and } u'(0) = y \end{cases}$$

in a Banach space X . Here $\{A(t) : t \in [0, T]\}$ is a family of closed linear operators in X with the common domain D of $A(t)$ and $\{B(t) : t \in [0, T]\}$ is a family of linear operators in X with $D \subset D(B(t))$ for $t \in [0, T]$.

In this case, as pointed out in [5], (SE) is closely related to the first order abstract Cauchy problem

$$(CP; u_0, f) \quad \begin{cases} u'(t) = A(t)u(t) + f(t) \text{ for } t \in [0, T] \\ u(0) = u_0. \end{cases}$$

Recently, Tanaka [10] has studied the problem $(CP; u_0, f)$ above in the case where a family $\{A(t) : t \in [0, T]\}$ of closed linear operators in X satisfies all conditions which are usually referred to as the "hyperbolic" case except for the density of the common domain D of $A(t)$ and shown the existence and uniqueness result of classical solutions of $(CP; u_0, f)$.

The purpose of this paper is to prove the existence and uniqueness of classical solutions of (SE) on the basis of the results on $(CP; u_0, f)$ in [10]. Our result obtained extends Neubrander's one [7] on the special case of (SE) where $A(t) = A$ is independent of t and A is the infinitesimal generator of a (C_0) -semigroup on X (and so D is dense in X).

In what follows, for a Banach space X_i ($i = 1, 2$), the associated norm is denoted by $\|\cdot\|_{X_i}$ and $L(X_1, X_2)$ denotes the Banach space of all bounded linear operators from X_1 into X_2 . The norm in $L(X_1, X_2)$ is denoted by $\|\cdot\|_{X_1, X_2}$.

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2. THE RESULTS ON $(CP; u_0, f)$

In this section we recall the fundamental results obtained by Tanaka [10] on the problem

$$(CP; u_0, f) \quad \begin{cases} u'(t) = A(t)u(t) + f(t) \text{ for } t \in [0, T] \\ u(0) = u_0 \end{cases}$$

and introduce the notion of a D -integral solution of $(CP; u_0, f)$.

Here D is another Banach space with norm $\|\cdot\|_D$ which is continuously embedded in X . Assume the following three conditions on a family $\{A(t) : t \in [0, T]\}$ of closed linear operators in X .

(a1) $D(A(t)) = D$ is independent of t , and there exists a $c_0 > 0$ such that

$$(2.1) \quad c_0^{-1} \|y\|_D \leq \|y\|_X + \|A(t)y\|_X \leq c_0 \|y\|_D$$

for $y \in D$ and $t \in [0, T]$.

(a2) There are constants $M \geq 1$ and $\omega \in (-\infty, \infty)$ such that

$$(\omega, \infty) \subset \rho(A(t)) \text{ for } t \in [0, T]$$

and

$$(2.2) \quad \left\| \prod_{j=1}^k (\lambda I - A(t_j))^{-1} \right\| \leq M(\lambda - \omega)^{-k} \text{ for } \lambda > \omega$$

and every finite sequence $\{t_j\}_{j=1}^k$ with $0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq T$ and $k = 1, 2, \dots$.

Then by (2.1) and (2.2) for $\lambda_0 > \omega_0 := \max(0, \omega)$ there exists a $c > 0$ such that

$$(2.3) \quad \|(A(t) - \lambda_0)^{-1}x\|_D \leq c\|x\|_X \text{ for } x \in X.$$

(a3) For $y \in D$, $A(t)y$ is continuously differentiable in X .

First recall the results in [10] on the problem $(CP; u_0, f)$.

By [10, Theorem 4.2] we obtain the following existence and uniqueness result of classical solutions of the problem $(CP; u_0, f)$.

Theorem 2.1. *Suppose conditions (a1)–(a3). Let $f \in W^{1,1}([0, T] : X)$ and suppose that $u_0 \in D$ satisfies the compatibility condition that $A(0)u_0 + f(0) \in \overline{D}$. Then $(CP; u_0, f)$ has a unique classical solution $u \in C([0, T] : D) \cap C^1([0, T] : X)$ given by*

$$(2.4) \quad u(t) = \lim_{\lambda \rightarrow 0^+} \left(U_\lambda(t, 0)u_0 + \int_0^{[t/\lambda]\lambda} U_\lambda(t, s)f(s)ds \right)$$

for $t \in [0, T]$, where $U_\lambda(t, s) = \prod_{i=[s/\lambda]+1}^{[t/\lambda]} (I - \lambda A(i\lambda))^{-1}$ for $0 \leq s \leq t \leq T$.

Moreover u satisfies the integral equation

$$(2.5) \quad \begin{aligned} (A(t) - \lambda_0)u(t) + f(t) &= U(t, 0)((A(0) - \lambda_0)u_0 + f(0)) \\ &+ \lim_{\lambda \rightarrow 0^+} \int_0^{[t/\lambda]\lambda} U_\lambda(t, s)(\dot{A}(s)u(s) + f'(s) - \lambda_0 f(s))ds \end{aligned}$$

for $t \in [0, T]$, where $\lambda_0 > \omega_0$ and $U(t, 0)z = \lim_{\lambda \rightarrow 0^+} U_\lambda(t, 0)z$ for $z \in \overline{D}$ and $t \in [0, T]$ (see (4.2) and (4.5) in [10]).

Remark 2.1. The combination of (2.4) and (2.5) gives

$$(2.6) \quad A(t)u(t) + f(t) = U(t, 0)(A(0)u_0 + f(0)) + \lim_{\lambda \rightarrow 0^+} \int_0^{[t/\lambda]\lambda} U_\lambda(t, s)(\dot{A}(s)u(s) + f'(s))ds$$

for $t \in [0, T]$.

One integrates $(CP; u_0, f)$ formally and then an integration by parts yields

$$\begin{aligned} u(t) - u_0 &= \int_0^t A(s)u(s)ds + \int_0^t f(s)ds \\ &= A(t) \int_0^t u(s)ds - \int_0^t \dot{A}(s) \int_0^s u(r)drds + \int_0^t f(s)ds. \end{aligned}$$

This consideration leads us to the following definition.

Definition 2.1. Let $u_0 \in X$ and $f \in L^1([0, T] : X)$. A function $u : [0, T] \rightarrow X$ is called a *D-integral solution* of $(CP; u_0, f)$ if $u \in C([0, T] : X)$, $\int_0^\cdot u(s)ds \in C([0, T] : D)$ and

$$(2.7) \quad u(t) - u_0 = A(t) \int_0^t u(s)ds - \int_0^t \dot{A}(s) \int_0^s u(r)drds + \int_0^t f(s)ds.$$

Remark 2.2. The notion of a *D-integral solution* of $(CP; u_0, f)$ coincides with that of an integral solution introduced by Da Prato and Sinestrari [1] in the case where $A(t) = A$ is independent of t .

Let $u_0 \in \overline{D}$ and $f \in L^1([0, T] : X)$. Then it is shown in [10, Theorem 3.1] that the limit (2.4) exists uniformly in $t \in [0, T]$. Then we have

Theorem 2.2. *Suppose conditions (a1)–(a3). Let $u_0 \in \overline{D}$ and $f \in L^1([0, T] : X)$. A function u defined by (2.4) is a unique D-integral solution of $(CP; u_0, f)$. Moreover we have the following estimates :*

$$(2.8) \quad \|u(t)\|_X \leq C_1 \left(\|u_0\|_X + \int_0^t \|f(s)\|_X ds \right);$$

$$(2.9) \quad \left\| \int_0^t u(s)ds \right\|_D \leq C_1 \left(\|u_0\|_X + \int_0^t \|f(s)\|_X ds \right)$$

for $t \in [0, T]$, where C_1 is a constant independent of u_0 and f .

Proof. Let $v_0(t) = 0$ on $t \in [0, T]$. Since $u_0 \in \overline{D}$, by Theorem 2.1 we can define $v_n \in C^1([0, T] : X) \cap C([0, T] : D)$ inductively by the unique classical solution of the problem

$$\begin{cases} v'_n(t) = A(t)v_n(t) + u_0 - \int_0^t \dot{A}(s)v_{n-1}(s)ds + \int_0^t f(s)ds \text{ for } t \in [0, T] \\ v_n(0) = 0 \end{cases}$$

for $n \geq 1$.

Then from (2.5) it follows that

$$\begin{aligned} & (A(t) - \lambda_0)v_n(t) + u_0 - \int_0^t \dot{A}(s)v_{n-1}(s)ds + \int_0^t f(s)ds \\ &= U(t, 0)u_0 + \lim_{\lambda \rightarrow 0^+} \int_0^{[t/\lambda]\lambda} U_\lambda(t, s) \left[\dot{A}(s)v_n(s) - \dot{A}(s)v_{n-1}(s) + f(s) \right. \\ & \quad \left. - \lambda_0 \left(u_0 - \int_0^s \dot{A}(r)v_{n-1}(r)dr + \int_0^s f(r)dr \right) \right] ds \end{aligned}$$

for $t \in [0, T]$. This together with (2.3) shows that there exists a constant $K > 0$ such that

$$\|v_{n+1}(t) - v_n(t)\|_D \leq K \int_0^t (\|v_{n+1}(s) - v_n(s)\|_D + \|v_n(s) - v_{n-1}(s)\|_D) ds,$$

which implies by Gronwall's inequality that

$$\|v_{n+1}(t) - v_n(t)\|_D \leq K \int_0^t e^{K(t-s)} \|v_n(s) - v_{n-1}(s)\|_D ds$$

for $t \in [0, T]$ and $n \geq 1$. By standard arguments we see that there exists an element $v \in C([0, T] : D)$ such that $\sup\{\|v_n(t) - v(t)\|_D : t \in [0, T]\}$ converges to zero as $n \rightarrow \infty$. Thus $v'_n(t) \left(= A(t)v_n(t) + u_0 - \int_0^t \dot{A}(s)v_{n-1}(s)ds + \int_0^t f(s)ds \right)$ converges to $A(t)v(t) + u_0 - \int_0^t \dot{A}(s)v(s)ds + \int_0^t f(s)ds$ uniformly in $t \in [0, T]$ as $n \rightarrow \infty$ and hence v' is a D -integral solution of $(CP; u_0, f)$.

Let w be a D -integral solution of $(CP; u_0, f)$ and let $w_1(t) = \int_0^t w(s)ds$. Then since $w'_1(t) = A(t)w_1(t) + u_0 - \int_0^t \dot{A}(s)w_1(s)ds + \int_0^t f(s)ds$ and $w_1(0) = 0$, by (2.6) we have

$$\begin{aligned} w(t) &= w'_1(t) \\ &= U(t, 0)u_0 + \lim_{\lambda \rightarrow 0^+} \int_0^{[t/\lambda]\lambda} U_\lambda(t, s) \{ \dot{A}(s)w_1(s) + (-\dot{A}(s)w_1(s) + f(s)) \} ds, \end{aligned}$$

which is equal to a function u defined by (2.4). This proves the uniqueness of D -integral solutions of $(CP; u_0, f)$. This together with (2.4) shows (2.8).

Finally we shall show (2.9). By (2.1), (2.7) and (2.8) we get

$$\begin{aligned} c_0^{-1} \left\| \int_0^t u(s)ds \right\|_D &\leq \left\| \int_0^t u(s)ds \right\|_X + \left\| A(t) \int_0^t u(s)ds \right\|_X \\ &\leq \int_0^t \|u(s)\|_X ds + \left\| u(t) - u_0 + \int_0^t \dot{A}(s) \int_0^s u(r)dr ds - \int_0^t f(s)ds \right\|_X \\ &\leq C_2 \left(\|u_0\|_X + \int_0^t \|f(s)\|_X ds \right) + \int_0^t \|\dot{A}(s)\|_{D, X} \left\| \int_0^s u(r)dr \right\|_D ds \end{aligned}$$

for $t \in [0, T]$, where C_2 is a constant independent of u_0 and f . Thus Gronwall's inequality gives the desired estimate (2.9). \square

3. COMPLETE SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

Now we turn to the complete second order linear differential equations

$$(SE) \quad \begin{cases} u''(t) = A(t)u'(t) + B(t)u(t) + f(t) \text{ for } t \in [0, T] \\ u(0) = x \text{ and } u'(0) = y. \end{cases}$$

A family $\{A(t) : t \in [0, T]\}$ of closed linear operators in X is assumed to satisfy conditions (a1) – (a3) in Section 2. We assume the next two conditions on a family $\{B(t) : t \in [0, T]\}$ of linear operators in X :

(b1) For $t \in [0, T]$, $D \subset D(B(t))$ and $B(t) \in L(D, X)$.

(b2) For $y \in \cap_{0 \leq t \leq T} D(B(t))$, $B(t)y$ is continuously differentiable in X .

It should be noted that we do not assume the closedness of the operator $B(t)$ in X .

Definition 3.1. A function $u : [0, T] \rightarrow X$ is called a *D-classical solution* of (SE) if $u \in C^2([0, T] : X)$, $u(t) \in D(B(t))$, $u'(t) \in D$, the functions $B(t)u(t)$, $A(t)u'(t)$ are continuous in t and u satisfies (SE).

Let $x \in \cap_{0 \leq t \leq T} D(B(t))$. Consider the following first order differential equation:

$$(FE) \quad \begin{cases} u'(t) = A(t)u(t) + B(t)x + B(t) \int_0^t u(s)ds + f(t) \text{ for } t \in [0, T] \\ u(0) = y. \end{cases}$$

We say that u is a *D-classical solution* of (FE) if $u \in C^1([0, T] : X) \cap C([0, T] : D)$ and u satisfies (FE).

Remark 3.1. If $u \in C([0, T] : D)$, then $\int_0^t u(s)ds \in C([0, T] : D)$ by (a1) and the closedness of $A(0)$.

By assumptions (b1) and (b2) we obtain the following proposition immediately.

Proposition 3.1. *Suppose conditions (a1)–(a3) and (b1)–(b2) are fulfilled. A function $u : [0, T] \rightarrow X$ is a D-classical solution of (FE) if and only if $v(\cdot)$ defined by $v(t) = x + \int_0^t u(s)ds$ is a D-classical solution of (SE).*

Now we are in a position to state our main result in this paper.

Theorem 3.2. *Suppose conditions (a1)–(a3) and (b1)–(b2) are fulfilled. Let $f \in W^{1,1}([0, T] : X)$, $x \in \cap_{0 \leq t \leq T} D(B(t))$ and $y \in D$. If the compatibility condition that $B(0)x + A(0)y + f(0) \in \overline{D}$ is satisfied, then (SE) has a unique D-classical solution v which satisfies*

$$(3.1) \quad \|v(t)\|_X \leq C \left(\|x\|_X + \|y\|_X + \int_0^t \|B(s)x + f(s)\|_X ds \right)$$

for $t \in [0, T]$, where C is a constant independent of x, y and f .

Proof. By Proposition 3.1, it is sufficient to prove the existence and uniqueness of *D-classical solutions* of (FE).

Let $u_0(t) = y$ for $t \in [0, T]$. By Theorem 2.1 we can define

$$u_n \in C^1([0, T] : X) \cap C([0, T] : D)$$

inductively by the unique classical solution of the problem

$$\begin{cases} u'_n(t) = A(t)u_n(t) + B(t)x + B(t) \int_0^t u_{n-1}(s)ds + f(t) \text{ for } t \in [0, T] \\ u_n(0) = y \end{cases}$$

for $n \geq 1$.

Then by the same arguments used in proving Theorem 2.2 we see that u_n converges to an element u in $C([0, T] : D)$ as $n \rightarrow \infty$ in the topology of $C([0, T] : D)$ and that u is a D -classical solution of (FE).

To prove the uniqueness of D -classical solutions of (FE), let u_i ($i = 1, 2$) be D -classical solutions of (FE) and let $w = u_1 - u_2$. Then we have $w'(t) = A(t)w(t) + B(t) \int_0^t w(s)ds$ and $w(0) = 0$, and so by (2.5)

$$\begin{aligned} & (A(t) - \lambda_0)w(t) + B(t) \int_0^t w(s)ds \\ &= \lim_{\lambda \rightarrow 0^+} \int_0^{[t/\lambda]^\lambda} U_\lambda(t, s) \left[\dot{A}(s)w(s) \right. \\ & \quad \left. + B(s)w(s) + \dot{B}(s) \int_0^s w(r)dr - \lambda_0 B(s) \int_0^s w(r)dr \right] ds \end{aligned}$$

for $t \in [0, T]$. The estimation of this equality gives

$$\|w(t)\|_D \leq K' \int_0^t \|w(s)\|_D ds$$

for some constant $K' > 0$. By Gronwall's inequality we have $w = 0$.

Finally we shall show the estimate (3.1).

Since u is a D -integral solution of $(CP; y, B(\cdot)x + B(\cdot) \int_0^\cdot u(s)ds + f(\cdot))$, the estimate (2.9) gives

$$\begin{aligned} \left\| \int_0^t u(s)ds \right\|_D &\leq C_1 \left(\|y\|_X + \int_0^t \left\| B(s)x + B(s) \int_0^s u(r)dr + f(s) \right\|_X ds \right) \\ &\leq C_1 \left(\|y\|_X + \int_0^t \|B(s)x + f(s)\|_X ds \right) \\ &\quad + C_1 \int_0^t \|B(s)\|_{D, X} \left\| \int_0^s u(r)dr \right\|_D ds, \end{aligned}$$

which implies by Gronwall's inequality that

$$\left\| \int_0^t u(s)ds \right\|_D \leq C_3 \left(\|y\|_X + \int_0^t \|B(s)x + f(s)\|_X ds \right)$$

for some constant $C_3 > 0$. Combining this with (2.8) we obtain

$$\begin{aligned} \|u(t)\|_X &\leq C_1 \left(\|y\|_X + \int_0^t \left\| B(s)x + B(s) \int_0^s u(r)dr + f(s) \right\|_X ds \right) \\ &\leq C_4 \left(\|y\|_X + \int_0^t \|B(s)x + f(s)\|_X ds \right) \end{aligned}$$

for some constant $C_4 > 0$. Therefore Proposition 3.1 implies the desired estimate (3.1). \square

The rest of this section is devoted to an application of Theorem 3.2 to the following hyperbolic partial differential equation :

$$(P) \begin{cases} u_{tt} + \alpha(t, x)u_{tx} + \beta(t, x)u_t = \gamma(t, x)u_x + \delta(t, x)u + f(t, x), \\ \hspace{15em} (t, x) \in [0, T] \times [0, 1], \\ u(t, 0) = u(t, 1), \quad t \in [0, T], u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in [0, 1]. \end{cases}$$

We denote by X the Banach space $C[0, 1]$ with norm $\|u\|_\infty = \sup\{|u(x)| : x \in [0, 1]\}$ and by D the Banach space $\{u \in C^1[0, 1] : u(0) = u(1)\}$ with norm $\|u\|_\infty + \|u'\|_\infty$. Define three families $\{A(t) : t \in [0, T]\}$, $\{B(t) : t \in [0, T]\}$ and $\{C(t) : t \in [0, T]\}$ of linear operators in X by

$$\begin{cases} D(A(t)) = D \\ (A(t)u)(x) = -\alpha(t, x)u'(x) \quad \text{for } u \in D, \\ D(B(t)) = C^1[0, 1] \\ (B(t)u)(x) = \gamma(t, x)u'(x) + \delta(t, x)u(x) \quad \text{for } u \in C^1[0, 1], \text{ and} \\ D(C(t)) = X \\ (C(t)u)(x) = -\beta(t, x)u(x) \quad \text{for } u \in X, \end{cases}$$

respectively.

If α is a positive function of class C^1 , then the family $\{A(t) : t \in [0, T]\}$ of closed linear operators satisfies three conditions (a1) – (a3) (see [2, Theorem 6.1]). Also, it is well-known that the stability condition is preserved under the perturbation of a uniformly bounded family of bounded linear operators on X (see [4] or [9, Theorem 5.2.3]). Therefore, if β is of class C^1 , the family $\{A(t) + C(t) : t \in [0, T]\}$ of closed linear operators in X satisfies conditions (a1) – (a3) with $A(t)$ replaced by $A(t) + C(t)$. It is easy to see that conditions (b1) – (b2) are satisfied if γ and δ are of class C^1 . Since $\overline{D} = \{u \in C[0, 1] : u(0) = u(1)\}$, Theorem 3.2 asserts that if $f \in W^{1,1}([0, T] : C[0, 1])$, $u_0, u_1 \in C^1[0, 1]$, $u_1(0) = u_1(1)$ and the compatibility condition that $-\alpha(0, 0)u_1'(0) - \beta(0, 0)u_1(0) + \gamma(0, 0)u_0'(0) + \delta(0, 0)u_0(0) + f(0, 0) = -\alpha(0, 1)u_1'(1) - \beta(0, 1)u_1(1) + \gamma(0, 1)u_0'(1) + \delta(0, 1)u_0(1) + f(0, 1)$ is satisfied, then the problem (P) has a unique solution $u \in C^2([0, T] : C[0, 1]) \cap C([0, T] : C^1[0, 1])$ which satisfies the estimate :

$$\sup_{x \in [0, 1]} |u(t, x)| \leq C \left(\sup_{x \in [0, 1]} |u_0(x)| + \sup_{x \in [0, 1]} |u_1(x)| + \int_0^t \sup_{x \in [0, 1]} |\gamma(s, x)u_0'(x) + \delta(s, x)u_0(x) + f(s, x)| ds \right)$$

for $t \in [0, T]$, where C is a constant independent of u_0, u_1 and f .

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