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BOUNDARIES OF ROTATION SETS FOR HOMEOMORPHISMS OF THE n-TORUS

RICHARD SWANSON AND RUSSELL WALKER

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ABSTRACT. We construct a C^ω diffeomorphism of the 3-torus whose rotation set is not closed. We prove that the rotation set of a homeomorphism of the n-torus contains the extreme points of its closed convex hull. Finally, we show that each pseudo-rotation set is closed for torus homeomorphisms.

1. Introduction

H. Poincaré introduced the idea of the rotation set of an orientation preserving circle homeomorphism and proved that it consists of a single real number ([P]). In 1979 S. Newhouse, J. Palis, and F. Takens introduced the idea of a rotation set for maps of the circle (homotopic to the identity) and proved that this set is always a closed interval ([N-P-T]). M. Handel has shown that the analogous rotation set for an annulus homeomorphism (isotopic to the identity) must be closed ([H]). It follows from work of J. Franks (but see also [M-Z]) that the rotation set of a 2-torus homeomorphism must contain the extreme points and the (2-dimensional) interior of its closed convex hull ([F]). (We extend this extreme point containment result to arbitrary n-tori in Theorem 4.8, using a functional analytic approach.)

Whether or not the rotation set of a 2-torus homeomorphism must be closed remains an open question. But M. Misiurewicz and K. Ziemian have shown that vectors on the boundary of the rotation set of a homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ need not be the expected value of the displacement by f for any ergodic invariant probability measure ([M-Z]). We caution the reader that the definition of rotation set used here corresponds to the "point rotation set" in [M-Z].

Our main result is the construction of a C^ω diffeomorphism $F : \mathbb{T}^3 \to \mathbb{T}^3$ isotopic to the identity with a lift $\widetilde{F} : \mathbb{R}^3 \to \mathbb{R}^3$ with rotation set $\rho(\widetilde{F})$ which is not closed (Theorem 3.2). (Note: throughout this article lifts of a map G to the universal cover will be written $\widetilde{G}$.) The rotation set of $\widetilde{F}$ is found explicitly which is unusual for non-trivial analytic examples of rotation behavior.

Next we analyze $\rho(\widetilde{F})$ by proving two theorems about the general nature of boundaries of rotation sets for homeomorphisms of $\widetilde{G} : \mathbb{R}^n \to \mathbb{R}^n$ $(n \geq 1)$ which are lifts of toral maps. First we use the Krein-Milman theorem to show that $\rho(\widetilde{G})$ contains the extreme points of its closed convex hull (Theorem 4.8). And second

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we show that the "pseudo-rotation set" (defined rigorously below) $\rho_\psi(\widetilde{G})$ (which contains $\rho(\widetilde{G})$) must be closed (Theorem 5.1). Roughly speaking, the latter set measures the rotation set that a computer might calculate. Such pseudo-rotation vectors are easy to compute and behave well on chain transitive sets, where they constitute a closed convex set. See [B-S] for more on pseudo rotation sets for annulus homeomorphisms. Lastly, we exhibit the pseudo rotation set, $\rho_\psi(\widetilde{F})$, for the example constructed above. Our work with rotation sets of 3-torus homeomorphisms has led us to the following conjecture:

Conjecture 1.1. *The rotation set of a lift of a* 3*-torus homeomorphism, isotopic to the identity, contains the closure of its* (3*-dimensional*) *interior.*

2. Definitions

For general definitions consult an introductory text on dynamical systems such as [G-H]. We will take as our model of the n-torus the space $\mathbb{T}^n = \overbrace{S^1 \times \cdots \times S^1}^{n}$, where $S^1 = \{z \in \mathbb{C} : \ |z| = 1\}$. As universal cover we will use the map $\Pi : \mathbb{R}^n \to \mathbb{T}^n$ given by $\Pi(x_1, \ldots, x_n) = (\exp\, 2\pi i x_1, \ldots, \exp\, 2\pi i x_n)$.

Now assume that $G : \mathbb{T}^n \to \mathbb{T}^n$ is isotopic to the identity and $\widetilde{G} : \mathbb{R}^n \to \mathbb{R}^n$ is a (continuous) lift of G. Then $\rho(\widetilde{G}, x)$ denotes the *rotation set of* $x \in \mathbb{R}^n$ *under* $\widetilde{G}$, and is given by

$$\rho(\widetilde{G}, x) = \mathrm{LIM}\left\{ \frac{\widetilde{G}^n(x) - x}{n} \middle| n \in \mathbb{N} \right\}.$$

Here we use "LIM" to denote "the set of limit points". The *rotation set of* $\widetilde{G}$ is the union

$$\rho(\widetilde{G}) = \bigcup_{x \in \mathbb{R}^n} \rho(\widetilde{G}, x).$$

For any map $f : X \to X$ on a compact metric space, let $\mathcal{M}_I(f)$ denote the set of f-invariant Borel probability measures, which forms a compact convex subset in the topological vector space of signed (regular Borel) measures on X. If $G : \mathbb{T}^n \to \mathbb{T}^n$, with a lift $\widetilde{G}$, define $\mathcal{M}_I(\widetilde{G})$ to be the set of lifts of regular Borel probability measures on $\mathbb{T}^n$. A Borel measure $\tilde{\mu}$ on $\mathbb{R}^n$ is the *lift* of the probability measure μ on $\mathbb{T}^n$ if for each fundamental domain Q and Borel set $E \subset \mathbb{R}^n$, $\tilde{\mu}(E \cap Q) = \mu(\Pi(E \cap Q))$. For example, the Haar measure on $\mathbb{T}^2$ lifts to Lebesgue measure on $\mathbb{R}^2$.

Define the *mean rotation number* by

$$\rho_{\tilde{\mu}}(\widetilde{G}) = \int_Q (\widetilde{G} - Id)\, d\tilde{\mu},$$

where Q is an arbitrary fundamental domain in the covering space of $\mathbb{T}^n$. When μ is ergodic for G, we will speak of the *ergodic mean rotation number*. Note that if μ is a G-invariant Borel probability measure on $\mathbb{T}^n$, then $\tilde{\mu}$ is a $\widetilde{G}$-invariant Borel (infinite!) measure on $\mathbb{R}^n$ that is invariant under deck translations.

Now define the *mean rotation set*

$$\rho_{\mathrm{mes}}(\widetilde{G}) = \bigcup_{\mu \in \mathcal{M}_I(\widetilde{G})} \rho_{\tilde{\mu}}(\widetilde{G}).$$

Define an ϵ *pseudo-orbit* to be an infinite sequence $\{z_k\}_{k\geq 0} \subset \mathbb{R}^n$ such that $||z_{k+1} - \widetilde{G}(z_k)|| < \epsilon$ for $k \geq 0$. An ϵ-*chain* is a finite sequence $z_0, z_1, \ldots, z_m$ with $||z_{k+1} - \widetilde{G}(z_k)|| < \epsilon$ for all k with $0 \leq k < m$. Define the *pseudo-rotation set of* the ϵ pseudo-orbit $\{z_k\}$ by $\rho_\psi(\{z_k\}) = \text{LIM}\{(z_k - z_0)/k : \ k \in \mathbb{N}\}$. The union over all such pseudo-rotation sets, for a given ϵ, is the ϵ *pseudo-rotation set*, $\rho_\psi(\widetilde{G}, \epsilon)$. Then $r \in \mathbb{R}^n$ is a *pseudo-rotation vector* if it lies in $\rho_\psi(\widetilde{G}, \epsilon)$ for each ϵ. The analogous pseudo-rotation set for an annulus homeomorphism, in fact, equals the closure of the rotation set ([B-S]). Thus, $\rho_\psi(\widetilde{G}) = \rho(\widetilde{G})$, since M. Handel has shown that the true rotation set is closed ([H]). It is not hard to find examples of lifts, $\widetilde{G}$, of homeomorphisms of $\mathbb{T}^2$ for which $\rho(\widetilde{G}) \neq \rho_\psi(\widetilde{G})$.

We need these next definitions to state our theorem concerning extreme points. Let X be a subset of a topological vector space $\mathbb{E}$. An *extreme point* $p \in X$ is any point which meets the following criterion: if $p = tx + (1-t)y$ for points x and y in X and $0 < t < 1$, then $x = p = y$. A subset C of a topological vector space is *convex* if whenever $x, y \in C$ then $tx + (1-t)y \in C$, for all $0 \leq t \leq 1$. A topological vector space $\mathbb{E}$ is *locally convex* if zero has a neighborhood base of convex sets. The *closed convex hull* $\langle X \rangle$ of X is the intersection of all closed convex sets containing X. The closed convex hull equals the closure of the convex hull. The convex hull of X equals the set of all finite linear combinations of the form $\sum t_i\, x_i$, where $x_i \in X$ and $\sum t_i = 1$. If the linear space $\mathbb{E}$ is a metric space, then $\langle X \rangle$ consists of all convergent infinite series of the form $\sum t_i\, x_i$ with $x_i \in X$ for $i = 1, 2, \ldots$ and $\sum t_i = 1$.

3. A rotation set which is not closed

We first construct a 1-parameter family of C^ω diffeomorphisms $\{f_\lambda : \ 0 \leq \lambda \leq 1\}$ of $\mathbb{T}^2$ with lifts $\{\widetilde{f}_\lambda : \mathbb{R}^2 \to \mathbb{R}^2\}$, for which the diameter of the rotation set is not continuous in the parameter. (No such family of circle maps or annulus homeomorphisms exists.) At $\lambda = \frac{1}{2}$, $\big\{f_\lambda |\ \lambda \in [0, 1/2]\big\}$ undergoes a double "blue sky" bifurcation. For $0 \leq \lambda < \frac{1}{2}$, f_λ has no periodic orbits. And the rotation set of the lift $\rho(\tilde{f}_\lambda) = \big\{(\frac{1}{4}, \tau(\lambda))\big\}$, where $\tau(\lambda) > 0$ and $\tau(\lambda) \to 0$ as $\lambda \to \frac{1}{2}^-$. The map $f_{\frac{1}{2}}$ has a semi-stable orbit of period two and a parallel semi-stable fixed circle, and $\rho(\tilde{f}_{\frac{1}{2}}) = \big\{(0,0)\,, (\frac{1}{2}, 0)\big\}$.

Each member of our family of lifts, $\{\tilde{f}_\lambda\}$, is the time-one map of a flow φ^t_λ generated by the C^ω vector field $X_\lambda = (X^1_\lambda, X^2_\lambda)$ on $\mathbb{R}^2$, where

$$X^1_\lambda(x,y) = \frac{1}{4}\big(1 - \cos(2\pi y)\big) \quad \text{and} \quad X^2_\lambda(x,y) = 2 - \cos(4\pi y) - 2\lambda.$$

Clearly the vectorfields $\{X_\lambda\}$ satisfy the following:

3.1) $X_\lambda(x,y)$ is independent of x.
3.2) $X^1_\lambda(x,y) = X^1_\lambda(y)$ is independent of λ, is an even function of y, and $X^1_\lambda(y) = \frac{1}{2} - X^1_\lambda\big(\frac{1}{2} - y\big)$.
3.3) $X^2_\lambda(y) = X^2_\lambda\big(y + \frac{1}{2}\big)$.

Lemma 3.1. *For all* $0 \leq \lambda < \frac{1}{2}$, $\rho(\widetilde{f}_\lambda) = \big\{(\frac{1}{4}, \tau(\lambda))\big\}$, *where* $\tau : [0, \frac{1}{2}] \to \mathbb{R}$ *is a continuous non-increasing function such that* $\tau(\lambda) = 0$ *if and only if* $\lambda = \frac{1}{2}$. *And when* $\lambda = \frac{1}{2}$, $\rho(\widetilde{f}_\lambda) = \big\{(0,0), (\frac{1}{2}, 0)\big\}$.

Proof. We will need the notation

$$\rho_x(\widetilde{f}_\lambda) = \mathrm{proj}_x\, \rho(\widetilde{f}_\lambda) \quad \text{and} \quad \rho_y(\widetilde{f}_\lambda) = \mathrm{proj}_y\, \rho(\widetilde{f}_\lambda),$$

where proj_x, $\mathrm{proj}_y : \mathbb{R}^2 \to \mathbb{R}$ are projections onto the x- and y-axes. $\rho(\widetilde{f}_\lambda)(x, y) = \rho(\varphi^t_\lambda)(x, y)$ for all (x, y) ([FM]). Thus by 3.1) $\rho(\widetilde{f}_\lambda)(x, y)$ is independent of x. Fix (x_0, y_0) and let $\varphi^t_\lambda(x_0, y_0) = (x(t), y(t))$. So $(x'(t), y'(t)) = (X^1_\lambda, X^2_\lambda)$.

First assume $0 \leq \lambda < \frac{1}{2}$. Because X^2_λ is a *positive* function only of y, by 3.1) $\rho(\widetilde{f}_\lambda)(x, y)$ is independent of (x, y). We first show $\rho_x(\widetilde{f}_\lambda) = \frac{1}{4}$ for λ in this range. It follows from 3.3) that there exists $T > 0$ (depending on y_0) such that

$$y(t + T) \quad = \quad y + \tfrac{1}{2}$$

for all t. Using 3.2) the reader should check that

$$x(t) = x_0 + \int_0^t X^1_\lambda(y(s))\, ds = x_0 + \frac{1}{2}t - \int_T^{t+T} X^1_\lambda(y(s))\, ds.$$

Thus

$$x(t) = x_0 + \frac{1}{2}\, t - x(t + T) + x(t)$$

and

$$\begin{aligned} \rho_x(\widetilde{f}_\lambda) = {} & \mathrm{LIM}\left\{ \tfrac{x(t)-x_0}{t} \,|\, t \geq 0 \right\} \\ = {} & \mathrm{LIM}\left\{ \tfrac{1}{2} - \tfrac{x(t+T)}{t} + \tfrac{x(T)}{t} | t \geq 0 \right\} \\ = {} & \tfrac{1}{2} - \mathrm{LIM}\left\{ \tfrac{x(t+T)-x_0}{t} | t \geq 0 \right\} \\ = {} & \tfrac{1}{2} - \rho_x(\widetilde{f}_\lambda). \end{aligned}$$

So $\rho_x(\widetilde{f}_\lambda) = \frac{1}{4}$ for λ in this range.

Next we consider $\rho_y(\widetilde{f}_\lambda)$. The equation $y'(t) = X^2_\lambda(y)$ generates a flow on $\mathbb{R}$. The time one map of this flow covers a circle homeomorphism with well-defined rotation number ([P]). In fact this rotation number is $\rho_y(\widetilde{f}_\lambda)$. In the statement of this lemma we call it $\tau(\lambda)$. But rotation numbers of circle homeomorphisms depend continuously on the homeomorphism. So τ is continuous. Now for each y, $X^2_\lambda(y)$ is a positive strictly decreasing function of λ in this range. However, $X^2_{\frac{1}{2}}$ has a zero at $y = 0$ (as well as many others). This means that $\rho_y(\widetilde{f}_{\frac{1}{2}}) = 0$ and consequently $\tau(\lambda) \to 0$ as $\lambda \to \frac{1}{2}$.

Now fix $\lambda = \frac{1}{2}$. Notice $X^1_{\frac{1}{2}}(x, y) > 0$ and $X^2_{\frac{1}{2}}(x, y)$ has singularities only at $\{\frac{n}{2} | n \in \mathbb{Z}\}$. In terms of the torus coordinates $(\underline{x}, \underline{y}) = (e^{2\pi\, i\, x}, e^{2\pi\, i\, y})$ on $\mathbb{T}^2$, the ω-limit set under $f_{\frac{1}{2}}$ of each point in $\mathbb{T}^2$ is either contained in $\{\underline{y} = 1\}$ or $\{\underline{y} = -1\}$. But, for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, $X^1_{\frac{1}{2}}(x, n) = 0$ and $X^1_{\frac{1}{2}}\left(x, n + \frac{1}{2}\right) = \frac{1}{2}$. Thus, $\rho(\widetilde{f}_{\frac{1}{2}}) = \left\{(0, 0), (\frac{1}{2}, 0)\right\}$ as desired. □

We now define our C^ω 3-torus diffeomorphism: Let $\widetilde{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$\widetilde{F}(x, y, z) = (\widetilde{F}^1_{\tilde{h}(z)}(x, y), \tilde{f}^2_{\tilde{h}(z)}(x, y), z),$$

where $\widetilde{f}_\lambda = (\widetilde{f}^1_\lambda, \widetilde{f}^2_\lambda)$ was defined above and $\tilde{h} : \mathbb{R} \to \mathbb{R}$ is given by

$$\tilde{h}(z) = \tfrac{1}{4}\left[1 - \cos(2\pi z)\right].$$

Notice $\widetilde{F}$ covers a C^ω map F on $\mathbb{T}^3$.

Theorem 3.2. *There exists a C^ω diffeomorphism of $\mathbb{T}^3$, isotopic to the identity, with lift $\widetilde{F} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\rho(\widetilde{F})$ is not closed.*

Proof. On each 2-torus, $\left\{\underline{z} = e^{2\pi i z} = \text{const}\right\}$, F is the time one map of the $X_{h(z)}$-flow. Thus F is a C^ω diffeomorphism.

We consider two cases, $z = \frac{1}{2}$ and $z \neq \frac{1}{2}$. When $z = \frac{1}{2}$, $\tilde{h}(z) = \frac{1}{2}$ and by Lemma 3.1, $\rho(\tilde{f}_{\tilde{h}(z)}) = \{(0,0), (\frac{1}{2}, 0)\}$. So since F is the identity in the third coordinate,

$$\rho(\widetilde{F})(x,y,z) = \left\{(0,0,0), \left(\tfrac{1}{2},0,0\right)\right\}.$$

When $z \neq \frac{1}{2}$, $\tilde{h}(z) \in [0, \frac{1}{2})$. So by Lemma 3.1, $\rho(\widetilde{F}_{\tilde{h}(z)}) = \left(\frac{1}{4}, \tau(\tilde{h}(z))\right)$. Thus,

$$\rho(\widetilde{F})(x,y,z) = \left\{\left(\tfrac{1}{4}, \tau(\tilde{h}(z)), 0\right)\right\}.$$

We now combine the two cases. Recall that τ is non-increasing, $\tau(\lambda) > 0$, and $\tau(\lambda) \to 0$ as $\lambda \to \frac{1}{2}$. Thus

$$\rho(\widetilde{F}) = \{(0,0,0)\} \bigcup \left\{\left(\tfrac{1}{2},0,0\right)\right\} \bigcup \left\{\left(\tfrac{1}{4}, y, 0\right) \middle| 0 < y < \tau(0)\right\},$$

which is not closed. □

Comments: By compactness, $\tau(0) > 0$. A computer computation shows $\tau(0) \approx 1$. However, by rescaling the vector fields $\{X_\lambda\}$, we can arrange for $\tau(0)$ to equal any desired positive real number.

4. Extreme points of rotation sets

In this section we prove that rotation sets of lifts of n-torus homeomorphisms ($n \geq 1$) contain their extreme points. We need several theorems from functional analysis.

Theorem 4.1 (Krein-Milman)**.** *Suppose $\mathbb{E}$ is a locally convex topological vector space. A compact convex subset X is the closed convex hull of the set of its extreme points.*

Let $\mathcal{M}_I(\widetilde{G})$ denote the set of lifts of regular Borel G-invariant probability measures on $\mathbb{T}^n$ (as defined in Section 2).

Note that $\mathcal{M}_I(\widetilde{G})$ is simply the space of $\widetilde{G}$-invariant regular Borel measures on $\mathbb{R}^n$ that are invariant under deck translations and have the property $\tilde{\mu}(Q) = 1$ for each fundamental domain Q.

There is considerable awkwardness in attempting to transport dynamical and measure theory arguments back and forth between a manifold and its covering space. One solution is to make all computations in the covering space. Towards this end, we shall identify the topological vector space $\mathcal{S}$ of signed measures on $\mathbb{T}^n$ with the space $\tilde{\mathcal{S}}$ of their lifted counterparts on $\mathbb{R}^n$. Topologize the measures via the customary separable and metrizable "weak* topology": a sequence $\{\tilde{\mu}_n\}$ in $\tilde{\mathcal{S}}$ converges to a measure $\tilde{\mu}$ if and only if the sequence of integrals $\int_Q \tilde{k}\, d\tilde{\mu}_i$ converges to $\int_Q \tilde{k}\, d\tilde{\mu}$ for each lift $\tilde{k} : \mathbb{R}^n \to \mathbb{R}$ of a map $k : \mathbb{T}^n \to \mathbb{R}$. By Alaoglu's Theorem, in this topology the space of probability measures is compact (e.g. [D-S], page 424).

We may summarize this discussion as the following lemma:

Lemma 4.2. *Given a torus homeomorphism $G : \mathbb{T}^n \to \mathbb{T}^n$, the correspondence $\mu \mapsto \tilde{\mu}$ defines a canonical topological vector space isomorphism from the topological vector space $\mathcal{S}$ of signed (regular Borel) measures on $\mathbb{T}^n$ onto the space $\widetilde{\mathcal{S}}$ (of lifts of such signed measures) which carries $\mathcal{M}_I(G)$ onto $\mathcal{M}_I(\widetilde{G})$.*

A proof of the following lemma may be found in [D-S, p. 428], for example.

Lemma 4.3. *The space of signed measures $\mathcal{S}$, with the weak* topology is locally convex. Therefore, the space of lifts $\widetilde{\mathcal{S}}$ is locally convex.*

Theorem 4.4 (Choquet's Theorem)**.** *A G-invariant Borel probability measure is an extreme point of $\mathcal{M}_I(G)$ if and only if it is ergodic. Hence, the extreme points of $\mathcal{M}_I(\widetilde{G})$ are lifts of ergodic measures on $\mathbb{T}^n$.*

For a simple proof of the proceeding theorem in any compact metric space, consult [M, p. 104]. Define the lifted measure $\tilde{\mu}$ to be *ergodic* with respect to $\widetilde{G}$ if and only if μ is ergodic with respect to G.

Our next lemma is then a consequence of Theorems 4.1 and 4.4 and Lemmas 4.2 and 4.3.

Lemma 4.5. *The set of $\widetilde{G}$-invariant measures, $\mathcal{M}_I(\widetilde{G})$, is the closed convex hull of the (extreme point) set of ergodic $\widetilde{G}$-invariant measures.*

Now suppose again that $\widetilde{G} : \mathbb{R}^n \to \mathbb{R}^n$ is a lift of an n-torus homeomorphism, G, and let Q be a fundamental domain of $\mathbb{R}^n$. The following (as in [B-S]) is an easy application of the powerful Birkhoff Ergodic Theorem (for a proof, see [M, p. 89]):

Lemma 4.6. *If $\tilde{\mu} \in \mathcal{M}_I(\widetilde{G})$ is ergodic, then*

$$\rho(\widetilde{G}, x) = \rho_{\tilde{\mu}}(\widetilde{G}) = \int_Q (\widetilde{G} - Id)\, d\tilde{\mu}$$

for $\tilde{\mu}$-almost all $x \in \mathbb{R}^n \cap Q$.

Proof. Apply the Birkhoff Ergodic Theorem to each component function of the vector field $\widetilde{G} - Id$ using the measure $\tilde{\mu}$ and the measure-preserving transformation $\widetilde{G}$. □

Recall that $\rho_{\text{mes}}(\widetilde{G})$ is the union of all mean rotation vectors $\rho_{\tilde{\mu}}(\widetilde{G})$.

Lemma 4.7. *The mean rotation set $\rho_{\text{mes}}(\widetilde{G})$ is closed, convex and contains the rotation set, $\rho(\widetilde{G})$.*

Proof. That $\rho_{\text{mes}}(\widetilde{G})$ is closed and convex follows from the fact that $\mathcal{M}_I(\widetilde{G})$ is closed and convex in the space of lifted signed measures on $\mathbb{R}^n$ and the map $\mu \mapsto \rho_\mu$ is linear and continuous. Now let $r \in \rho(\widetilde{G})(x)$. Then

$$r = \lim_{i \to \infty} (\widetilde{G}^{n_i}(x) - x)/n_i$$

for some subsequence of natural numbers, $n_i \to \infty$. Let $\tilde{\delta}(x)$ denote the Borel measure that assigns unit mass to each lattice point $x + \vec{n}$ for all $\vec{n} \in \mathbb{Z}^n$. Let $\tilde{\mu}$ be a limit point of the sequence of measures

$$\tilde{\mu}_i = \sum_{j=1}^{n_i} \frac{1}{n_i} \tilde{\delta}(\widetilde{G}^j(x)).$$

Recall that the set $\mathcal{M}_I(\widetilde{G})$ of invariant measures is compact (hence, sequentially compact in the metrizable weak* topology). Then $\tilde{\mu}$ is $\widetilde{G}$-invariant and $\lim_{i\to\infty} \rho_{\tilde{\mu}_i}(\widetilde{G}) = \rho_{\tilde{\mu}}(\widetilde{G}) = r$. □

Theorem 4.8. *The rotation set $\rho(\widetilde{G})$ contains the extreme points of the mean rotation set, $\rho_{\mathrm{mes}}(\widetilde{G})$. In particular, $\rho_{\mathrm{mes}}(\widetilde{G})$ equals the closed convex hull of $\rho(\widetilde{G})$. Furthermore, $\rho(\widetilde{G})$ contains the extreme points of its closure. Finally, these extreme points correspond to ergodic mean rotation numbers.*

Proof. Let r be an extreme point of $\rho_{\mathrm{mes}}(\widetilde{G})$. Then $r = \rho_{\tilde{\mu}}(\widetilde{G})$ for some $\widetilde{G}$-invariant probability measure μ. By Lemma 4.5, μ lies in the closed convex hull of the $\widetilde{G}$-invariant ergodic measures. In particular, we can write $\mu = \sum t_i \mu_i$, where $\sum t_i = 1$, $0 \le t_i \le 1$, and each μ_i is ergodic. (This sum may be infinite.) And $r = \rho_{\tilde{\mu}}(\widetilde{G}) = \sum t_i \rho_{\mu_i}(\widetilde{G})$.

If μ is not ergodic, we may assume, by reordering the index if necessary, that $0 < t_1 < 1$. In such case define the probability measures $\nu_1 = \mu_1$ and $\nu_2 = \frac{1}{M}\sum_{i\ge 2} t_i \mu_i$, where $M = \sum_{i\ge 2} t_i$. Then $r = \rho_{\tilde{\mu}}(\widetilde{G}) = t_1 \rho_{\widetilde{\nu_1}} + (1-t_1)\rho_{\widetilde{\nu_2}}$. But, by definition of extreme point, this only happens if $\rho_{\widetilde{\nu_1}} = \rho_{\tilde{\mu}} = \rho_{\widetilde{\nu_2}}$. So $\rho_{\tilde{\mu}} = \rho_{\widetilde{\mu_1}}$, where $\widetilde{\mu_1}$ is ergodic and by Lemma 4.6 $\rho_{\widetilde{\mu_1}}(\widetilde{G}) \in \rho(\widetilde{G})$. The last two statements in the theorem follow from Lemma 4.7. □

5. Pseudo-rotation sets are closed

The pseudo-rotation set $\rho_\psi(\widetilde{G}) \supset \rho(\widetilde{G})$ ([B-S]). In this section we prove that $\rho_\psi(\widetilde{G})$ is closed.

Theorem 5.1. *Let $\widetilde{G} : \mathbb{R}^n \to \mathbb{R}^n$ $(n \ge 1)$ be the lift of an n-torus homeomorphism. Then the pseudo-rotation set $\rho_\psi(\widetilde{G})$ is closed.*

Proof. Assume that

$$r \in \overline{\rho_\psi(\widetilde{G})} = \overline{\bigcap_{\epsilon>0} \rho_\psi(\widetilde{G}, \epsilon)}.$$

Then r is in $\overline{\rho_\psi(\widetilde{G}, \epsilon)}$ for all $\epsilon > 0$. So for each $\epsilon > 0$, there exist ϵ pseudo-orbits $\mathcal{O}_k$ such that $r = \lim_{k\to\infty} r_k(\epsilon)$, for some $r_k(\epsilon) \in \rho_\psi(\mathcal{O}_k)$.

We will construct a 2ϵ pseudo-orbit $\{z_m\}$ for $\widetilde{G}$ such that $\rho_\psi(\{z_m\}) = r$. Let $Q = [0,1]^n$ as before. By passing to a subsequence we may assume $\{\overline{\mathcal{O}_{n_k}} \bigcap Q\}$ converges (in the Hausdorff topology on compact subsets). By periodicity of the lift, the convergence holds for each choice Q of fundamental domain, and, therefore, we can choose k sufficiently large that each vector in $\mathcal{O}_{n_k}$ lies within ϵ of some vector in $\mathcal{O}_{n_{k+1}}$.

Now choose $\{z_0, z_1, \ldots, z_{m_1}\} \subset \mathcal{O}_{n_1}$ with m_1 large enough that

$$\left\| \frac{z_{m_1} - z_0}{m_1} - r_{n_1} \right\| < \frac{1}{2}.$$

Choose $z_{m_1+1} \in \mathcal{O}_{n_2}$ such that $\|z_{m_1} - z_{m_1+1}\| < \epsilon$. Next choose $m_2 \gg m_1$ and an ϵ-chain $\{z_{m_1+1}, \dots, z_{m_2}\} \subset \mathcal{O}_{n_2}$ such that

$$\left\| \frac{z_{m_2} - z_0}{m_2} - r_{n_2} \right\| < \frac{1}{4}.$$

Notice $\{z_0, z_1, \dots, z_{m_2}\}$ is a 2ϵ-chain for $\widetilde{G}$.

Continuing recursively, define $\{z_{m_k}\}$ and a 2ϵ pseudo-orbit $\{z_0, z_1, \dots\}$ such that for all k,

$$\left\| \frac{z_{m_k} - z_0}{m_k} - r_{n_k} \right\| < \frac{1}{2k}.$$

Thus $\rho_\psi(\{z_m\}) = r$ and $r(\epsilon) \in \rho_\psi(\widetilde{G}, 2\epsilon)$.

So, $r \in \rho_\psi(\widetilde{G})$ as desired. □

To complete our investigation of the lift $\widetilde{F}$ of a 3-torus diffeomorphism constucted in Section 3, we now locate its pseudo-rotation set. An easy proposition is that the pseudo-rotation set of the restriction of $\widetilde{G}$ to the lift $\widetilde{\Lambda}$ of a chain transitive set Λ in $\mathbb{T}^n$ is closed and convex. The reason is that one can construct pseudo-orbit itineraries for each $\epsilon > 0$ that spend an arbitrary number of iterates on any desired true orbit. On the other hand an ϵ-chain only requires a bounded number of iterates to join any two points of Λ. Thus, we have

Proposition 5.2.

$$\rho_\psi(\widetilde{F}) = \langle \rho(\widetilde{F}) \rangle.$$

Notes: 1) Notice $\rho(\widetilde{F})$ contains its extreme points (as required in Theorem 4.8). But $\rho(\widetilde{F})$ does not have any (non-empty) 3-dimensional interior nor the (non-empty) 2-dimensional interior of its convex hull.

2) Consider the Cartesian product of $\widetilde{F}$ with the identity map on $\mathbb{R}^{n-3}$ $(n \geq 4)$. This map covers a C^ω diffeomorphism of $\mathbb{T}^n$. And the geometry of its rotation set is identical to that of $\widetilde{F}$.

3) Any compact subset of $\mathbb{R}^n$ can be realized as the rotation set of some diffeomorhism $F : \mathbb{T}^{2n} \to \mathbb{T}^{2n}$, using methods similar to the construction in this paper.

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Department of Mathematical Sciences, Montana State University, Bozeman, Montana 59717-0240

E-mail address: dswanson@math.montana.edu

E-mail address: walker@math.montana.edu