ON THE FAILURE OF CLOSE-TO-NORMAL STRUCTURE TYPE CONDITIONS AND PATHOLOGICAL KANNAN MAPS

MICHAEL A. SMYTH

(Communicated by Palle E. T. Jorgensen)

Abstract. We consider the failure of close-to-normal structure type conditions and show that a Banach space can be renormed to fail close-to-weak normal structure exactly when it contains a norm inseparable weakly compact subset. Included is an example of a particularly pathological fixed point free Kannan map.

1.

Throughout $X$ will denote a real Banach space. We recall that $X$ is said to have (weak) normal structure if whenever $C$ is a closed (weak compact) bounded convex subset of $X$ with $\text{diam } C > 0$, then $\text{rad } C < \text{diam } C$ where

$$\text{diam } C := \sup \{ \|x - y\| : x, y \in C\} \quad \text{and} \quad \text{rad } C := \inf_{x \in C} \sup \{\|x - y\| : y \in C\}$$

are the diameter and radius of the set $C$. We will denote normal structure and weak normal structure by, respectively, ns and w-ns. If $X$ is a dual space, it has weak star normal structure (w*-ns) if we require the set $C$ of the above definition to be weak star compact. A Banach space $X$ has uniform normal structure if

$$\sup \left\{ \frac{\text{rad } C}{\text{diam } C} : C \text{ nonempty nonsingleton closed bounded convex subset of } X \right\} < 1.$$

The above normal structure type conditions have been useful in the fixed point theory of nonexpansive maps (see [2] for example). A Banach space $X$ has the fixed point property (weak fixed point property) if, given a nonempty closed (weak compact) bounded convex subset $C$ of $X$ that is self-mapped by a nonexpansive map $T$, then $T$ has a fixed point in $C$ (recall that $T : C \rightarrow C$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$). We abbreviate the two properties to fpp and w-fpp. If $X$ is a dual space, then it has the weak star fixed point property (w*-fpp) if we require the set $C$ of the above definition to be weak star compact. It is known that w-ns implies the w-fpp and that w*-ns implies the w*-fpp.

A Banach space $X$ is said to have close-to-normal structure (close-to-ns) if, given a nonempty nonsingleton closed bounded convex subset $C$ of $X$, there exists

Received by the editors February 23, 1995.
1991 Mathematics Subject Classification. Primary 46B20, 47H09.

©1996 American Mathematical Society

3063
This is seen by the use of a convex series. Suppose that strictly convex have close-to-ns and that the KK property implies close-to-w-ns. It is shown in [11] that spaces which are separable or close-to-w-ns. The results stated in [6] give that close-to-w*-ns is equivalent to the C Kannan maps if, given a nonempty convex weak compact subset of a Banach space, of a Banach space, x of C A nonempty and convex subset of a Banach space and that "weak star compact" in the above. Self-mapped by a Kannan map T then C. Wong [12] showed that has the w-fpp for Kannan maps if and only if it has close-to-w-ns. The results stated in [6] give that close-to-w*-ns is equivalent to the C Kannan maps if, given a nonempty convex weak compact subset C of X that is self-mapped by a Kannan map T, then T has a fixed point. The w*-fpp for Kannan maps can be defined when X is a dual space by replacing "weak compact" with "weak star compact" in the above.

Wong [12] showed that X has the w-fpp for Kannan maps if and only if it has close-to-w-ns. The results stated in [6] give that close-to-w*-ns is equivalent to the w*-fpp for Kannan maps. It is shown in [11] that spaces which are separable or strictly convex have close-to-ns and that the KK property implies close-to-w-ns. This is seen by the use of a convex series. Suppose that C is a closed bounded nonempty and convex subset of a Banach space and that A is a countable subset of C. Then CA is a separable subset of C. Let \{x_n\}_n\in\mathbb{N} be a countable dense subset of CA. Consider the point x = ∑x_n/2^n. Since C is a closed bounded convex subset of a Banach space, x exists and is in C. Now it is easily seen that if y ∈ C (so \|x_n - y\| ≤ diam C for all n) and \|x - y\| = diam C, then \|x_n - y\| = diam C for all n. But this also gives that \text{dist}(y, CA) = diam C. This procedure can be used to establish all of the results from [11] that were stated above.

In Section 2 we give examples of spaces that fail the close-to-normal structure conditions defined above, covering what appears to be known so far.

In Section 3 we concern ourselves with the equivalence stated in the abstract and give an example of a fixed point free idempotent Kannan map.

2.

Perhaps the simplest example of a set violating close-to-w-ns (and thus close-to-ns) is in c0(Γ) for an uncountable Γ. Indeed, if the x_i are the usual unit basis elements, then C := \overline{co}\{x_i\}_{i∈Γ} is weak compact, convex, and, for any x ∈ c0(Γ) and i ∈ Γ, \|x_i - x\| ≥ 1 if i /∈ supp x. Of course diam C = 1.

It is not hard to give nonconstructive subsets of l_∞ violating close-to-ns, as in the following example.

**Example 1.** Let \mathcal{U} be a free ultrafilter on \mathbb{N}. Put

\[ A := \{x ∈ l_∞ : x(n) ∈ \{0, 1\} \text{ for } n ∈ \mathbb{N} \text{ and } \{n : x(n) = 0\} ∈ \mathcal{U} \} \]

and

\[ C := \overline{co} A. \]

Clearly diam C = 1. We show that for any y ∈ C there exists x ∈ A so that \|y - x\| = 1. Indeed, suppose that y ∈ C and m ∈ \mathbb{N}. Define

\[ B_m := \{n ∈ \mathbb{N} : y(n) ≤ 1/m\}. \]

Note that if z ∈ co A, z = λ_1x_1 + λ_2x_2 + ⋯ + λ_px_p, say, with x_i ∈ A, \sum λ_i = 1, λ_i ≥ 0, then there exists U ∈ \mathcal{U} so that z(n) = 0 for all n ∈ U. This will imply that the complement of B_m is not in \mathcal{U}, giving B_m ∉ \mathcal{U}.
It follows that there exists an infinite subsequence \((n_m)_{m \in \mathbb{N}}\) of \(\mathbb{N}\) so that \(y(n_m) \leq 1/m\) for all \(m \in \mathbb{N}\). But every infinite subset of \(\mathbb{N}\) contains an infinite subset that is not in \(\mathcal{U}\). Thus, we can extract a further sequence \((n_m)_{m \in \mathbb{N}}\) so that \(y(n_m) \rightarrow 0\) and \(M := \{n_m\}_{m \in \mathbb{N}} \notin \mathcal{U}\). Then \(\chi_M \in A\) and \(\|y - \chi_M\| = 1\), giving the result. \(\square\)

In [6] it is shown that \(l_\infty\) fails close-to-w*-ns. Indeed their result is that if \((\Omega, \Sigma, \mu)\) is a sigma finite measure space (so that the dual of \(L_1(\Omega, \Sigma, \mu)\) is \(L_\infty(\Omega, \Sigma, \mu)\)), then \(L_\infty(\Omega, \Sigma, \mu)\) fails close-to-w*-ns if it is inseparable. Of course, \(l_\infty\) has close-to-w-ns by the result of Wong given earlier, since all of its weak compacts are (norm) separable. In [4] the above result from [6] is used to show that if \(X\) is an infinite dimensional Hilbert space, then \(B(X)\), the space of bounded linear operators on \(X\), fails close-to-w*-ns. It is also shown that the space of compact operators on \(X\) has close-to-w-ns if and only if \(X\) is inseparable but that the space of trace class operators always has close-to-w*-ns. It was subsequently shown in [5] that the trace class has w*-ns.

Suppose that \(\Omega\) is a compact Hausdorff space. Then \(C(\Omega)^*\), the space of continuous real valued functions on \(\Omega\) with the supremum norm, fails close-to-w*-ns exactly when \(\Omega\) is uncountable (that is, when \(C(\Omega)^*\) is inseparable). To verify this, we first identify \(C(\Omega)^*\) with \(M(\Omega)\), the space of radon measures on \(\Omega\) with total variation norm, the actions on \(C(\Omega)\) being integration. If \(\Omega\) is countable, to show that \(C(\Omega)^*\) has close-to-w*-ns we can assume that \(\Omega\) is infinite. Then \(M(\Omega) \equiv l_1\), a separable space which thus has close-to-w*-ns from above. If \(\Omega\) is uncountable, consider

\[
C := \{\mu \in M(\Omega) : \|\mu\| \leq 1, \mu(\Omega) = 1\}.
\]

\(C\) is the intersection of the w* compact unit ball and a w* closed hyperplane and so is thus convex and w* compact. Clearly \(\text{diam}\, C = 2\). If \(\mu \in C\), then \(\{x \in \Omega : \mu(\{x\}) = 0\} \neq \emptyset\) since \(\Omega\) is uncountable. Now if \(\delta_x\) denotes the dirac measure at \(x\), then \(\|\mu - \delta_x\| = 2\). Thus \(C(\Omega)^*\) fails close-to-w*-ns.

We make a slight digression here on the failure of w*-ns and the w*-fpp. Suppose that \(K\) is a locally compact Hausdorff space with \(C_0(K)\) denoting the space of real valued continuous functions on \(K\) vanishing at infinity. In [4] it was asked when does \(C_0(K)^*\) fail w*-ns? We note here that \(C_0(K)^*\) fails w*-ns (and also the w*-fpp) exactly when \(K\) is non-discrete. To verify this, we identify \(C_0(K)^*\) with \(M(K)\), the space of radon measures on \(K\). As in [9], if \(\Omega\) is a compact subset of \(K\), then \(C(\Omega)^*\) is isometrically isomorphic to a w* closed subspace of \(C_0(K)^*\) via a w* homeomorphism (namely, the map which extends a measure to be identically zero outside \(\Omega\)). Now if \(K\) is non-discrete, then, by the local compactness, it contains an infinite compact subset \(\Omega\). But \(C(\Omega)^*\) fails the w*-fpp by a result from [9], and thus so does \(C_0(K)^*\). Otherwise, if \(K\) has the discrete topology, then \(C_0(K)^* \equiv c_0(K)^*\), well known to have w*-ns (and thus the w*-fpp).

We now give a general method for producing spaces which fail close-to-ns before considering renorming results. Suppose that the Banach space \(X\) fails uniform normal structure. That is, for every \(m \in \mathbb{N}\) there exists a closed convex subset \(C_m\) of \(X\) so that \(\text{diam}\, C_m = 1\) and \(\text{rad}\, C_m \geq 1 - 1/m\). We can also assume that \(0 \in C_m\) for all \(m\). Put \(Y := l_\infty(X)\) and

\[
C := \prod_m C_m = \{(x_m) \in l_\infty(X) : x_m \in C_m \text{ for all } m\}.
\]
Clearly $C$ is closed, convex and $\text{diam} C = 1$. Now suppose that $(x_m) \in C$. For any $m$ there exists $y_m \in C_m$ so that $\|x_m - y_m\| > 1 - 2/m$. Then $\|(x_m) - (y_m)\| = 1$, showing that $Y$ fails close-to-ns.

The above example can be further refined using ultrapowers. For material on ultrapowers see, for example, [3], [7] or [2]. Suppose $X$ is as above and $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$. Let $\tilde{X}$ be the associated ultrapower of $X$, so $\tilde{X} = l_\infty(X)/\mathcal{N}(\mathcal{U})$, where

$$\mathcal{N}(\mathcal{U}) := \{(x_n) \in l_\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0\}.$$ 

Now define

$$\tilde{C} := \left(\prod_m C_m\right)_{\mathcal{U}} = \{(x_m)_{\mathcal{U}} \in \tilde{X} : x_m \in C_m \text{ for all } m\}.$$ 

Then $\tilde{C}$ is closed, convex and bounded with $\text{diam} \tilde{C} = 1$. Also if $(x_m)_{\mathcal{U}} \in \tilde{C}$ and we choose the $y_m$ as above, then $\|x_m - y_m\| \to 1$, so $\|(x_m)_{\mathcal{U}} - (y_m)_{\mathcal{U}}\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_m - y_m\| = 1$, showing that $\tilde{X}$ fails close-to-ns.

Of course if $X$ was originally superreflexive (so every ultrapower of $X$ is also superreflexive), then $\tilde{X}$ is a superreflexive space that fails close-to-ns. We note that superreflexive spaces that fail uniform normal structure are easily given using a result from [1]: Every infinite-dimensional Banach space can be renormed to fail normal structure. More examples of superreflexive spaces failing close-to-ns are given below, where we are concerned with renorming spaces to fail close-to-ns or close-to-w-ns. The pathological sets so obtained will be similar to the example in $c_0(I)$ given above.

We start by recalling that a class of pairs $(x_i, x_i^*)_{i \in I}$, where $x_i \in X$ and $x_i^* \in X^*$, is called a biorthogonal system if $x_i^*(x_j) = \delta_{ij}$ for any $i, j \in I$. The following proposition uses an adaptation of the technique for renorming to fail normal structure used in [1].

**Proposition 2.** Suppose that $X$ admits an uncountable biorthogonal system. Then $X$ can be renormed to fail close-to-ns.

**Proof.** Suppose $(x_i, x_i^*)_{i \in I}$ is the uncountable system. We can assume that the set $\{x_i\}_{i \in I}$ is bounded. Now, since $I$ is uncountable, there exists $N \in \mathbb{N}$ and an uncountable subset $J$ of $I$ so that $\|x_i^*\| \leq N$ for all $i \in J$. For simplicity we will assume that $I = J$. Now for $x \in X$ define

$$\|x\|' := \max \left\{ \frac{\|x\|}{\text{diam}\{x_i\}_{i \in I}}, \sup_{i \in I} |x_i^*(x)| \right\}.$$ 

Obviously $(X, \|\|')$ is isomorphic to $X$. Put $C := \overline{\text{co}}\{x_i\}_{i \in I}$. Clearly $\text{diam} C = 1$. Now suppose that $x \in \overline{\text{span}}\{x_i\}_{i \in I}$. Then there exists a countable subset $A$ of $I$ so that $x \in \overline{\text{span}}\{x_i\}_{i \in A}$. Thus, if we choose $j \in I \setminus A$, then $\|x_j - x\|' \geq |x_j^*(x_j - x)| = 1$. This obviously implies that $C$ violates close-to-ns. \hfill $\square$

If $X = l_2(I)$, $I$ uncountable, with the $x_i$ the usual unit basis elements and $x_i^*$ the coordinate functionals, then, in the above,

$$\|x\|' = \max \left\{ \frac{\|x\|_2}{\sqrt{2}^j}, \|x\|_\infty \right\},$$
giving \( Y := (X, \| \cdot' \|) \) isometric to an example used in [10], the norms differing by a multiplication by \( 2^{1/2} \). As in [10], the set \( \{ x \in Y : x(i) \geq 0 \text{ for all } i \text{ and } \| x \|_2 \leq 1 \} \) can be used instead of \( \overline{\text{co}}\{x_i \} \) to show failure of close-to-ns.

Also, note that if \( X = c_0(I), (x_i, x_i^*)_{i \in I} \) the usual biorthogonal system, then \( \| \cdot' \| = \| \cdot \|_\infty \).

Not every inseparable Banach space has an uncountable biorthogonal system (see pg. 861 of [8]). It is unknown whether every inseparable Banach space can be renormed to fail close-to-ns. For the analogous close-to-w-ns problem the answer is given in the next section.

3.

First we recall some further material. A set \( \{ x_i \}_{i \in I} \) is said to be an M-basis of a Banach space \( X \) if \( \text{span}\{x_i\}_{i \in I} = X \) and there exists a total family \( \{x_i^*\}_{i \in I} \subseteq X^* \) so that \( (x_i, x_i^*)_{i \in I} \) is a biorthogonal system. Let \( \delta(X) \) denote the density character of \( X \). That is,

\[
\delta(X) = \min\{ k : X \text{ has a dense subset of cardinality } \leq k \}.
\]

In general, if \( (x_i, x_i^*)_{i \in I} \) is a biorthogonal system, then \( |I| \leq \delta(X) \) (see pg. 673 of [8]), and if \( \{x_i\}_{i \in I} \) is an M-basis of \( X \), then clearly \( |I| = \delta(X) \) (assuming \( X \) is infinite dimensional).

**Theorem 3.** A Banach space \( X \) can be renormed to fail close-to-w-ns if and only if it contains a (norm) inseparable weak compact subset.

**Proof.** The forward implication is clear since any (norm) isomorphism is a weak homeomorphism.

For the reverse implication suppose that \( X \) contains an inseparable weak compact set \( C \). Then \( Y := \text{span} C \) is an inseparable weakly compactly generated subspace of \( X \). By a well-known result, \( Y \) can then be generated by a weak compact balanced convex set \( K \).

Theorem 20.5(a) on pg. 693 of [8] will now give that \( Y \) has an M-basis \( \{ x_i \}_{i \in I} \) with \( x_i \in K \) for all \( i \).

Since \( \delta(Y) = |I| \), \( I \) is uncountable. Since \( K \) is weak compact, \( \{x_i\}_{i \in I} \) is bounded. We can also extend the functionals associated with the M-basis to all of \( X \) by Hahn-Banach and use the method of proof of the above proposition to produce an uncountable subset \( J \) of \( I \) and a renorming of \( X \) so that the weak compact set \( \overline{\text{co}}\{x_i\}_{i \in J} \) violates close-to-w-ns.

The following corollary is now immediate.

**Corollary 4.** If \( X \) is an inseparable weakly compactly generated space, then it can be renormed to fail close-to-w-ns.

A self-map \( T \) of a nonempty set \( C \) is periodic if there exists \( n \in \mathbb{N} \) so that \( T^n = I \).

If \( C \) is a nonempty subset of a Banach space \( X \) and \( T : C \to C \) is nonexpansive, then \( T \) is said to be rotative if there exist \( n \in \mathbb{N} \) and \( a < n, a \in \mathbb{R} \), so that

\[
\| x - T^n x \| \leq a \| x - Tx \| \text{ for all } x \in C.
\]

Clearly a periodic nonexpansive map is rotative. Using the fixed point result concerning rotative maps that is given on pg. 177 of [2], we have that if \( C \) is a nonempty closed convex subset of \( X \), then any periodic nonexpansive self-map of \( C \) has a fixed point. However, this is not true for periodic Kannan mappings.
Proposition 5. Suppose that $X$ admits a biorthogonal system $(x_i, x_i^*)_{i \in I}$ with $|I| = 2^{\aleph_0}$. Then $X$ can be renormed so that the resulting space contains a nonempty bounded closed convex subset $C$ on which a fixed point free self-mapping Kannan map $T$ is defined satisfying $T^2 = I$.

Proof. Since the cofinality of $2^{\aleph_0}$ is uncountable, there exists a subset $J$ of $I$ so that $|J| = 2^{\aleph_0}$ and $N \in \mathbb{N}$ so that $\|x_i\| \leq N$ and $\|x_i^*\| \leq N$ for all $i \in J$.

We renorm $X$ as in the proof of Proposition 2, with

$$\|x\|^\prime = \max \left\{ \frac{\|x\|}{\text{diam}\{x_i\}_{i \in J}}, \sup_{i \in J} |x_i^*(x)| \right\}.$$

With $C := \overline{\text{co}}\{x_i\}_{i \in J}$ we note that the proof of Proposition 2 gives that if $x \in C$, then $|\{i \in J : \|x_i - x\| = 1\}| = 2^{\aleph_0}$. Also, since $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$, it follows that $|C| = 2^{\aleph_0}$. Write $C = \{y_i\}_{i < 2^{\aleph_0}}$. Define $T : C \to C$ recursively as follows. Suppose that $T$ has been defined on $\{y_i\}_{i < \alpha}$ for some ordinal $\alpha < 2^{\aleph_0}$.

Case 1. There exists $j < \alpha$ so that $Ty_j = y_\alpha$. Then define $Ty_\alpha := y_j$.

Case 2. Otherwise. Put $F := \{y_i\}_{i < \alpha} \cup \{Ty_i\}_{i < \alpha}$. Then $|F| < 2^{\aleph_0}$. From above, there exists $z \in C \setminus F$ so that $\|y_i - z\| = 1$. Now define $Ty_\alpha := z$.

Note that in the above procedure the $j$ in Case 1 is unique. Also $\|T(x - z)\| = 1$ for all $x \in C$ and $\text{diam} C = 1$, so that $T$ is a fixed point free Kannan self-map of $C$. Finally, $T^2 = I$. Indeed, suppose that $\alpha < 2^{\aleph_0}$. First, suppose $Ty_\alpha = y_j$ for some $j < \alpha$. Then $Ty_j = y_\alpha$ from the definition of $T$, so $T^2 y_\alpha = y_\alpha$. Otherwise, $Ty_\alpha = y_j$ for some $j > \alpha$. Then, again from the definition of $T$, $T(y_j) = y_\alpha$ also giving $T^2 y_\alpha = y_\alpha$. \qed

By combining the methods of proof of Theorem 3 and Proposition 5, we obtain the following.

Corollary 6. Suppose $X$ contains a weak compact subset $K$ with $\delta(K) \geq 2^{\aleph_0}$. Then $X$ can be renormed so that the resulting space contains a nonempty weak compact convex subset $D$ on which a self-mapping fixed point free Kannan map is defined satisfying $T^2 = I$.

References

1. D. van Dulst, Equivalent norms and the fixed point property for nonexpansive mappings, J. London Math. Soc. 25 (1982), 139–144. MR 83e:47040

Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand

Current address: 1 Frost Rd., Mt. Roskill, Auckland, New Zealand