

A GENERAL COINCIDENCE THEOREM ON CONTRACTIBLE SPACES

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ABSTRACT. We obtain a general coincidence theorem for multifunctions in very large classes defined on contractible spaces. Our theorem generalizes a recent result due to Tarafdar and Yuan (1994) and many other earlier works including the Fan-Browder fixed point theorem.

0. INTRODUCTION

Recently, Tarafdar and Yuan [TY] obtained a very interesting coincidence theorem for compact upper semicontinuous maps with contractible values defined on contractible spaces. This generalizes results on convex-valued maps defined on convex spaces.

In the present paper, we show that this result holds for non-compact maps in very general classes. Our new theorem includes a large number of known results. Our argument is based on an earlier work of the first author and H. Kim [PK2, Theorem 1], which simplifies the proof in [TY].

A *multifunction* or *map* $F : X \multimap Y$ is a function from a set X into the power set of a set Y . Note that $F^{-}y = \{x \in X : y \in Fx\}$ for $y \in Y$.

A triple $(X, D; \Gamma)$ is called an *H-space* [P1], [PK2] if X is a topological space, D a nonempty subset of X , and $\Gamma = \{\Gamma_A\}$ a family of contractible subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$; here $\langle D \rangle$ denotes the set of all nonempty finite subsets of D . The triple is called a *c-space* [H] whenever $X = D$. A convex space X in the sense of Lassonde [L] is an example of *c-spaces* with $\Gamma_A = \text{co}A$, the convex hull of $A \in \langle X \rangle$.

Given a class \mathbb{X} of maps, $\mathbb{X}(X, Y)$ denotes the set of all maps $F : X \multimap Y$ belonging to \mathbb{X} , and \mathbb{X}_c the set of all finite composites of maps in \mathbb{X} .

A class \mathfrak{A} of maps is one satisfying:

- (i) \mathfrak{A} contains the class \mathbb{C} of (single-valued) continuous functions;
- (ii) each $F \in \mathfrak{A}_c$ is u.s.c. and nonempty compact-valued; and

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- (iii) each $F \in \mathfrak{A}_c(P, P)$ has a fixed point, where P is a convex hull of any $N \in \langle E \rangle$ for a topological vector space E .

Examples of \mathfrak{A} are \mathbb{C} , the Kakutani maps \mathbb{K} (with convex values), the Aronszajn maps \mathbb{M} (with R_δ -values), the acyclic maps \mathbb{V} (with acyclic values), the Powers map \mathbb{V}_c , the O'Neill maps \mathbb{N} (continuous with values consisting of one or m acyclic components, where m is fixed), the approachable maps in topological vector spaces due to Ben-El-Mechaiekh, admissible maps of Górniewicz, and the class of permissible maps of Dzedzej. See [P2]–[P5], [PK1], [PK2].

Further, we define the following:

$F \in \mathfrak{A}_c^\sigma(X, Y) \iff$ for any σ -compact subset K of X , there is a $T \in \mathfrak{A}_c(K, Y)$ such that $Tx \subset Fx$ for each $x \in K$.

$F \in \mathfrak{A}_c^\kappa(X, Y) \iff$ for any compact subset K of X , there is a $T \in \mathfrak{A}_c(K, Y)$ as above.

Note that \mathbb{K}_c^σ is due to Lassonde and \mathbb{V}_c^σ to Park, Singh, and Watson. See [PSW].

Moreover, the class of approximable maps due to Ben-El-Mechaiekh and Idzik [BI] is an example of \mathfrak{A}_c^κ . The functional values of approximable maps can be convex, contractible, decomposable, or ∞ -proximally connected whenever the domains of the maps are convex subsets of a locally convex Hausdorff topological vector space [BI].

Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\sigma \subset \mathfrak{A}_c^\kappa$. Any map belonging to a class in \mathfrak{A}_c^κ is called *admissible*. For details, see [P2]–[P5], [PK1], [PK2].

The following is due to the first author and H. Kim [PK2, Theorem 1]:

Theorem 1. *Let $(X, D; \Gamma)$ be an H -space, Y a Hausdorff space, K a nonempty compact subset of Y , $F \in \mathfrak{A}_c^\kappa(X, Y)$, $S : D \rightarrow Y$ and $T : X \rightarrow Y$. Suppose that*

- (1) *for each $x \in D$, Sx is compactly open in Y ;*
- (2) *for each open subset O of Y , $M \in \langle S^{-1}O \rangle$ implies $\Gamma_M \subset O$;*
- (3) *$\overline{F(X)} \cap K \subset S(D)$; and*
- (4) *$Y \setminus K \subset S(N)$ for some $N \in \langle D \rangle$.*

Then F and T have a coincidence point $\bar{x} \in X$; that is, $F\bar{x} \cap T\bar{x} \neq \emptyset$.

From Theorem 1, we deduce the following:

Theorem 2. *Let X be a contractible space, Y a Hausdorff space, K a nonempty compact subset of Y , and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that a map $G : X \rightarrow Y$ satisfies the following:*

- (a) *Gx is open in Y for each $x \in X$;*
- (b) *for each open subset O of Y , the set $\bigcap_{y \in O} G^{-1}y$ is empty or contractible;*
- (c) *$\overline{F(X)} \cap K \subset G(X)$; and*
- (d) *$Y \setminus K \subset G(N)$ for some $N \in \langle X \rangle$.*

Then F and G have a coincidence point.

Proof. Since $\overline{F(X)} \cap K$ is compact, by (a) and (c), there exists an $L \in \langle X \rangle$ such that $\overline{F(X)} \cap K \subset G(L)$. Let $D = L \cup N$. Now we define an H -space $(X, D; \Gamma)$ as follows: For any $J \in \langle D \rangle$, let

$$\Gamma_J = \begin{cases} \bigcap \{G^{-1}y : y \in \bigcap_{x \in J} Gx\} & \text{if } \bigcap_{x \in J} Gx \neq \emptyset, \\ X & \text{otherwise,} \end{cases}$$

as in [TY]. Note that if $y \in \bigcap_{x \in J} Gx$, then $J \in \langle G^{-}y \rangle$. Therefore, if $O = \bigcap_{x \in J} Gx \neq \emptyset$, then $\Gamma_J = \bigcap_{y \in O} G^{-}y$ is nonempty and contractible by (b). It is clear that $\Gamma_J \subset \Gamma_{J'}$ whenever $J \subset J' \in \langle D \rangle$.

We now apply Theorem 1 with $S = G|_D$ and $T = G$. Then (1) and (3) follow from (a) and (c), respectively. We show that (2) holds. In fact, for each $y \in Y$ and $M \in \langle S^{-}y \rangle = \langle D \cap G^{-}y \rangle$, we have $y \in \bigcap_{x \in M} Gx \neq \emptyset$. Hence $\Gamma_M = \bigcap \{G^{-}z : z \in \bigcap_{x \in M} Gx\} \subset G^{-}y = T^{-}y$. Finally, (4) is same as (d). Therefore, by Theorem 1, F and G have a coincidence point. \square

Remarks. 1. The open sets in (a) and (b) can be replaced by compactly open sets without affecting the conclusion of Theorem 2.

2. If X is a convex space in Theorem 2, then condition (b) can be replaced by (b)' for each $y \in Y$, $G^{-}y$ is convex.

In this case, Theorem 2 reduces to a result equivalent to Park [P4, Theorem 5], which extends many known theorems and has numerous applications in the KKM theory, as shown in [P4].

3. If F is a compact map, then by putting $Y = K = \overline{F(X)}$, condition (d) holds trivially. In this case, we have the following:

Theorem 3. *Let X be a contractible space, Y a Hausdorff compact space, and $A \in \mathfrak{A}_c^c(X, Y)$. Suppose that $B : Y \multimap X$ is such that*

- (a) $B^{-}x$ is open for each $x \in X$ and By is nonempty for each $y \in Y$; and
- (b) for each open set O in Y , the set $\bigcap_{y \in O} By$ is empty or contractible.

Then there exist an $x_0 \in X$ and a $y_0 \in Y$ such that $y_0 \in Ax_0$ and $x_0 \in By_0$.

Remarks. 1. When A is an u.s.c. map with nonempty compact contractible values, Theorem 3 reduces to the result of [TY].

2. As we observed in [PK2], if $F = f$ is a single-valued continuous map, the Hausdorffness assumption on Y in Theorems 1-3 is not necessary. Moreover, if $X = Y$ and $A = F = 1_X$, the identity map on X , each of Theorems 1-3 reduces to fixed point theorems. The following is a simple consequence of Theorem 3.

Theorem 4. *Let X be a compact contractible space and $G : X \multimap X$ a map satisfying*

- (a) Gx is open for each $x \in X$ and $G^{-}y$ is nonempty for each $y \in Y$; and
- (b) for each open set O in X , the set $\bigcap_{y \in O} G^{-}y$ is empty or contractible.

Then G has a fixed point.

Remark. If X is a convex space in Theorem 4 and if we replace (b) by (b)', then Theorem 4 reduces to the well-known Fan-Browder fixed point theorem [F], [Br], which has numerous applications.

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