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GENERALIZED ALUTHGE TRANSFORMATION ON p-HYPONORMAL OPERATORS

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(Communicated by Palle E. T. Jorgensen)

*Dedicated to Professor C. R. Putnam on his retirement
with respect and affection*

Abstract. We shall introduce a generalized Aluthge transformation on p-hyponormal operators and also, by using the Furuta inequality, we shall give several properties on this generalized Aluthge transformation as further extensions of some results of Aluthge.

1. Introduction

An operator means a bounded linear operator on a Hilbert space. It is well known that an operator T can be decomposed into $T = U|T|$ where U is a partial isometry with $N(U) = N(|T|)$, where $N(X)$ denotes the kernel of an operator X and $T = U|T|$ is said to be the polar decomposition of an operator T if this kernel condition $N(U) = N(|T|)$ is satisfied.

An operator T is said to be hyponormal if $T^*T \geq TT^*$ and also T is said to be semihyponormal if $(T^*T)^{1/2} \geq (TT^*)^{1/2}$ and semihyponormal was introduced by Xia [12]. It is known in Xia [12] that there exists an example which is semihyponormal but not hyponormal, that is, the class of semihyponormal operators properly contains the one of hyponormal operators. An operator T on a Hilbert space H is said to be *p-hyponormal* if

$$(T^*T)^p \geq (TT^*)^p \quad \text{for a positive number } p.$$

The class of p-hyponormal has been defined as an extension of semihyponormal and also it has been studied by many authors, mainly Aluthge [1], [2], Duggal [3] and Xia [13].

For a p-hyponormal operator $T = U|T|$, Aluthge [1] introduced the operator $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ which is called Aluthge transformation and Aluthge [1] showed very interesting results on $\widetilde{T}$. As an extension of $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$, we shall consider $\widetilde{T} = |T|^qU|T|^q$ for a positive number q which is not necessarily $1/2$ and by using the Furuta inequality, we shall give several properties on $\widetilde{T} = |T|^qU|T|^q$ for a positive number q, which can be considered as further extensions of some results in Aluthge [1] and [2].

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Our main result is as follows: let $T = U|T|$ be the polar decomposition of p-hyponormal for $1 \geq p > 0$ with $N(T) = N(T^*)$. Then $\widetilde{T} = |T|^q U|T|^q$ is $\frac{1}{2}(1+\frac{p}{q})$-hyponormal for any q such that $q \geq p$. This result implies the following one by Aluthge [1]: Let $T = U|T|$ be p-hyponormal for $0 < p < \frac{1}{2}$ and U be unitary. Then $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ is $(p+\frac{1}{2})$-hyponormal.

Let $T = U|T|$ be the polar decomposition of p-hyponormal for $1 \geq p > 0$. Then $\widetilde{T} = |T|^q U|T|^q$ is hyponormal for any q such that $p \geq q > 0$.

Also we have the following corollary: Let $T = U|T|$ be the polar decomposition of an invertible p-hyponormal operator for $1 \geq p > 0$, where U is a unitary operator and $|T| > 0$. Let q and r be any positive numbers such that $q \geq p$ and $\frac{1}{2}(1+\frac{p}{q}) \geq r$. Also let $\widetilde{T} = \widetilde{U}|\widetilde{T}|$ be the polar decomposition of an operator $\widetilde{T} = |T|^q U|T|^q$. Then $\widetilde{\widetilde{T}} = |\widetilde{T}|^r \widetilde{U}|\widetilde{T}|^r$ is hyponormal. Also this corollary can be considered as an extension of Corollary 3.3 in Aluthge [2].

2. Statement of the results

Theorem 1. *Let $T = U|T|$ be the polar decomposition of p-hyponormal for $1 \geq p > 0$ with $N(T) = N(T^*)$. Then $\widetilde{T} = |T|^q U|T|^q$ is $\frac{1}{2}(1+\frac{p}{q})$-hyponormal for any q such that $q \geq p$.*

Remark 1. $\widetilde{T} = |T|^q U|T|^q$ in Theorem 1 turns out to be semihyponormal because $1 \geq \frac{1}{2}(1+\frac{p}{q}) > \frac{1}{2}$ holds for $1 \geq p > 0$ and $q \geq p$ and we have only to apply the Löwner-Heinz theorem to this inequality.

We need the following result to give a proof of Theorem 1.

Theorem A (Furuta inequality [6]). *If $A \geq B \geq 0$, then for each $r \geq 0$*

(i) $$(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q}$$

and

(ii) $$(A^r A^p A^r)^{1/q} \geq (A^r B^p A^r)^{1/q}$$

hold for each p and q such that $p \geq 0$, $q \geq 1$ and $(1+2r)q \geq p+2r$.

We remark that Theorem A yields the Löwner-Heinz theorem [9], [11] when we put $r = 0$ in (i) or (ii) stated above: if $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$. Alternative proofs of Theorem A are given in [4], [7], [10] and an elementary proof is shown in [8].

Proof of Theorem 1. We have only to employ the ingenious proof of Theorem 2 in Aluthge [1] based on Theorem A. Firstly we recall that $U^*U|T|^q = |T|^q$ holds for any $q > 0$ since U^*U is the initial projection onto $\overline{R(|T|)}$, so that T is p-hyponormal for $p > 0$ is equivalent to the following

(1) $$|T|^{2p} \geq U|T|^{2p}U^* \quad \text{holds for } p > 0.$$

By (1) we have

(2) $$U^*|T|^{2p}U \geq U^*U|T|^{2p}U^*U = |T|^{2p} \quad \text{for any } p > 0.$$

By (1) and (2), we have

(3) $$U^*|T|^{2p}U \geq |T|^{2p} \geq U|T|^{2p}U^* \quad \text{for any } p > 0.$$

Next we shall show the following equation for any $r > 0$

(4) $(U^*|T|U)^r = U^*|T|^r U$ holds under the hypothesis $N(T) = N(T^*)$.

The hypothesis $N(T) = N(T^*)$ is equivalent to $N(T)^\perp = N(T^*)^\perp$ and also this means that the initial projection coincides with the final projection, that is, $UU^* = U^*U$. Therefore we have

$$\begin{aligned}(U^*|T|U)^2 &= U^*|T|UU^*|T|U\\ &= U^*|T|U^*U|T|U^* \quad \text{by } UU^* = U^*U\\ &= U^*|T|^2U \quad \text{since } U^*U \text{ is the initial projection}\end{aligned}$$

and similarly $(U^*|T|U)^{n/m} = U^*|T|^{n/m}U$ holds by induction for any natural number n and m, so that the continuity of an operator yields $(U^*|T|U)^r = U^*|T|^rU$ by attending $n/m \to r$, so we have (4).

Let $A = U^*|T|^{2p}U$, $B = |T|^{2p}$ and $C = U|T|^{2p}U^*$. Then for any $p > 0$ and $q > 0$

$$A^{2q/2p} = (U^*|T|^{2p}U)^{2q/2p} = (U^*|T|^{2q}U) \quad \text{holds by (4)}$$

and also $C^{2q/2p} = (U|T|^{2p}U^*)^{2q/2p} = C|T|^{2q}U^*$ holds in general.

Applying Theorem A to (3) since $(1 + 2\frac{q}{2p})\frac{2q}{p+q} = \frac{2q}{2p} + 2\frac{q}{2p}$ and $\frac{2q}{p+q} \geq 1$ for $1 \geq p > 0$ and $q \geq p$, we get

$$\begin{aligned}(\widetilde{T}^*\widetilde{T})^{(p+q)/2q} &= (|T|^qU^*|T|^{2q}U|T|^q)^{(p+q)/2q}\\ &= (B^{q/2p}A^{2q/2p}B^{q/2p})^{(p+q)/2q}\\ &\geq B^{(2q/2p+2q/2p)((p+q)/2q)} = B^{1+q/p}\end{aligned}$$

and

$$\begin{aligned}(\widetilde{T}\widetilde{T}^*)^{(p+q)/2q} &= (|T|^qU|T|^{2q}U^*|T|^q)^{(p+q)/2q}\\ &= (B^{q/2p}C^{2q/2p}B^{q/2p})^{(p+q)/2q}\\ &\leq B^{(2q/2p+2q/2p)((p+q)/2q)} = B^{1+q/p}.\end{aligned}$$

Hence $(\widetilde{T}^*\widetilde{T})^{(p+q)/2q} \geq (\widetilde{T}\widetilde{T}^*)^{(p+q)/2q}$, that is, $\widetilde{T}$ is $\frac{1}{2}(1+\frac{p}{q})$-hyponormal.

Theorem 2. *Let $T = U|T|$ be the polar decomposition of p-hyponormal for $1 \geq p > 0$. Then $\widetilde{T} = |T|^qU|T|^q$ is hyponormal for any q such that $p \geq q > 0$.*

Proof. As T is p-hyponormal for $p > 0$, T is q-hyponormal for q such that $p \geq q > 0$ by the Löwner-Heinz theorem, then by (3) we have

(5) $U^*|T|^{2q}U \geq |T|^{2q} \geq U|T|^{2q}U^*$ for any q such that $p \geq q > 0$,

therefore (5) implies

$$(\widetilde{T}^*\widetilde{T}) - (\widetilde{T}\widetilde{T}^*) = |T|^q(U^*|T|^{2q}U - U|T|^{2q}U^*)|T|^q \geq 0$$

for any q such that $p \geq q > 0$, that is, $\widetilde{T}$ is hyponormal, so the proof is complete.

Corollary 3. *Let $T = U|T|$ be the polar decomposition of an invertible p-hyponormal operator for $1 \geq p > 0$, where U is a unitary operator and $|T| > 0$. Let q and r be any positive numbers such that $q \geq p$ and $\frac{1}{2}(1+\frac{p}{q}) \geq r$. Also let $\widetilde{T} = \widetilde{U}|\widetilde{T}|$ be the polar decomposition of an operator $\widetilde{T} = |T|^qU|T|^q$. Then $\widetilde{\widetilde{T}} = |\widetilde{T}|^r\widetilde{U}|\widetilde{T}|^r$ is hyponormal.*

Proof. As T is invertible p-hyponormal, $1 \geq p > 0$, T can be decomposed into $T = U|T|$, where U is unitary and $|T| > 0$, so that Theorem 1 ensures that $\widetilde{T} = |T|^q U|T|^q$ is $\frac{1}{2}(1+\frac{p}{q})$-hyponormal for q such that $q \geq p$. Then Theorem 2 yields that $\widetilde{\widetilde{T}} = |\widetilde{T}|^r \widetilde{U} |\widetilde{T}|^r$ is hyponormal for any r such that $\frac{1}{2}(1+\frac{p}{q}) \geq r$, so the proof is complete.

We remark that our Theorem 1 yields the following Theorem B in Aluthge [1] because U in Theorem 1 can be extended to become unitary under the hypothesis $N(T) = N(T^*)$ by Theorem 4 in Furuta [5].

Theorem B (Aluthge [1])**.** *Let $T = U|T|$ be p-hyponormal for $0 < p < \frac{1}{2}$ and U be unitary. Then $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ is $(p+\frac{1}{2})$-hyponormal.*

Also we remark that Corollary 3 can be considered as an extension of Corollary 3.3 in Aluthge [2].

ADDENDUM

We have the following slight extension of Theorem 1 by slightly modifying the proof of Theorem 1.

Theorem 1′. *Let $T = U|T|$ be the polar decomposition of p-hyponormal for $1 \geq p > 0$ with $N(T) = N(T^*)$. Then $\widetilde{T} = |T|^s U|T|^t$ is $(\frac{p+s}{s+t})$-hyponormal for any $s \geq 0$ and $t \geq \operatorname{Max}\{p, s\}$.*

Professor A. Aluthge has kindly sent his excellent preprint in which one of them is overlapped with ours.

After reading our preprint, Professor T. Huruya kindly pointed out that $N(T) = N(T^*)$ in Theorem 1 is unnecessary and his proof is quite ingenious based on Xia's lemma.

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