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BOUNDING FAMILIES OF RULED SURFACES

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Abstract. In this paper we provide a sharp bound for the dimension of a family of ruled surfaces of degree d in $\mathbf{P}^3_{\mathbf{K}}$. We also find the families with maximal dimension: the family of ruled surfaces containing two unisecant skew lines, when $d \geq 9$ and the family of rational ruled surfaces, when $d \leq 9$.

The first tool we use is a Castelnuovo-type bound for the irregularity of ruled surfaces in $\mathbf{P}^n_{\mathbf{K}}$. The second tool is an exact sequence involving the normal sheaf of a curve in the grassmannian. This sequence is analogous to the one constructed by Eisenbud and Harris in 1992, where they deal with the problem of bounding families of curves in projective space. However, our construction is more general since we obtain the mentioned sequence by purely algebraic means, studying the geometry of ruled surfaces and of the grassmannian.

A classical subject in algebraic geometry is the study of ruled surfaces in projective space. If one is interested in studying families of ruled surfaces of a given degree d, the first question that one has to answer is what the maximum possible dimension is.

As ruled surfaces of degree d in $\mathbf{P}^3_{\mathbf{K}} = \mathbf{P}(V)$ correspond to curves of degree d in the grassmannian $G = G(2,4)$, embedded in $\mathbf{P}^5_{\mathbf{K}}$ by Plücker embedding, the problem we are dealing with is equivalent to that of bounding the dimension of the *Chow variety* of curves of degree d in G, i.e. the connected component of the reduced structure of the *Chow scheme* (representing the Chow functor) parametrizing curves of degree d in G. Throughout this paper, $\mathbf{K}$ is an algebraically closed field of characteristic zero, and Σ will denote a family of (integral) ruled surfaces of degree d in $\mathbf{P}^3_{\mathbf{K}}$.

In this paper we provide a sharp bound to the dimension of Σ. We assume that the general point C of Σ corresponds to an *integral* curve, equally denoted C. We also assume $d \geq 3$ since for $d = 1, 2$ one can easily see that the dimension is 5, 9 respectively.

The arithmetic genus of this curve is the irregularity q of the corresponding ruled surface S birationally immersed in $\mathbf{P}^3$ as

$$\mathbf{P}(E_C'^{\vee}) \hookrightarrow \mathbf{P}^3 \times \mathcal{O}_C \to \mathbf{P}^3$$

where

$$0 \to E \to V^* \otimes \mathcal{O}_G \to E'^{\vee} \to 0$$

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is the dual of the universal sequence

$$0 \to E' \to V \otimes \mathcal{O}_G \to E^\vee \to 0.$$

Let us review some properties of G. The tangent bundle of G is $\mathrm{Hom}(E', E^\vee) = E'^\vee \otimes E^\vee = E'(1) \otimes E(1)$. A section $s \in H^0(E^\vee) = V$,

$$0 \to \mathcal{O} \xrightarrow{s} E^\vee \to \mathcal{I}_{\alpha_p}(1) \to 0,$$

vanishes along the α-plane $\alpha_p \subset G$ parametrizing all lines in $\mathbf{P}^3$ passing through $p = \langle s \rangle$. Analogously, a section $s' \in H^0(E'^\vee) = V^\vee$,

$$0 \to \mathcal{O} \xrightarrow{s'} E'^\vee \to \mathcal{I}_{\alpha'_\pi}(1) \to 0,$$

vanishes along the α'-plane (or *β-plane*) $\alpha'_\pi \subset G$ parametrizing all lines in $\mathbf{P}^3$ contained in π.

It is easy to check that $E \otimes \mathcal{O}_{\alpha_p} = \Omega_{\alpha_p}(1) = T_{\alpha_p}(-2)$ and analogously for $E' \otimes \mathcal{O}_{\alpha'_\pi}$.

Recall that an integral ruled surface is called *regular* if the lift $E'^\vee_{\widetilde{C}}$ of $E'^\vee$ by

$$\widetilde{C} \xrightarrow{\varphi} C \subset G$$

has $H^1(E'^\vee_{\widetilde{C}}) = 0$ (as noted in [ASP], this vanishing is equivalent to the fact that the corresponding ruled surface is a projection of a linearly normal surface of $\mathbf{P}^{d+1-2q}$, i.e. it is *regular* in the terminology of the classics).

Proposition 1. *For an irreducible component Σ of the Chow variety whose generic element corresponds to a regular ruled surface of irregularity q we have*

$$\dim \Sigma \leq 4d - g + 1$$

where g ($\leq q$) is the geometric genus of the general curve $C \in \Sigma$. This is an equality if C is smooth.

Proof. The dimension of Σ is that of the tangent space to the Hilbert scheme $\mathcal{H}_{d,g}$, whose irreducible component H at C is dense in Σ and smooth at C.

If C is smooth, the proof is easy: The normal bundle has presentation

$$0 \to T_C \to T_G|_C \to N_{C,G} \to 0,$$

thus $\deg N_{C,G} = \deg \omega_C(4) = 2g - 2 + 4d$. From $V \otimes E'^\vee_C \to E^\vee \otimes E'^\vee_C = T_G|_C \to 0$ we obtain that $H^1(E'^\vee_C) = 0$ implies $H^1(T_G|_C) = 0$, thus $H^1(N_{C,G}) = 0$. Therefore

$$\dim \Sigma = h^0(N_{C,G}) = \deg N_{C,G} + 3(1-g) = 4d - g + 1.$$

Assume now C is not smooth. Let $\varphi : \widetilde{C} \to C \hookrightarrow G$ be the desingularization of C and define its normal sheaf as cokernel

$$0 \to T_{\widetilde{C}} \to \varphi^* T_G = E^\vee_{\widetilde{C}} \otimes E'^\vee_{\widetilde{C}} \to N_\varphi \to 0.$$

From $0 = V \otimes H^1(E'^\vee_{\widetilde{C}}) \to H^1(E^\vee_{\widetilde{C}} \otimes E'^\vee_{\widetilde{C}}) \to 0$ we conclude $H^1(N_\varphi) = 0$. A vector in $T_C\Sigma = T_C H$ corresponds to an infinitesimal deformation (i.e. a flat family over the spectrum of the dual numbers) of C in G, which is equisingular, since curves in H equisingular to the general curve C of H form a dense open neighborhood of C. This means (cf. [AC], for instance) that such a vector does not lie in the subspace $H^0((N_\varphi)_{\mathrm{tor}}) \subset H^0(N_\varphi)$ of infinitesimal deformations partially desingularizing C. (A singular branch of C corresponds to a point of $\widetilde{C}$ where $(N_\varphi)_{\mathrm{tor}}$ has nonzero stalk and it will not be desingularized by the infinitesimal deformation if it does

not lie in this stalk). We can make the monomorphism of sheaves into a *bundle* inclusion

$$0 \to T_{\widetilde{C}} \otimes \mathcal{O}_{\widetilde{C}}(Z) \to \varphi^* T_G \to N_\varphi/(N_\varphi)_{\mathrm{tor}} \to 0$$

after twisting $T_{\widetilde{C}}$ with the *divisor* Z where it fails to be so. Since $N_\varphi/(N_\varphi)_{\mathrm{tor}}$ is nonspecial we end up with

$$\begin{aligned}\dim \Sigma = \dim H \le h^0(N_\varphi/(N_\varphi)_{\mathrm{tor}}) &= \deg N_\varphi/(N_\varphi)_{\mathrm{tor}} + 3(1-g)\\ &= 2g - 2 + 4d - \deg Z + 3(1-g) \le 4d - g + 1. \quad \square\end{aligned}$$

Recall that a ruled surface is called developable if all tangent lines to the corresponding curve in G lie in G (as a surface of $\mathbf{P}^3$ it consists of all tangents to a curve in $\mathbf{P}^3$—the so-called tangent developable surfaces—or it is a cone, i.e. consists of lines passing through a point in $\mathbf{P}^3$).

Proposition 2. *For an irreducible component Σ of the Chow variety whose generic element C has geometric genus g and corresponds to a nondevelopable and nonregular ruled surface of irregularity q ($\ge g$), we have*:

$$\dim \Sigma \le 3d + q + r - 1$$

where $r = h^0(E_C^\vee) - 1$.

Proof. Let $p \in \mathbf{P}^3$ and let

$$0 \to E'^\vee \to E'^\vee \otimes E^\vee \to E'^\vee \otimes \mathcal{I}_{\alpha_p}(1) \to 0$$

be the sequence presenting the ideal of the corresponding α-plane α_p, after tensoring with $E'^\vee$. If we blow up G with center in α_p, the ideal $\mathcal{I}_{\alpha_p}$ becomes the ideal of the exceptional divisor, thus a line bundle, so the inclusion at the left of the previous sequence becomes a bundle inclusion, which we want to identify. Consider the two natural projections of this blow-up

$$\begin{tikzcd}[column sep=small] & \tilde{G} = \tilde{G}(\alpha_p) = \{(\pi,\, L) \in \mathbf{P}^{3\vee} \times G \mid p \in \pi,\, L \subset \pi\} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \mathbf{P}^2 = \{\pi \in \mathbf{P}^{3\vee} \mid p \in \pi\} & & G \end{tikzcd}$$

Take as generators of $\operatorname{Pic}\widetilde{G} = \mathbf{Z} \oplus \mathbf{Z}$ the pullback by $\mathrm{pr}_1, \mathrm{pr}_2$ of the generators of the Picard groups of $\mathbf{P}^2$ and G, so we denote $\mathrm{pr}_1^* \mathcal{O}_{\mathbf{P}^2}(n) \otimes \mathrm{pr}_2^* \mathcal{O}_G(m)$ by $\mathcal{O}_{\widetilde{G}}(n,m)$.

The above monomorphism of sheaves $E'^\vee \hookrightarrow E^\vee \otimes E'^\vee$, failing to be a bundle inclusion in α_p, lifts to a monomorphism of sheaves which becomes a bundle inclusion

$$E'^\vee_{\widetilde{G}}(D) = E'_{\widetilde{G}}(-1,2) \hookrightarrow E^\vee_{\widetilde{G}} \otimes E'^\vee_{\widetilde{G}}$$

after twisting with the exceptional divisor $\mathcal{O}_{\widetilde{G}}(D) = \mathcal{O}_{\widetilde{G}}(-1,1)$ of the blow-up, where it fails to be so. The restriction of $E'^\vee_{\widetilde{G}}(D)$ to each pr_1-fibre $\mathrm{pr}_1^{-1}(\pi) = \alpha'_\pi$ is its tangent bundle $E'(2) \otimes \mathcal{O}_{\alpha'_\pi} = T_{\alpha'_\pi}$. From this we can conclude that $E'^\vee_{\widetilde{G}}(D)$ is the relative tangent bundle $T_{\widetilde{G}/\mathbf{P}^2}$, after computing the first Chern class and checking that we obtain $(-2,3)$ in both cases. We also note that this is just the *inclusion of bundles* $T_{\widetilde{G}/\mathbf{P}^2} \hookrightarrow (T_G)_{\widetilde{G}}$ obtained as composition of the bundle inclusion $T_{\widetilde{G}/\mathbf{P}^2} \hookrightarrow T_{\widetilde{G}}$ with the monomorphism of sheaves $T_{\widetilde{G}} \hookrightarrow (T_G)_{\widetilde{G}}$.

Consider C in the statement of the proposition. First we assume it to be smooth. Since we assume the corresponding ruled surface S of $\mathbf{P}^3$ to be nondevelopable, there is only a finite number of planes which are tangent to S along a line of the ruling. Pick $p \in S$, general in S, not in those planes. Out of p, the image of $E_C'^\vee \hookrightarrow E_C \otimes E_C'^\vee = T_G|_C$ has null intersection with the bundle $T_C \hookrightarrow E_C^\vee \otimes E_C'^\vee$ tangent to C. This is clear after our previous geometrical interpretation of the monomorphism of sheaves $E'^\vee \hookrightarrow E^\vee \otimes E'^\vee$: it is the inclusion of the bundle tangent to the projection $\widetilde{G} \to \mathbf{P}^2$ in the tangent bundle, pushed down to G. Thus it fails to be a bundle inclusion only at α_p. Therefore, the composition $E_C'^\vee \hookrightarrow T_G|_C \to N_C$ fails to be a bundle inclusion only at $r_p = C \cap \alpha_p$, where it is zero. We thus get a bundle inclusion

$$0 \to E_C'^\vee(r_p) \hookrightarrow N_C \to \omega_C(3-2r_p) \to 0$$

whose rank 1 cokernel has been computed by comparing the first Chern classes.

Observe that $h^0(E_C'^\vee(r_p)) - h^0(E_C'^\vee) \leq 1$. Indeed, since p is general, the map

$$0 \neq H^1(E_C'^\vee)^\vee = H^0(E_C' \otimes \omega) \to H^0(E_C' \otimes \omega \otimes \mathbf{K}(r_p)) = \mathbf{K}^2$$

is not zero, thus

$$h^0(E_C' \otimes \omega_C(-r_p)) \leq h^0(E_C' \otimes \omega_C) - 1$$

so the inequality follows from the Riemann-Roch formula for rank 2 bundles.

Denoting $r = h^0(E_C^\vee) - 1$ as usual, we obtain from the short sequence

$$h^0(N_{C,G}) \leq (r+2) + h^0(\omega_C(3-2r_p)) = 3d + g + r - 1.$$

If C is singular we pick $p \in S$ general, and proceed as before: We lift the bundle inclusion $E'^\vee(r_p) \to T_G$ to $\widetilde{C}$ via φ and obtain a bundle inclusion $E_{\widetilde{C}}'^\vee(r_p) \hookrightarrow (T_G)_{\widetilde{C}}$, where we still denote r_p in $\widetilde{C}$. From our geometric description of the inclusion in $\widetilde{G}$, it becomes clear that the inclusion in $\widetilde{C}$ does not factorize through

$$0 \to T_{\widetilde{C}} \hookrightarrow (T_G)_{\widetilde{C}} \to N_\varphi \to 0,$$

thus we obtain

$$0 \to E_{\widetilde{C}}'^\vee(r_p) \to N_\varphi \to M \to 0$$

with $(N_\varphi)_{\mathrm{tor}} \cong M_{\mathrm{tor}}$. The rank one cokernel

$$0 \to E_{\widetilde{C}}'^\vee(r_p) \to (N_\varphi)/(N_\varphi)_{\mathrm{tor}} \to M/M_{\mathrm{tor}} \to 0$$

has degree $(2g-2+4d-\deg Z)-(d+2) \leq 3d+2g-4$, thus $h^0(M/M_{\mathrm{tor}}) = 3d+g-3$ if M/M_{tor} is nonspecial, and $h^0(M/M_{\mathrm{tor}}) \leq g-1 \leq 3d+g-3$ otherwise.

By a remark as in the proof of the preceding proposition

$$\begin{aligned}\dim \Sigma &\leq h^0(N_\varphi/(N_\varphi)_{\mathrm{tor}}) \leq h^0(E_C'^\vee(r_p)) + h^0(M/M_{\mathrm{tor}}) \\ &\leq (r+2) + (3d+g-3) = 3d+g+r-1. \quad \square\end{aligned}$$

Remark 3. Let us show now the relation between the inclusion of bundles that we have introduced in the proof of Proposition 2 and the field tangent to the flow in [EH]. This flow is used there to get a sequence involving the normal sheaf to a curve in $\mathbf{P}^3$ and from this sequence an estimate of the dimension of the component of the Hilbert scheme in which that curve is a general point.

Let H be a plane of $\mathbf{P}^3$ not passing through p and let $\mathcal{O}_G \overset{H}{\hookrightarrow} E^\vee$ be the section of $E^\vee$ which vanishes in the corresponding α'-plane, denoted α'_H, of G. Lifting it to $\widetilde{G}$ and composing

$$\mathcal{O}_{\widetilde{G}} \overset{H}{\hookrightarrow} E^\vee_{\widetilde{G}} \overset{D}{\hookrightarrow} E^\vee_{\widetilde{G}}(D) \hookrightarrow T_{\widetilde{G}}$$

we get a monomorphism of sheaves failing to be a bundle inclusion just at D and α'_H. If we apply now pr_{2_*} we get a tangent field

$$\mathcal{O}_G \to T_G$$

that vanishes only at α_p and α'_H. This is analogous to the field tangent to the flow in [EH] vanishing only at zero and infinity.

Lemma 4. *Let S be a ruled surface of $\mathbf{P}^n$ of degree d not contained in a hyperplane and not a cone. Its irregularity is sharply bounded by Castelnuovo's bound $\pi(d,n)$ for nondegenerate curves of $\mathbf{P}^n$ of degree d.*

Proof. If $d < n$, then necessarily $d = n-1$ and $g = 0$. So we assume $d \geq n$. It suffices to prove that the curve C in $G(1,n)$ which is the image of S after Plücker embedding spans at least $\mathbf{P}^n$, because in that case, the irregularity of S, which is nothing but the genus of C, will be bounded by Castelnuovo's bound $\pi(d,n)$. We prove that C spans at least a $\mathbf{P}^n$ by induction on n. We will show the sharpness of the bound later on.

We start with the case $n = 3$. In this case $G(1,3)$ is embedded by the Plücker embedding as a smooth quadric hypersurface in $\mathbf{P}^5$. If C lies in a general $\mathbf{P}^2 \subset \mathbf{P}^5$, then C necessarily is of degree 2, which is a contradiction to the assumption that the ruled surface has degree greater than or equal to 3.

It cannot be in an α or α'-plane, because then the ruled surface would be a cone or would lie on a plane.

Let us suppose that it holds for $n-1$. We will prove it for n. First of all, we choose a point $p \in S$ that is not on a line that meets all the lines of S. We choose a general $\mathbf{P}^{n-1} \subset \mathbf{P}^n$ such that $p \notin \mathbf{P}^{n-1}$ and define S' to be the projection of S from p into $\mathbf{P}^{n-1}$. S' is a new ruled surface of degree one less than S. We get any fiber of S' as the intersection with $\mathbf{P}^{n-1}$ of the plane spanned by p and a line of S. Therefore S' is not a cone and does not lie in a subspace of dimension $n-2$. Then we consider the following diagram

$$\begin{array}{ccc} G(1,n-1) & \xrightarrow{\iota'} & \mathbf{P}^M \\ \downarrow & & \downarrow \\ G(1,n) & \xrightarrow{\iota} & \mathbf{P}^N \end{array}$$

Where ι and ι' are the Plücker embeddings, $M = \binom{n}{2} - 1$ and $N = \binom{n+1}{2} - 1$. Hence $\mathbf{P}^M$ is a linear subspace of codimension n.

We will show that C spans a $\mathbf{P}^n$. Indeed, the projection can be seen in $\mathbf{P}^N$ as a linear projection from the linear subspace of dimension $n-1$ consisting of all the lines in the original $\mathbf{P}^n$ passing through p, that we will denote by $\alpha(p)$, to the distinguished $\mathbf{P}^M$ in which $G(1,n-1)$ lies. Obviously, the projection maps a point $q \in \mathbf{P}^N$ outside $\alpha(p)$ to the point at which the linear subspace that it spans with $\alpha(p)$ meets $\mathbf{P}^M$. The image of C under the projection is the curve C' which is also the image of S' after Plücker embedding. If C lies in a $\mathbf{P}^{n-1}$, the dimension of the

space that this subspace spans with $\alpha(p)$ is less than or equal to $2n-2$. Call it W. Clearly $C' \subseteq W \cap \mathbf{P}^M = V$. If $\dim V = n-1$, then $V \cap \alpha(p) \neq \varnothing$, which is a contradiction. Then $\dim V \leq n-2$ and C' lies in an $(n-2)$-dimensional linear subspace, but this is impossible by the induction hypothesis.

For the sharpness of the bound, we recall that Castelnuovo's bound for curves in $\mathbf{P}^n$ is achieved by curves in $F_e = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-e))$ $(e \leq n-3)$ embedded by $\mathcal{O}(C_0 + \frac{n-1+e}{2}F)$ or $\mathbf{P}^2$ embedded in $\mathbf{P}^5$ by $\mathcal{O}_{\mathbf{P}^2}(2)$. Now we just observe that those surfaces are embedded in $G(1,n)$ by $E = \mathcal{O}(F) \oplus \mathcal{O}(C_0 + \frac{n-3+e}{2}F)$ and $\mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2}(1)$ respectively. □

The curves of bidegree (d_1, d_2) in a fixed smooth quadric surface of $\mathbf{P}^3$ form a family of dimension $(d_1+1)(d_2+1)-1$. This number attains a maximum

$$\delta_{Q_2}(d) \stackrel{\text{def}}{=} \begin{cases} (\frac{d^2}{4} + d) & \text{if } d \text{ is even,} \\ (\frac{d^2}{4} + d - \frac{1}{4}) & \text{if } d \text{ is odd} \end{cases}$$

depending on whether d is even or odd.

The number

$$\delta_{\mathbf{P}^3}(d) \stackrel{\text{def}}{=} \delta_{Q_2}(d) + 9$$

has been shown in [EH] to be the maximum dimension for a family of curves of $\mathbf{P}^3$ of degree d, which are nondegenerate, i.e. which do not lie in a $\mathbf{P}^2$. The maximum for curves in $\mathbf{P}^2$ is

$$\delta_{\mathbf{P}^2}(d) = \frac{d(d+3)}{2}.$$

The theorem below studies the analogous bound for ruled surfaces. Ruled surfaces having two unisecant skew lines of $\mathbf{P}^3$, i.e. curves of G lying in a smooth quadric surface $Q_2 = \mathbf{P}^3 \cap Q_4$ form a family Σ_1 of dimension $\delta_{Q_2}(d) + 8$.

On the other hand, the smooth rational curves of G of degree d form a family Σ_2 of dimension $4d+1$. Indeed, for such curves $C \cong \mathbf{P}^1$ the bundle $E_C'^{\vee} \cong \mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(b)$ is generated by global sections

$$V^* \otimes \mathcal{O}_C \to E_C'^{\vee}$$

so has splitting numbers $a, b \geq 0$, thus $H^1(E_C'^{\vee}) = 0$. This vanishing shows, according to [ASP], that Σ_2 is irreducible and gives us the estimation of the dimension from Proposition 1.

Theorem 5. *The dimension of an irreducible component Σ of the Chow variety of ruled surfaces of degree $d \geq 3$ whose generic element C is not developable is sharply bounded by*

$$\delta_G(d) \stackrel{\text{def}}{=} \max\{4d+1, \delta_{Q_2}(d) + 8\}$$

(which is $\delta_{Q_2}(d)+8$ if $d \geq 9$, and $4d+1$ if $d \leq 9$). Furthermore, if Σ has dimension δ_G, then it is

$$\begin{cases} \Sigma_1 & \textit{if } d \geq 10, \\ \Sigma_2 & \textit{if } d < 9, \\ \Sigma_1 \textit{ or } \Sigma_2 & \textit{if } d = 9. \end{cases}$$

TABLE 1

d	$\delta_{Q_2}(d)+8$	$\pi(d,4)+3d+3$
10	43	42
11	49	48
12	56	54
13	63	60
14	71	67
15	79	74
16	88	81
17	97	89
18	107	97

Proof of the theorem in case $d \geq 10$. We can assume C is not regular, since otherwise

$$\dim \Sigma \leq 4d - g + 1 \leq 4d + 1 < \delta_G(d).$$

Assume first that the surface is linearly normal, i.e. $r = h^0(E_C'^{\vee}) - 1 = 3$, so that by Proposition 2

$$\dim \Sigma \leq g + 3d + 2.$$

We know C spans at least a $\mathbf{P}^4$. If it spans a $\mathbf{P}^3$, then C is contained in a smooth quadric or in a quadric cone. In the first case its dimension is at most $\delta_{Q_2}(d)+8$ and $\Sigma = \Sigma_1$ in case of equality. In the second case, the dimension is strictly less than $\delta_{Q_2}(d)+8$. If C spans at least a $\mathbf{P}^4$, then

$$\dim \Sigma \leq \pi(d,4) + 3d + 2 \leq \frac{d^2-d}{6} + 3d + 2.$$

This last number is smaller than $\delta_{Q_2}(d)+8$ if $d \geq 19$. If $d \leq 18$, the inequality $\pi(d,4)+3d+2 < \delta_{Q_2}(d)+8$ can be checked by hand.

Assume now that C is not linearly normal, i.e. $r > 3$. Then we can use Proposition 2 and Lemma 4:

$$\begin{aligned}\dim \Sigma &\leq q + 3d + r - 1 \leq \pi(d,r) + 3d + r - 1\\ &\leq \pi(d,4) + 3d + 3 \leq \frac{d^2-d}{6} + 3d + 3.\end{aligned}$$

Again this number is smaller than $\delta_{Q_2}(d)+8$ if $d \geq 19$ and for $d \leq 18$ the inequality $\pi(d,4)+3d+3 < \delta_{Q_2}(d)+8$ is checked by consulting Table 1. □

Proof in case $d = 9$ and C is not regular. Now $\delta_G(9) = 37$. Assume first $r = 3$. If C spans a $\mathbf{P}^3$, then C is contained in a quadric surface; thus $\dim \Sigma \leq \delta_{Q_2}(d)+8$, with $\Sigma = \Sigma_1$ in case of equality. If C spans at least a $\mathbf{P}^4$, then

$$\dim \Sigma \leq \pi(9,4) + 3 \cdot 9 + 2 = 36.$$

Assume now that $r = 4$. Then

$$\dim \Sigma \leq \pi(9,4) + 3 \cdot 9 + 3 = 37.$$

Let us see why this cannot be an equality. If so, the curve C has maximal genus and therefore it is a smooth element of the linear system $|3H|$ on a smooth rational scroll of $\mathbf{P}^4$ (Theorem 3.11 in [H], for example). The dimension of the family of smooth rational scrolls of $\mathbf{P}^4$ lying in G is 11, as computed in [AS]. Thus

$$\dim(\Sigma) = \dim|3H| + 11 = h^0(\mathrm{Sym}^3(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(2))) + 10 = 32$$

which is a contradiction.

If $r \geq 5$, then $\dim \Sigma \leq \pi(9,5) + 3 \cdot 9 + 4 = 35$. □

Proof in the case $d = 9$ and C is regular. By Proposition 1

$$\dim \Sigma \leq -g + 37 \leq 37$$

and in case of equality, C is rational and smooth, so $\Sigma = \Sigma_2$. □

Proof in case $d \leq 8$. If $d = 3$, the theorem is trivial since the variety of twisted cubics in G is easily seen to have dimension 13, which is $\delta_G(3)$. We can thus assume $4 \leq d \leq 8$ (see Table 2). If C spans a $\mathbf{P}^3$ it is a curve in a quadric and $\dim \Sigma \leq \delta_{Q_2}(d) + 8 < \delta_G(d)$, so we can assume that C spans at least a $\mathbf{P}^4$. If C is nonregular, then by Proposition 2

$$\dim \Sigma \leq \pi(d,4) + 3d + 3$$

which is strictly smaller than $\delta_G(d) = 4d + 1$.

Table 2

d	$4d+1$	$\pi(d,4)+3d+3$
4	17	15
5	21	19
6	25	23
7	29	27
8	33	32

If C is regular, then by Proposition 1

$$\dim \Sigma \leq -g + 4d + 1 \leq 4d + 1$$

with equality only in case $g = 0$, i.e. $\Sigma = \Sigma_2$. □

Remark 6. If the generic element of a component is a tangent developable surface, its dimension is sharply bounded by $\delta_{\mathbf{P}^3}(d)$ and in case it is a cone, it is sharply bounded by $\delta_{\mathbf{P}^2}(d) + 3$.

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