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COMMUTING HOLOMORPHIC FUNCTIONS AND HYPERBOLIC AUTOMORPHISMS

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Abstract. We give a complete classification of the holomorphic self-maps of the unit ball of $\mathbf{C}^n$ into itself which commute with a given hyperbolic automorphism.

Introduction

Let Δ be the unit disc of $\mathbf{C}$, and let γ be a hyperbolic automorphism of Δ. In 1941 M.H. Heins (see [6]) proved that, if a holomorphic map $f \in Hol(\Delta, \Delta)$ from the unit disc Δ into itself commutes with γ (under composition), then f is either the identity map on Δ, or it is a hyperbolic automorphism of Δ with the same fixed points of γ.

If one considers the unit ball Δ_n of $\mathbf{C}^n$ for $n > 1$, then the study of the class of all holomorphic maps $f \in Hol(\Delta_n, \Delta_n)$ which commute with a given hyperbolic automorphism γ of Δ_n is still open. The author, together with Gentili (see [4]), contributed to this subject by obtaining information on the "structure" of the maps f which commute with γ, under the hypothesis of "regularity" at one of the fixed points of γ in $\partial\Delta_n$.

In this paper a complete classification of all the holomorphic maps of Δ_n into itself which commute with a given hyperbolic automorphism γ of Δ_n, for $n > 1$, is obtained (Theorem 2.5). In dimension greater than one, the results turn out to be very different from those obtained by Heins for the unit disc of $\mathbf{C}$. A map $f \in Hol(\Delta_n, \Delta_n)$ which commutes with a hyperbolic automorphism γ need not be an automorphism of Δ_n; instead a large class of non-automorphisms which commute with γ is found and classified (Theorem 2.5 and Corollary 2.7).

In dimension 2, the results obtained also provide information on the fixed points set of a map f which commutes with γ; still the results differ much from those obtained in the one-dimensional case (Proposition 2.9).

Preliminaries and notation can be found in [10], [1] and [4].

1. Preliminary results

In this section we recall some results which will be useful in the sequel. The proof of the following theorem can be found, *e.g.*, in [1].

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Theorem 1.1. *Each element* γ *of the group* $Aut\Delta_n$ *can be extended holomorphically to an open neighborhood of* $\bar{\Delta}_n$ *and, if* $\gamma \neq id_{\Delta_n}$*, then either* γ *has at least one fixed point in* Δ_n*, or it has no fixed points in* Δ_n *and it has one or two fixed points in* $\partial\Delta_n$*.*

The following definition is also classical.

Definition 1.1. If γ has some fixed points in Δ_n, then it is called *elliptic*; if γ has no fixed points in Δ_n and one fixed point in $\partial\Delta_n$, then it is called *parabolic*; if γ has no fixed points in Δ_n and two fixed points in $\partial\Delta_n$, then it is called *hyperbolic.*

In 1941 M.H. Heins proved the following

Theorem 1.2. *Let* γ *be a hyperbolic automorphism of* Δ*, and let* $f \in Hol(\Delta, \Delta)$ *be such that* $f \circ \gamma = \gamma \circ f$*. Then either* $f = id_\Delta$ *or* f *is a hyperbolic automorphism of* Δ *with the same fixed points of* γ*.*

A proof of the above theorem can be found in [6] (for a more recent exposition of this and related results, see [1]): the proof relies upon the existence of the derivative of f at the Wolff point.

From now on γ will be a hyperbolic element of $Aut\Delta_n$. Since $Aut\Delta_n$ acts doubly transitively on $\partial\Delta_n$, we can suppose, up to conjugation in $Aut\Delta_n$, that the fixed points of γ in $\partial\Delta_n$ are e_1 and $-e_1$, where e_j denotes the j-th element of the standard basis of $\mathbf{C}^n$. Such a hyperbolic automorphism γ of Δ_n can be expressed, up to conjugation in $Aut\Delta_n$, by

$$\gamma(z) = \frac{(\cosh t_0 z_1 + \sinh t_0, e^{i\theta_2} z_2, \dots, e^{i\theta_n} z_n)}{\sinh t_0 z_1 + \cosh t_0}, \tag{1.1}$$

where $t_0 \in \mathbf{R} - \{0\}$ and $\theta_2, \dots, \theta_n \in \mathbf{R}$. Therefore, if γ is a hyperbolic automorphism of Δ_n and if $f \in Hol(\Delta_n, \Delta_n)$, then in the search for the solutions of equation $f \circ \gamma = \gamma \circ f$, we can suppose that γ is given by (1.1).

The following result is due to de Fabritiis and Gentili (see [4]).

Proposition 1.3. *Let* $\gamma \in Aut\Delta_n$ *be a hyperbolic automorphism as in* (1.1)*, and let* $f = (f_1, \dots, f_n) \in Hol(\Delta_n, \Delta_n)$*. If* $f \circ \gamma = \gamma \circ f$*, then there exists* $t_1 \in \mathbf{R}$ *such that*

$$f_1(z_1, 0, \dots, 0) = \frac{\cosh t_1 z_1 + \sinh t_1}{\sinh t_1 z_1 + \cosh t_1}. \tag{1.2}$$

The next result also appears in [4] and completely determines the behaviour of f on the disc $\Delta \times \{0\}$.

Proposition 1.4. *Let* $\gamma \in Aut\Delta_n$ *be a hyperbolic automorphism as in* (1.1)*, and let* $f = (f_1, \dots, f_n) \in Hol(\Delta_n, \Delta_n)$ *be such that* $f \circ \gamma = \gamma \circ f$*. Then* $f_2(z_1, 0, \dots, 0) = \dots = f_n(z_1, 0, \dots, 0) = 0$ *for all* $z_1 \in \Delta$*.*

Now, if $f \in Hol(\Delta_n, \Delta_n)$ is a map which commutes with the holomorphic automorphism γ defined by (1.1), we want to study the behaviour of f outside the disk $\Delta \times \{0\}$. At first we “transfer” the problem to the Siegel upper half-space $H_n = \{w \in \mathbf{C}^n : \operatorname{Im} w_1 > |w_2|^2 + \dots + |w_n|^2\}$ via the Cayley transform $\mathcal{C}$ from Δ_n to H_n given by

$$\mathcal{C}(z) = \left(i\frac{1 + z_1}{1 - z_1}, \frac{iz_2}{1 - z_1}, \dots, \frac{iz_n}{1 - z_1} \right).$$

If $F = \mathcal{C} \circ f \circ \mathcal{C}^{-1}$ and $\mu = \mathcal{C} \circ \gamma \circ \mathcal{C}^{-1}$, then $\mu \in AutH_n$ and the fact that f and γ commute is equivalent to the fact that F and μ commute.

An expression for μ is easily recovered from the form of γ (see (1.1)); it turns out that

$$\mu(w) = (\lambda^2 w_1, e^{i\theta_2}\lambda w_2, \dots, e^{i\theta_n}\lambda w_n), \tag{1.3}$$

where $\lambda = e^{t_0}$ (hence, by our assumptions, $\lambda \neq 1$). By Proposition 1.4 and by the definition of the Cayley transform we obtain:

Corollary 1.5. *Let μ be a hyperbolic automorphism of H_n given by* (1.3)*, and let $F : H_n \to H_n$ be holomorphic and such that $F \circ \mu = \mu \circ F$. Then there exists $k > 0$ such that*

$$F_1(w_1, 0, \dots, 0) = k^2 w_1 \tag{1.4}$$

and that

$$F_2(w_1, 0, \dots, 0) = \dots = F_n(w_1, 0, \dots, 0) = 0. \tag{1.5}$$

2. Main results

In this section we study the family of all holomorphic self-maps of Δ_n which commute with a given hyperbolic automorphism of Δ_n $(n \geq 1)$ without any condition on the "regularity" of the self-maps. As we have seen, via the Cayley transform, this is equivalent to studying the family of all holomorphic self-maps F of the Siegel upper half-space H_n which commute with a hyperbolic automorphism μ of H_n given by (1.3).

We know that there exists $k \in \mathbf{R}^+$ such that $F_1(w_1, 0, \dots, 0) = k^2 w_1$ and $F_j(w_1, 0, \dots, 0) = 0$ for all $j \geq 2$ and all $w_1 \in H_1$ (see Corollary 1.5). Now equation (1.5) has very strong consequences on the form of F_2 if $n = 2$: since $F_2(w_1, 0) = 0$, for all $w_1 \in H_1$, then we can find a function h, holomorphic on H_2, such that $F_2(w) = w_2 h(w)$. The fact that F and μ commute yields that $e^{i\theta_2} w_2 h\big(\mu(w)\big) = e^{i\theta_2} w_2 h(w)$ for all $w \in H_2$. Therefore

$$h\big(\mu(w)\big) = h(w) \quad \forall w \in H_2. \tag{2.1}$$

In fact, if $w_2 \neq 0$, equation (2.1) is obviously satisfied and, by continuity, it holds for all $w \in H_2$.

Equation (2.1) suggests the investigation of the action of the subgroup generated by μ on H_2, or on H_n for $n \geq 1$. Let $\Gamma = \{\mu^m, m \in \mathbf{Z}\}$ be the subgroup of $AutH_n$ generated by μ.

Proposition 2.1. *Γ acts freely and properly discontinuously on H_n.*

Proof. The fact that $\lambda \neq 1$ in (1.3) implies that Γ acts freely on H_n. To prove that Γ acts properly discontinuously, we can consider the case in which $\lambda > 1$ (otherwise we consider μ^{-1} instead of μ). Let $\tilde{w} \in H_n$ and set

$$U(\tilde{w}) = \overline{B(\tilde{w}, \rho)} \cap \{w \in \mathbf{C}^n : |\operatorname{Im} \tilde{w}_1 - \operatorname{Im} w_1| \leq \frac{1}{4}(1 - \frac{1}{\lambda^2}) \operatorname{Im} \tilde{w}_1\},$$

where $B(\tilde{w}, \rho)$ is the ball of center $\tilde{w}$ and euclidean radius ρ in $\mathbf{C}^2$. If $\rho \ll 1$, then $U(\tilde{w})$ is a compact neighborhood of $\tilde{w}$ contained in H_n.

Now we prove that, if there exists $s \in \mathbf{Z}$ such that $\mu^s(U(\tilde{w})) \cap U(\tilde{w}) \neq \emptyset$, then $s = 0$. Let $w \in U(\tilde{w})$ be such that $\mu^s(w) \in U(\tilde{w})$. Since $w \in U(\tilde{w})$, we have

$$\operatorname{Im} w_1 \geq \big(1 - \frac{1}{4}(1 - \frac{1}{\lambda^2})\big) \operatorname{Im} \tilde{w}_1 \geq \frac{3}{4} \operatorname{Im} \tilde{w}_1. \tag{2.2}$$

The fact that $\mu^s(w)$ belongs to $U(\tilde{w})$ implies now that

$$|\operatorname{Im} \tilde{w}_1 - \lambda^{2s} \operatorname{Im} w_1| \leq \frac{1}{4}(1 - \frac{1}{\lambda^2}) \operatorname{Im} \tilde{w}_1$$

and hence

$$\begin{aligned} |\lambda^{2s} - 1| \operatorname{Im} w_1 &= |\operatorname{Im} w_1 - \lambda^{2s} \operatorname{Im} w_1| \\ &\leq |\operatorname{Im} w_1 - \operatorname{Im} \tilde{w}_1| + |\lambda^{2s} \operatorname{Im} w_1 - \operatorname{Im} \tilde{w}_1| \leq \frac{1}{2}(1 - \frac{1}{\lambda^2}) \operatorname{Im} \tilde{w}_1. \end{aligned}$$

This in turn implies that

$$\operatorname{Im} w_1 \leq \frac{1}{2}(1 - \frac{1}{\lambda^2}) \operatorname{Im} \tilde{w}_1 |\lambda^{2s} - 1|^{-1}.$$

If $s > 0$, since $\lambda > 1$, then $\lambda^{2s} - 1 \geq \lambda^2 - 1$, and therefore we obtain that

$$\operatorname{Im} w_1 \leq \frac{1}{2}(1 - \frac{1}{\lambda^2}) \operatorname{Im} \tilde{w}_1 (\lambda^2 - 1)^{-1} = \frac{1}{2\lambda^2} \operatorname{Im} \tilde{w}_1 < \frac{3}{4} \operatorname{Im} \tilde{w}_1,$$

which contradicts (2.2).

If $s < 0$, then $|\lambda^{2s} - 1| = 1 - \lambda^{2s} \geq 1 - \lambda^{-2}$, therefore we have

$$\operatorname{Im} w_1 \leq \frac{1}{2}(1 - \frac{1}{\lambda^2}) \operatorname{Im} \tilde{w}_1 (1 - \frac{1}{\lambda^2})^{-1} = \frac{1}{2} \operatorname{Im} \tilde{w}_1 < \frac{3}{4} \operatorname{Im} \tilde{w}_1,$$

which again contradicts (2.2). In conclusion $s = 0$ and therefore Γ acts properly discontinuously on H_n. □

Let μ be a hyperbolic automorphism of H_n as in (1.3), and let $F \in Hol(H_n, H_n)$ be such that $F \circ \mu = \mu \circ F$. We will now prove a result on the structure of the last $n - 1$ components of F.

Let j be a natural number such that $1 \leq j \leq n$, and define Γ_j to be the subgroup of $Aut H_j$ generated by the holomorphic automorphism of H_j given by $(w_1, \dots, w_j) \mapsto (\lambda^2 w_1, \lambda e^{i\theta_2} w_2, \dots, \lambda e^{i\theta_j} w_j)$, *i.e.* by the "restriction" of μ to H_j. Then Proposition 2.1 implies that Γ_j acts freely and properly discontinuously on H_j for all $j \leq n$, and therefore we can endow $X_j = H_j/\Gamma_j$ with a complex structure such that the projection π_j from H_j to X_j is holomorphic.

We will find a suitable form of the m-th component F_m of F, for $2 \leq m \leq n$. Let $\ln : H_1 \to \mathbf{C}$ be a branch of the logarithm on the upper half-plane in $\mathbf{C}$ and recall that, by Corollary 1.5, $F_m(w_1, 0, \dots, 0) = 0$ for all $2 \leq m \leq n$.

Proposition 2.2. *Let $F : H_n \to H_n$ be a holomorphic map which commutes with the hyperbolic automorphism μ given by* (1.3)*. Then there exist $(n-1)^2$ holomorphic functions $\hat{g}_{jm} : X_j \to \mathbf{C}$ such that the m-th component F_m of F is given by*

$$F_m(w) = k \sum_{j=2}^{n} w_j e^{i(\theta_m - \theta_j) \log w_1 / 2 \log \lambda} \hat{g}_{jm}(\pi_j(w_1, \dots, w_j)).$$

Proof. First of all we prove that $F_m(w) = k \sum_{j=2}^{n} w_j g_{jm}(w_1, \dots, w_j)$, where g_{jm} are suitable holomorphic functions on H_j.

Since $F_m(w_1, 0, \dots, 0) = 0$, the map $w_2 \mapsto F_m(w_1, w_2, 0, \dots, 0)$ is equal to 0 if $w_2 = 0$, and hence there exists a holomorphic map on H_2, say g_{2m}, such that

$$F_m(w_1, w_2, 0, \dots, 0) = kw_2 g_{2m}(w_1, w_2).$$

Now the map $w_3 \mapsto F_m(w_1, w_2, w_3, 0, \dots, 0) - kw_2 g_{2m}(w_1, w_2)$ is equal to 0 if $w_3 = 0$, and hence there exists a holomorphic map on H_3, say g_{3m}, such that

$$F_m(w_1, w_2, w_3, 0, \dots, 0) - kw_2 g_{2m}(w_1, w_2) = kw_3 g_{3m}(w_1, w_2, w_3).$$

By a recursive procedure, we obtain the existence of the functions g_{jm} such that

$$F_m(w) = k \sum_{j=2}^{n} w_j g_{jm}(w_1, \dots, w_j).$$

Now we prove that such g_{jm}'s are unique. In fact, if there are two families of holomorphic functions, say g_{jm} and p_{jm}, from H_j to $\mathbf{C}$ such that

$$F_m(w) = k \sum_{j=2}^{n} w_j g_{jm}(w_1, \dots, w_j) = k \sum_{j=2}^{n} w_j p_{jm}(w_1, \dots, w_j),$$

then the difference $g_{jm} - p_{jm}$ satisfies the equation

$$\sum_{j=2}^{n} w_j (g_{jm} - p_{jm})(w_1, \dots, w_j) = 0 \quad \text{for all } w \in H_n.$$

Let $w_3 = \cdots = w_n = 0$. Then $g_{2m}(w_1, w_2) = p_{2m}(w_1, w_2)$ for all $(w_1, w_2) \in H_2$ (if $w_2 \neq 0$ the assertion is obvious, otherwise we use a continuity argument). We proceed in the same way, taking $w_4 = \cdots = w_n = 0$ and we obtain $g_{3m}(w_1, w_2, w_3) = p_{3m}(w_1, w_2, w_3)$ for all $(w_1, w_2, w_3) \in H_3$. In conclusion we obtain the uniqueness of the functions g_{jm} recursively.

Now we will get information about the behaviour of the functions g_{jm}. In order to simplify notation we will write $\mu(w_1, \dots, w_j)$ to denote $(\lambda^2 w_1, \lambda e^{i\theta_2} w_2, \dots, \lambda e^{i\theta_j} w_j)$. The fact that $F \circ \mu = \mu \circ F$ implies that (for any $2 \leq m \leq n$)

$$\sum_{j=2}^{n} w_j \big(e^{i\theta_m} g_{jm}(w_1, \dots, w_j) - e^{i\theta_j} g_{jm}(\mu(w_1, \dots, w_j))\big) = 0.$$

Then, for any $2 \leq j, m \leq n$,

$$g_{jm}(w_1, \dots, w_j) = e^{i(\theta_j - \theta_m)} g_{jm}(\mu(w_1, \dots, w_j))$$

(for all $(w_1, \dots, w_j) \in H_j$). Having defined

$$\tilde{g}_{jm}(w_1, \dots, w_j) = e^{i(\theta_j - \theta_m)\log w_1 / 2\log\lambda} g_{jm}(w_1, \dots, w_j),$$

it is easy to see that $\tilde{g}_{jm}$ is automorphic under the action of the "restriction" of μ to H_j. Therefore there exist holomorphic functions $\hat{g}_{jm} : X_j \to \mathbf{C}$ such that $\tilde{g}_{jm}(w_1, \dots, w_j) = \hat{g}_{jm}(\pi_j(w_1, \dots, w_j))$, and this concludes the proof. $\square$

We turn our attention now to the investigation of the behaviour of the first component of F. We already proved that there exists a positive k such that $F_1(w_1, 0, \dots, 0) = k^2 w_1$. Then we can write

$$F_1(w) = k^2\big(w_1 + \sum_{j=2}^{n} w_j \alpha_j(w_1) + P(w)\big), \tag{2.3}$$

where $P(w_1, 0, \ldots, 0) = 0$ and $\dfrac{\partial P}{\partial w_j}(w_1, 0, \ldots, 0) = 0$ for all $w_1 \in H_1$ and for $j = 2, \ldots, n$. First of all we want to prove that the α_j's vanish identically.

Proposition 2.3. *Let F be a holomorphic map from H_n into itself which commutes with μ given by* (1.3). *Write F_1 as in* (2.3). *Then $\alpha_j \equiv 0$ for $j = 2, \ldots, n$.*

Proof. Since $\operatorname{Im} F_1(w) > |F_2(w)|^2 + \cdots + |F_n(w)|^2$ for all $w \in H_n$, it follows that $\operatorname{Im} F_1(w) > 0$ for all $w \in H_n$. Take $j \in \{2, \ldots, n\}$ and consider F_1 on $(w_1, 0, \ldots, 0, w_j, 0, \ldots, 0)$. Writing w_2 instead of w_j, it is enough to prove that $\alpha_j = 0$ when $j = 2$. Therefore it is enough to prove the statement when $n = 2$. From now on we will denote α_2 by α. The fact that $P(w_1, 0) = 0$ and $\dfrac{\partial P}{\partial w_2}(w_1, 0) = 0$ for all $w_1 \in H_1$ implies that there exists a function η, holomorphic on H_2, such that $P(w_1, w_2) = w_2^2 \eta(w_1, w_2)$. Since $\operatorname{Im} F_1(w_1, w_2) > 0$ for all $(w_1, w_2) \in H_2$, we have

$$\operatorname{Im} w_1 > -\operatorname{Im}(\alpha(w_1) w_2 + \eta(w) w_2^2) \tag{2.4}$$

for all $(w_1, w_2) \in H_2$. Take $w_1^0 \in \mathbf{C}$ such that $\operatorname{Im} w_1^0 > 0$. Then, for any $w_2 \in \mathbf{C}$ such that $|w_2|^2 < \operatorname{Im} w_1^0$, the point (w_1^0, w_2) belongs to H_2. Let $\varepsilon \in \mathbf{R}$ be such that $0 < \varepsilon < \operatorname{Im} w_1^0$ and set $R = \sqrt{\operatorname{Im} w_1^0 - \varepsilon}$, $r = R/2$. The Borel-Carathéodory theorem and inequality (2.4) imply now that

$$\begin{aligned}
&\max_{w_2 \in \Delta_r} |\alpha(w_1^0) w_2 + w_2^2 \eta(w_1^0, w_2)| \\
&\qquad \le \frac{2r}{R - r} \max_{w_2 \in \partial\Delta_r} \operatorname{Im}\big(-(\alpha(w_1^0) w_2 + \eta(w_1^0, w_2) w_2^2)\big) \\
&\qquad \le \frac{2r}{R - r} \max_{w_2 \in \partial\Delta_r} \operatorname{Im} w_1^0 = 2 \operatorname{Im} w_1^0.
\end{aligned} \tag{2.5}$$

We pass to the evaluation of the maximum modulus of the function $w_2 \mapsto w_2^2 \eta(w_1^0, w_2) + w_2 \alpha(w_1^0)$ on Δ_r, and obtain

$$\begin{aligned}
2 \operatorname{Im} w_1^0 &\ge \max_{w_2 \in \Delta_r} |w_2^2 \eta(w_1^0, w_2) + w_2 \alpha(w_1^0)| = \max_{w_2 \in \partial\Delta_r} |w_2^2 \eta(w_1^0, w_2) + w_2 \alpha(w_1^0)| \\
&= \frac{1}{2}\sqrt{\operatorname{Im} w_1^0 - \varepsilon} \max_{w_2 \in \partial\Delta_r} |w_2 \eta(w_1^0, w_2) + \alpha(w_1^0)| \\
&= \frac{1}{2}\sqrt{\operatorname{Im} w_1^0 - \varepsilon} \max_{w_2 \in \Delta_r} |w_2 \eta(w_1^0, w_2) + \alpha(w_1^0)|.
\end{aligned}$$

Taking the limit for $\varepsilon \to 0^+$ we get

$$|w_2 \eta(w_1^0, w_2) + \alpha(w_1^0)| \le 4 (\operatorname{Im} w_1^0)^{1/2} \tag{2.6 }$$

for all $w_2 \in \mathbf{C}$ such that $|w_2|^2 < \operatorname{Im} w_1^0$. The number $\alpha(w_1^0)$ is the value at 0 of the holomorphic function $w_2 \mapsto w_2 \eta(w_1^0, w_2) + \alpha(w_1^0)$ and therefore inequality (2.6) implies that

$$|\alpha(w_1^0)| \le 4 (\operatorname{Im} w_1^0)^{1/2} \tag{2.7 }$$

for all $w_1^0 \in \mathbf{C}$ such that $\operatorname{Im} w_1^0 > 0$.

Inequality (2.7) implies now that, for any $\tau \in \mathbf{R}$,

$$\limsup_{w_1 \to \tau \quad \operatorname{Im} w_1 > 0} |\alpha(w_1)| = 0,$$

and the reflection principle yields that there exists an entire function $\tilde{\alpha}$ which extends α. Inequality (2.7) entails that $\tilde{\alpha}(w) = 0$ for all $w \in \mathbf{R}$. Therefore $\tilde{\alpha} \equiv 0$ and hence $\alpha \equiv 0$, which proves the proposition. □

Thus far we have proved that the function F_1 has the form

$$F_1(w) = k^2(w_1 + P(w)),$$

where $P(w_1, 0, \dots, 0) = 0$ and $\dfrac{\partial P}{\partial w_j}(w_1, 0, \dots, 0) = 0$ for all $w_1 \in H_1$ and for $j = 2, \dots, n$. Now we want to give a "standard" form to the function P and use this form to give a complete classification of the holomorphic self-maps of H_n which commute with μ given by (1.3).

Proposition 2.4. *Let P be a holomorphic function on H_n such that $P(w_1, 0, \dots, 0) = 0$ and $\dfrac{\partial P}{\partial w_j}(w_1, 0, \dots, 0) = 0$ for all $w_1 \in H_1$ and for $j = 2, \dots, n$. Then there exist $n(n-1)/2$ holomorphic functions $\beta_{jl} : H_j \to \mathbf{C}$ for $2 \le j \le l \le n$ such that $P(w) = \sum_{2 \le j \le l \le n} w_j w_l \beta_{jl}(w_1, \dots, w_j)$. Moreover the family $\{\beta_{jl}\}$ $(2 \le j \le l \le n)$ is unique.*

Proof. First of all we prove the assertion on uniqueness: if there are two families of functions with the required properties, then there exists a family $\{d_{jl}\}$ such that

$$\sum_{2 \le j \le l \le n} w_j w_l d_{jl}(w_1, \dots, w_j) = 0$$

on H_n. Let $w_3 = \dots = w_n = 0$. Then $w_2^2 d_{22}(w_1, w_2) = 0$ for all $(w_1, w_2) \in H_2$ and therefore $d_{22} \equiv 0$. Let $w_3 = \cdots \widehat{w}_j \cdots = w_n = 0$; then $w_2 w_j d_{2j}(w_1, w_2) + w_j^2 d_{jj}(w_1, w_2, 0, w_j, 0, 0) = 0$ and therefore $d_{2j} \equiv 0$ on H_2. By iterating the above procedure, we obtain the uniqueness of the family $\{\beta_{jl}\}$.

We pass now to the proof of the existence of such a family of functions $\{\beta_{jl}\}$ as in the statement.

Since $P(w_1, 0, \dots, 0) = 0$ and $\dfrac{\partial P}{\partial w_j}(w_1, 0, \dots, 0) = 0$ (for $j = 2, \dots, n$ and for all $w_1 \in H_1$), then there exist $\beta_{2j} : H_2 \to \mathbf{C}$, holomorphic, such that $w_2^2 \beta_{22}(w_1, w_2) = P(w_1, w_2, 0, \dots, 0)$ and that $w_2 \beta_{2j}(w_1, w_2) = \dfrac{\partial P}{\partial w_j}(w_1, w_2, 0, \dots, 0)$ (for $j = 3, \dots, n$ and for all $(w_1, w_2) \in H_2$).

Set $P_1(w) = P(w) - w_2^2 \beta_{22}(w_1, w_2) - w_2 w_3 \beta_{23}(w_1, w_2) - \cdots - w_2 w_n \beta_{2n}(w_1, w_2)$. It is easy to see that $P_1(w_1, w_2, 0, \dots, 0) = 0$ and $\dfrac{\partial P_1}{\partial w_j}(w_1, w_2, 0, \dots, 0) = 0$ for $j = 3, \dots, n$ and for all $(w_1, w_2) \in H_2$. Then there exist $\beta_{3j} : H_3 \to \mathbf{C}$, holomorphic, such that $w_3^2 \beta_{33}(w_1, w_2, w_3) = P_1(w_1, w_2, w_3, 0, \dots, 0)$ and $w_3 \beta_{3j}(w_1, w_2, w_3) = \dfrac{\partial P_1}{\partial w_j}(w_1, w_2, w_3, 0, \dots, 0)$ for $j = 4, \dots, n$ and for all $(w_1, w_2, w_3) \in H_3$. By iterating this procedure, we end with a holomorphic function $\beta_{nn} : H_n \to \mathbf{C}$ such that $P_{n-2}(w) = w_n^2 \beta_{nn}(w)$. This proves the existence of the required family. $\square$

By gathering together all the results, we have proved that, if $F : H_n \to H_n$ is a holomorphic map which commutes with μ given by (1.3), then there exist: $k > 0$, $\beta_{jl} \in Hol(H_j, \mathbf{C})$ $(2 \le j \le l \le n)$ and $\hat{g}_{jm} \in Hol(X_j, \mathbf{C})$ $(j, m = 2, \dots, n)$ such that

$$F_1(w) = k^2\big(w_1 + \sum_{2 \le j \le l \le n} w_j w_l \beta_{jl}(w_1, \dots, w_j)\big)$$

and

$$F_m(w) = k \sum_{j=2}^{n} w_j e^{i(\theta j - \theta_m)\log w_1/2\log\lambda} \hat{g}_{jm}(\pi_j(w_1, \dots, w_j)).$$

The fact that F commutes with μ and the uniqueness of the family β_{jm} imply that

$$e^{i(\theta_j+\theta_l)} \beta_{jl}\big(\mu(w)\big) = \beta_{jl}(w) \quad \forall w \in H_j. \tag{2.8 }$$

Let us define

$$\tilde{\beta}_{jl}(w) = e^{i(\theta_j+\theta_l)\log w_1/2\log\lambda} \beta_{jl}(w);$$

then equation (2.8) entails that $\tilde{\beta}_{jl}$ is automorphic under the action of Γ_j. Therefore, if $F : H_n \to H_n$ is a holomorphic map which commutes with μ given by (1.3), we can find $k > 0$, $\hat{g}_{jl}, \hat{\beta}_{jl} \in Hol(X_j, \mathbf{C})$ such that

$$F_1(w) = k^2\big(w_1 + \sum_{2\le j\le l\le n} w_j w_l e^{-i(\theta_j+\theta_l)\log w_1/2\log\lambda} \hat{\beta}_{jl}\big(\pi_j(w_1, \dots, w_j)\big)\big) \qquad \text{and}$$

$$F_m(w) = k \sum_{j=2}^{n} w_j e^{i(\theta_m - \theta_j)\log w_1/2\log\lambda} \hat{g}_{jm}(\pi_j(w_1, \dots, w_j)),$$

for $m = 2, \dots, n$.

Moreover it is easy to see that, if F has the above form, then $F : H_n \to \mathbf{C}^n$ commutes with μ given by (1.3) (to be more precise, with the holomorphic extension of μ to $\mathbf{C}^n$).

Then we need only a "restriction of the image" to obtain a complete classification of the holomorphic maps from H_n into itself which commute with μ. Now we can state:

Theorem 2.5. *Let $F : H_n \to H_n$ be a holomorphic map which commutes with μ, the hyperbolic automorphism of H_n given by* (1.3)*. Then there exist $k > 0$, $\hat{g}_{jl} \in Hol(X_j, \mathbf{C})$ (for $j, l = 2, \dots, n$) and $\hat{\beta}_{jl} \in Hol(X_j, \mathbf{C})$ (for $2 \le j \le l \le n$) such that*

$$F_1(w) = k^2\big(w_1 + \sum_{2\le j\le l\le n} w_j w_l e^{-i(\theta_j+\theta_l)\log w_1/2\log\lambda} \hat{\beta}_{jl}\big(\pi_j(w_1, \dots, w_j)\big)\big), \tag{2.9}$$

$$F_m(w) = k \sum_{j=2}^{n} w_j e^{i(\theta_m - \theta_j)\log w_1/2\log\lambda} \hat{g}_{jm}(\pi_j(w_1, \dots, w_j)) \tag{2.10}$$

for $m = 2, \dots, n$ and that

$$\begin{aligned} &\operatorname{Im}\Big(w_1 + \sum_{2\le j\le l\le n} w_j w_l e^{-i(\theta_j+\theta_l)\log w_1/2\log\lambda} \hat{\beta}_{jl}\big(\pi_j(w_1, \dots, w_j)\big)\Big) \\ &\qquad > \sum_{m=2}^{n} \Big| \sum_{j=2}^{n} w_j e^{i(\theta_m-\theta_j)\log w_1/2\log\lambda} \hat{g}_{jm}(\pi_j(w_1, \dots, w_j)) \Big|^2 \quad \forall w \in H_n. \end{aligned} \tag{2.11}$$

Vice versa, let F be as in (2.9) *and* (2.10)*, where $\hat{g}_{jm}, \hat{\beta}_{jm} \in Hol(X_j, \mathbf{C})$ satisfy* (2.11)*. Then $F : H_n \to H_n$ commutes with μ. Moreover, the map which associates F to $(k, \hat{g}_{jm}, \hat{\beta}_{jm})$ is one-to-one.*

The above theorem gives a complete answer to the problem of finding all holomorphic self-maps of H_n which commute with the hyperbolic automorphism μ given by (1.3). By conjugation—as remarked at the beginning of this paper—it also gives an answer to the problem of finding all holomorphic self-maps of H_n which commute with a given hyperbolic automorphism of H_n. Notice that condition (2.11) is fulfilled if the modulus of the $\hat{g}_{jm}$ and $\hat{\beta}_{jm}$ is very small. Therefore an open neighborhood of 0 in $Hol(X_2, \mathbf{C}) \times \cdots \times Hol(X_n, \mathbf{C})$ satisfies condition (2.11). This shows how deep the difference between the one-dimensional case and the multidimensional case is (for the one-dimensional case see, *e.g.*, [1] or [6]).

Now we will consider, with particular attention, the case $n = 2$, in which some other results can be given. First of all we restate Theorem 2.5 for $n = 2$.

Theorem 2.6. *Let $F : H_2 \to H_2$ be a holomorphic map which commutes with the hyperbolic automorphism μ given by* (1.3). *Then there exist $k > 0$ and $\hat{g}, \hat{\beta} \in Hol(X_2, \mathbf{C})$ such that*

a) $F(w) = \Big(k^2\big(w_1 + w_2^2 e^{-i\theta_2 \log w_1/\log\lambda}\hat{\beta}(\pi(w))\big), kw_2\hat{g}\big(\pi(w)\big)\Big)$ *and*

b) $\operatorname{Im}\Big(w_1 + w_2^2 e^{-i\theta_2 \log w_1/\log\lambda}\hat{\beta}\big(\pi(w)\big)\Big) > |w_2\hat{g}\big(\pi(w)\big)|^2 \quad \forall w \in H_2.$

Vice versa, let F be as in a), where $\hat{g}, \hat{\beta} \in Hol(X_2, \mathbf{C})$ satisfy b). Then $F : H_2 \to H_2$ commutes with μ. Moreover, the map which associates F to $(k, \hat{g}, \hat{\beta})$ is one-to-one.

Notice that condition *b*) is satisfied if $e^{\theta_2\pi/|\log\lambda|}|\hat{\beta}(\pi(w))| + |\hat{g}(\pi(w))|^2 \leq 1$. Therefore an open neighborhood of 0 in $Hol(X_2, \mathbf{C}) \times Hol(X_2, \mathbf{C})$ consists of maps satisfying condition *b*).

There are "many" functions f which commute with γ: we will study in detail the "richness" of the class of functions given in Theorem 2.6, when $\hat{\beta} = 0$.

Corollary 2.7. *Let $F : H_2 \to H_2$ be a holomorphic map which commutes with the hyperbolic automorphism μ given by* (1.3). *If F_1 does not depend on w_2, then there exist $k > 0$ and $\hat{g} \in Hol(X_2, \mathbf{C})$ such that*

a) $F(w) = \Big(k^2 w_1, kw_2\hat{g}\big(\pi(w)\big)\Big)$ *and*

b) $\big|\hat{g}\big(\pi(w)\big)\big| \leq 1 \quad \forall w \in H_2.$

Vice versa, let F be as in a), where $\hat{g} \in Hol(X_2, \mathbf{C})$ satisfies b). Then $F : H_2 \to H_2$ commutes with μ.

Proof. By Theorem 2.6, if F is a holomorphic self-map of H_2 which commutes with μ, then $F_1(w_1, 0) = k^2 w_1$ for a suitable $k > 0$.

The fact that F_1 does not depend on w_2 implies that

$$F(w) = \Big(k^2 w_1, kw_2\hat{g}\big(\pi(w)\big)\Big).$$

Moreover, that fact that F maps H_2 into itself entails that

$$\operatorname{Im} k^2 w_1 > k^2 |w_2|^2 \left|\hat{g}\big(\pi(w)\big)\right|^2 \quad \forall w \in H_2. \tag{2.12}$$

Consider $w_1^0 \in \mathbf{C}$ such that $\operatorname{Im} w_1^0 > 0$, and let $\varepsilon \in \mathbf{R}$ be such that $0 < \varepsilon < \operatorname{Im} w_1^0$. Define $r(\varepsilon) = \sqrt{\operatorname{Im} w_1^0 - \varepsilon}$. Inequality (2.12) now implies that

$$\begin{aligned}\operatorname{Im} w_1^0 &> \max_{w_2 \in \Delta_{r(\varepsilon)}} |w_2|^2 \left|\hat{g}\big(\pi(w_1^0, w_2)\big)\right|^2 = \max_{w_2 \in \partial\Delta_{r(\varepsilon)}} |w_2|^2 \left|\hat{g}\big(\pi(w_1^0, w_2)\big)\right|^2 \\ &= (\operatorname{Im} w_1^0 - \varepsilon) \max_{w_2 \in \partial\Delta_{r(\varepsilon)}} \left|\hat{g}\big(\pi(w_1^0, w_2)\big)\right|^2 \\ &= (\operatorname{Im} w_1^0 - \varepsilon) \max_{w_2 \in \Delta_{r(\varepsilon)}} \left|\hat{g}\big(\pi(w_1^0, w_2)\big)\right|^2 .\end{aligned}$$

Taking the limit for $\varepsilon \to 0^+$ we obtain that $\left|\hat{g}\big(\pi(w_1^0, w_2)\big)\right|^2 \leq 1$ for all $w_2 \in \mathbf{C}$ such that $|w_2|^2 < \operatorname{Im} w_1^0$. This in turn implies that, for any $w \in H_2$, $\left|\hat{g}\big(\pi(w)\big)\right| \leq 1$.

The sufficiency of *a*) and *b*) is obvious. ☐

The above Corollary shows how large the family of all holomorphic self-maps of H_2 which commute with a given hyperbolic automorphism of H_2 is. In fact, if $\hat{g} \circ \pi$ does not depend on w_2, then $\hat{g} \circ \pi$ is a holomorphic map from the annulus $A(\rho, 1) = \{z \in \mathbf{C} : \rho < |z| < 1\}$ (where $\rho = \exp(-\pi^2/|\log \lambda|)$) to the closed unit disk $\bar{\Delta}$. It is well known that this space of functions is quite large.

Theorem 2.6 also makes it possible to study the fixed points set of a holomorphic self-map of Δ_2 which commutes with a given hyperbolic automorphism γ. As we are interested in the structure of $fixf = \{z \in \Delta_2 : f(z) = z\}$, up to the action of $Aut\Delta_2$, we can suppose that γ is given by (1.1). First of all we restate Theorem 2.6 on Δ_2 by means of the Cayley transform $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2) : \Delta_2 \to H_2$.

Corollary 2.8. *Let* $f : \Delta_2 \to \Delta_2$ *be a holomorphic map which commutes with* γ *given by* (1.1). *Then there exist* $t_1 \in \mathbf{R}$ *and* $\hat{g}, \hat{\beta} \in Hol(X, \mathbf{C})$ *such that*

a) $f(z) =$

$$\frac{\big(\cosh t_1 z_1 + \sinh t_1 + i e^{t_1} z_2^2 e^{-i\theta_2 \log \mathcal{C}_1(z)/\log\lambda} \hat{\beta}(\pi(\mathcal{C}(z)))/2(1 - z_1), z_2 \hat{g}(\pi(\mathcal{C}(z)))\big)}{\cosh t_1 + z_1 \sinh t_1 + i e^{t_1} z_2^2 e^{-i\theta_2 \log \mathcal{C}_1(z)/\log\lambda} \hat{\beta}(\pi(\mathcal{C}(z)))/2(1 - z_1)}$$

and

b) $\operatorname{Im}\big(\mathcal{C}_1(z) + \mathcal{C}_2^2(z) e^{-i\theta_2 \log \mathcal{C}_1(z)/\log\lambda} \hat{\beta}(\pi(\mathcal{C}(z)))\big) > \left|\mathcal{C}_2(z) \hat{g}\big(\pi(\mathcal{C}(z))\big)\right|^2 \quad \forall z \in \Delta_2.$

Vice versa, let f *be as in a*), *where* $\hat{g}, \hat{\beta} \in Hol(X, \mathbf{C})$ *satisfy b*). *Then* $f : \Delta_2 \to \Delta_2$ *commutes with* γ.

Now we study the possible fixed points sets of f: in sharp contrast with the one-dimensional case we prove that a map which commutes with a hyperbolic automorphism of Δ_2 can have fixed points in Δ_2 and we find an explicit form for the possible fixed points sets of f, giving also a necessary and sufficient condition for f to have fixed points.

Proposition 2.9. *Let* $f : \Delta_2 \to \Delta_2$ *be a holomorphic map which commutes with the hyperbolic automorphism* γ *given by* (1.1). *If* f *has fixed points in* Δ_2, *then either* $f = id_{\Delta_2}$ *or* $fixf = \Delta \times \{0\}$. *Moreover,* f *has fixed points in* Δ_2 *iff* $f_1(z_1, 0) = z_1$ *for all* $z_1 \in \Delta$.

Proof. We recall that the fixed points set of a holomorphic self-map of Δ_n is given by the intersection of a complex affine space with Δ_n; therefore, if $n = 2$, $fixf$ is always contained in a complex affine line, unless $f = id_{\Delta_2}$.

Let $a, b, c \in \mathbf{C}$ be such that the affine line defined by $az_1 + bz_2 + c = 0$ contains the fixed points set of f, and let $z^0 = (z_1^0, z_2^0)$ be a fixed point of f. Since γ and f commute, it is easy to see that $\gamma^m(z^0)$ belongs to $fixf$ for all $m \in \mathbf{Z}$. Calling in the form of γ given by (1.1) we obtain that

$$a\frac{\cosh mt_0 z_1^0 + \sinh mt_0}{\sinh mt_0 z_1^0 + \cosh mt_0} + b\frac{e^{im\theta_2} z_2^0}{\sinh mt_0 z_1^0 + \cosh mt_0} + c = 0 \quad \forall m \in \mathbf{Z}.$$

This is equivalent to

$$(az_1^0 + c)\cosh mt_0 + (cz_1^0 + a)\sinh mt_0 + be^{im\theta_2} z_2^0 = 0 \quad \forall m \in \mathbf{Z}.$$

Divide both members of the last equation by $\cosh mt_0$ and take the limit both for $m \to +\infty$ and for $m \to -\infty$. Since $\lim_{m\to\pm\infty} \cosh mt_0 = +\infty$, $\lim_{m\to\pm\infty} \tanh mt_0 = \pm 1$ and since the modulus of $be^{im\theta_2} z_2^0$ is bounded when m diverges, we obtain

$$(az_1^0 + c) + (cz_1^0 + a) = 0 \quad \text{and} \quad (az_1^0 + c) - (cz_1^0 + a) = 0.$$

Since $|z_1^0| < 1$, the last equations imply $a = c = 0$. Then the complex affine line containing $fixf$ is the complex line $z_2 = 0$, and hence $z_2^0 = 0$. Propositions 1.3 and 1.4 show that, on the complex line $z_2 = 0$, the function f is given by

$$f(z_1, 0) = \left(\frac{\cosh t_1 z_1 + \sinh t_1}{\sinh t_1 z_1 + \cosh t_1}, 0\right).$$

Therefore, if $(z_1^0, 0)$ is a fixed point for f, we obtain that $t_1 = 0$ and hence the set $\Delta \times \{0\}$ is contained in $fixf$. The assertion follows.

□

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