COMMUTING HOLOMORPHIC FUNCTIONS
AND HYPERBOLIC AUTOMORPHISMS

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(Communicated by Eric Bedford)

Abstract. We give a complete classification of the holomorphic self-maps of the unit ball of \(\mathbb{C}^n\) into itself which commute with a given hyperbolic automorphism.

Introduction

Let \(\Delta\) be the unit disc of \(\mathbb{C}\), and let \(\gamma\) be a hyperbolic automorphism of \(\Delta\). In 1941 M.H. Heins (see [6]) proved that, if a holomorphic map \(f \in Hol(\Delta, \Delta)\) from the unit disc \(\Delta\) into itself commutes with \(\gamma\) (under composition), then \(f\) is either the identity map on \(\Delta\), or it is a hyperbolic automorphism of \(\Delta\) with the same fixed points of \(\gamma\).

If one considers the unit ball \(\Delta_n\) of \(\mathbb{C}^n\) for \(n > 1\), then the study of the class of all holomorphic maps \(f \in Hol(\Delta_n, \Delta_n)\) which commute with a given hyperbolic automorphism \(\gamma\) of \(\Delta_n\) is still open. The author, together with Gentili (see [4]), contributed to this subject by obtaining information on the “structure” of the maps \(f\) which commute with \(\gamma\), under the hypothesis of “regularity” at one of the fixed points of \(\gamma\) in \(\partial \Delta_n\).

In this paper a complete classification of all the holomorphic maps of \(\Delta_n\) into itself which commute with a given hyperbolic automorphism \(\gamma\) of \(\Delta_n\), for \(n > 1\), is obtained (Theorem 2.5). In dimension greater than one, the results turn out to be very different from those obtained by Heins for the unit disc of \(\mathbb{C}\). A map \(f \in Hol(\Delta_n, \Delta_n)\) which commutes with a hyperbolic automorphism \(\gamma\) need not be an automorphism of \(\Delta_n\); instead a large class of non-automorphisms which commute with \(\gamma\) is found and classified (Theorem 2.5 and Corollary 2.7).

In dimension 2, the results obtained also provide information on the fixed points set of a map \(f\) which commutes with \(\gamma\); still the results differ much from those obtained in the one-dimensional case (Proposition 2.9).

Preliminaries and notation can be found in [10], [1] and [4].

1. Preliminary results

In this section we recall some results which will be useful in the sequel. The proof of the following theorem can be found, e.g., in [1].
Theorem 1.1. Each element $\gamma$ of the group $\text{Aut}\Delta_n$ can be extended holomorphically to an open neighborhood of $\Delta_n$ and, if $\gamma \neq \text{id}_{\Delta_n}$, then either $\gamma$ has at least one fixed point in $\Delta_n$, or it has no fixed points in $\Delta_n$ and it has one or two fixed points in $\partial\Delta_n$.

The following definition is also classical.

Definition 1.1. If $\gamma$ has some fixed points in $\Delta_n$, then it is called elliptic; if $\gamma$ has no fixed points in $\Delta_n$ and one fixed point in $\partial\Delta_n$, then it is called parabolic; if $\gamma$ has no fixed points in $\Delta_n$ and two fixed points in $\partial\Delta_n$, then it is called hyperbolic.

In 1941 M.H. Heins proved the following

Theorem 1.2. Let $\gamma$ be a hyperbolic automorphism of $\Delta$, and let $f \in \text{Hol}(\Delta, \Delta)$ be such that $f \circ \gamma = \gamma \circ f$. Then either $f = \text{id}_\Delta$ or $f$ is a hyperbolic automorphism of $\Delta$ with the same fixed points of $\gamma$.

A proof of the above theorem can be found in [6] (for a more recent exposition of this and related results, see [1]): the proof relies upon the existence of the derivative of $f$ at the Wolff point.

From now on $\gamma$ will be a hyperbolic element of $\text{Aut}\Delta_n$. Since $\text{Aut}\Delta_n$ acts doubly transitively on $\partial\Delta_n$, we can suppose, up to conjugation in $\text{Aut}\Delta_n$, that the fixed points of $\gamma$ in $\partial\Delta_n$ are $e_1$ and $-e_1$, where $e_j$ denotes the $j$-th element of the standard basis of $\mathbb{C}^n$. Such a hyperbolic automorphism $\gamma$ of $\Delta_n$ can be expressed, up to conjugation in $\text{Aut}\Delta_n$, by

\begin{equation}
\gamma(z) = \frac{(\cosh t_0 z_1 + \sinh t_0, e^{i\theta_2}z_2, \ldots, e^{i\theta_n}z_n)}{\sinh t_0 z_1 + \cosh t_0},
\end{equation}

where $t_0 \in \mathbb{R}\setminus\{0\}$ and $\theta_2, \ldots, \theta_n \in \mathbb{R}$. Therefore, if $\gamma$ is a hyperbolic automorphism of $\Delta_n$ and if $f \in \text{Hol}(\Delta_n, \Delta_n)$, then in the search for the solutions of equation $f \circ \gamma = \gamma \circ f$, we can suppose that $\gamma$ is given by (1.1).

The following result is due to de Fabritiis and Gentili (see [4]).

Proposition 1.3. Let $\gamma \in \text{Aut}\Delta_n$ be a hyperbolic automorphism as in (1.1), and let $f = (f_1, \ldots, f_n) \in \text{Hol}(\Delta_n, \Delta_n)$. If $f \circ \gamma = \gamma \circ f$, then there exists $t_1 \in \mathbb{R}$ such that

\begin{equation}
f_1(z_1, 0, \ldots, 0) = \frac{\cosh t_1 z_1 + \sinh t_1}{\sinh t_1 z_1 + \cosh t_1}.
\end{equation}

The next result also appears in [4] and completely determines the behaviour of $f$ on the disc $\Delta \times \{0\}$.

Proposition 1.4. Let $\gamma \in \text{Aut}\Delta_n$ be a hyperbolic automorphism as in (1.1), and let $f = (f_1, \ldots, f_n) \in \text{Hol}(\Delta_n, \Delta_n)$ be such that $f \circ \gamma = \gamma \circ f$. Then $f_2(z_1, 0, \ldots, 0) = \cdots = f_n(z_1, 0, \ldots, 0) = 0$ for all $z_1 \in \Delta$.

Now, if $f \in \text{Hol}(\Delta_n, \Delta_n)$ is a map which commutes with the holomorphic automorphism $\gamma$ defined by (1.1), we want to study the behaviour of $f$ outside the disk $\Delta \times \{0\}$. At first we “transfer” the problem to the Siegel upper half-space $H_n = \{w \in \mathbb{C}^n : \text{Im} w_1 > |w_2|^2 + \cdots + |w_n|^2\}$ via the Cayley transform $C$ from $\Delta_n$ to $H_n$ given by

$$C(z) = \left(\frac{1 + z_1}{1 - z_1}, \frac{iz_2}{1 - z_1}, \ldots, \frac{iz_n}{1 - z_1}\right).$$
If $F = C \circ f \circ C^{-1}$ and $\mu = C \circ \gamma \circ C^{-1}$, then $\mu \in AutH_n$ and the fact that $f$ and $\gamma$ commute is equivalent to the fact that $F$ and $\mu$ commute.

An expression for $\mu$ is easily recovered from the form of $\gamma$ (see (1.1)); it turns out that

$$\mu(w) = (\lambda^2 w_1, e^{i\theta_2} \lambda w_2, \ldots, e^{i\theta_n} \lambda w_n),$$

where $\lambda = e^{i\theta}$ (hence, by our assumptions, $\lambda \neq 1$). By Proposition 1.4 and by the definition of the Cayley transform we obtain:

**Corollary 1.5.** Let $\mu$ be a hyperbolic automorphism of $H_n$ given by (1.3), and let $F : H_n \to H_n$ be holomorphic and such that $F \circ \mu = \mu \circ F$. Then there exists $k > 0$ such that

$$F_1(w_1, 0, \ldots, 0) = k^2 w_1$$

and that

$$F_2(w_1, 0, \ldots, 0) = \cdots = F_n(w_1, 0, \ldots, 0) = 0.$$

## 2. Main results

In this section we study the family of all holomorphic self-maps of $\Delta_n$ which commute with a given hyperbolic automorphism of $\Delta_n$ ($n \geq 1$) without any condition on the “regularity” of the self-maps. As we have seen, via the Cayley transform, this is equivalent to studying the family of all holomorphic self-maps $F$ of the Siegel upper half-space $H_n$ which commute with a hyperbolic automorphism $\mu$ of $H_n$ given by (1.3).

We know that there exists $k \in \mathbb{R}^+$ such that $F_1(w_1, 0, \ldots, 0) = k^2 w_1$ and $F_j(w_1, 0, \ldots, 0) = 0$ for all $j \geq 2$ and all $w_1 \in H_1$ (see Corollary 1.5). Now equation (1.5) has very strong consequences on the form of $F_2$ if $n = 2$: since $F_2(w_1, 0) = 0$, for all $w_1 \in H_1$, then we can find a function $h$, holomorphic on $H_2$, such that $F_2(w) = w_2 h(w)$. The fact that $F$ and $\mu$ commute yields that $e^{i\theta_2} w_2 h(\mu(w)) = e^{i\theta_2} w_2 h(w)$ for all $w \in H_2$. Therefore

$$h(\mu(w)) = h(w) \quad \forall w \in H_2.$$

In fact, if $w_2 \neq 0$, equation (2.1) is obviously satisfied and, by continuity, it holds for all $w \in H_2$.

Equation (2.1) suggests the investigation of the action of the subgroup generated by $\mu$ on $H_2$, or on $H_n$ for $n \geq 1$. Let $\Gamma = \{\mu^n, m \in \mathbb{Z}\}$ be the subgroup of $AutH_n$ generated by $\mu$.

**Proposition 2.1.** $\Gamma$ acts freely and properly discontinuously on $H_n$.

**Proof.** The fact that $\lambda \neq 1$ in (1.3) implies that $\Gamma$ acts freely on $H_n$. To prove that $\Gamma$ acts properly discontinuously, we can consider the case in which $\lambda > 1$ (otherwise we consider $\mu^{-1}$ instead of $\mu$). Let $\tilde{w} \in H_n$ and set

$$U(\tilde{w}) = B(\tilde{w}, \rho) \cap \{w \in \mathbb{C}^n : |\text{Im } \tilde{w}_1 - \text{Im } w_1| \leq \frac{1}{4}(1 - \frac{1}{\lambda^2})|\text{Im } \tilde{w}_1|,$$

where $B(\tilde{w}, \rho)$ is the ball of center $\tilde{w}$ and euclidean radius $\rho$ in $\mathbb{C}^2$. If $\rho \ll 1$, then $U(\tilde{w})$ is a compact neighborhood of $\tilde{w}$ contained in $H_n$. 


Proposition 2.2. Let $H(F)$ be such that $\mu^s(U(\tilde{w})) \cap U(\tilde{w}) \neq \emptyset$, then $s = 0$. Let $w \in U(\tilde{w})$ be such that $\mu^s(w) \in U(\tilde{w})$.

Since $w \in U(\tilde{w})$, we have

\[
\text{Im } w_1 \geq (1 - \frac{1}{4}(1 - \frac{1}{\lambda^2})) \text{Im } \tilde{w}_1 \geq \frac{3}{4} \text{Im } \tilde{w}_1.
\]

The fact that $\mu^s(w)$ belongs to $U(\tilde{w})$ implies now that

\[
|\text{Im } \tilde{w}_1 - \lambda^{2s} \text{Im } w_1| \leq \frac{1}{4}(1 - \frac{1}{\lambda^2}) \text{Im } \tilde{w}_1
\]

and hence

\[
|\lambda^{2s} - 1| \text{Im } w_1 = |\text{Im } w_1 - \lambda^{2s} \text{Im } w_1|
\]

\[
\leq |\text{Im } w_1 - \text{Im } \tilde{w}_1| + |\lambda^{2s} \text{Im } w_1 - \text{Im } \tilde{w}_1| \leq \frac{1}{2}(1 - \frac{1}{\lambda^2}) \text{Im } \tilde{w}_1.
\]

This in turn implies that

\[
\text{Im } w_1 \leq \frac{1}{2}(1 - \frac{1}{\lambda^2}) \text{Im } \tilde{w}_1 |\lambda^{2s} - 1|^{-1}.
\]

If $s > 0$, since $\lambda > 1$, then $\lambda^{2s} - 1 \geq \lambda^2 - 1$, and therefore we obtain that

\[
\text{Im } w_1 \leq \frac{1}{2}(1 - \frac{1}{\lambda^2}) \text{Im } \tilde{w}_1 (\lambda^2 - 1)^{-1} = \frac{1}{2\lambda^2} \text{Im } \tilde{w}_1 < \frac{3}{4} \text{Im } \tilde{w}_1,
\]

which contradicts (2.2).

If $s < 0$, then $|\lambda^{2s} - 1| = 1 - \lambda^{2s} \geq 1 - \lambda^{-2}$, therefore we have

\[
\text{Im } w_1 \leq \frac{1}{2}(1 - \frac{1}{\lambda^2}) \text{Im } \tilde{w}_1 (1 - \frac{1}{\lambda^2})^{-1} = \frac{1}{2} \text{Im } \tilde{w}_1 < \frac{3}{4} \text{Im } \tilde{w}_1,
\]

which again contradicts (2.2). In conclusion $s = 0$ and therefore $\Gamma$ acts properly discontinuously on $H_n$. 

Let $\mu$ be a hyperbolic automorphism of $H_n$ as in (1.3), and let $F \in \text{Hol}(H_n, H_n)$ be such that $F \circ \mu = \mu \circ F$. We will now prove a result on the structure of the last $n - 1$ components of $F$.

Let $j$ be a natural number such that $1 \leq j \leq n$, and define $\Gamma_j$ to be the subgroup of $\text{Aut}H_j$ generated by the holomorphic automorphism of $H_j$ given by $(w_1, \ldots, w_j) \mapsto (\lambda w_1, \lambda e^{i\theta_2} w_2, \ldots, \lambda e^{i\theta_j} w_j)$, i.e. by the “restriction” of $\mu$ to $H_j$.

Then Proposition 2.1 implies that $\Gamma_j$ acts freely and properly discontinuously on $H_j$ for all $j \leq n$, and therefore we can endow $X_j = H_j/\Gamma_j$ with a complex structure such that the projection $\pi_j$ from $H_j$ to $X_j$ is holomorphic.

We will find a suitable form of the $m$-th component $F_m$ of $F$, for $2 \leq m \leq n$. Let $\ln : H_1 \to \mathbb{C}$ be a branch of the logarithm on the upper half-plane in $\mathbb{C}$ and recall that, by Corollary 1.5, $F_m(w_1, 0, \ldots, 0) = 0$ for all $2 \leq m \leq n$.

Proposition 2.2. Let $F : H_n \to H_n$ be a holomorphic map which commutes with the hyperbolic automorphism $\mu$ given by (1.3). Then there exist $(n-1)^2$ holomorphic functions $\hat{g}_{jm} : X_j \to \mathbb{C}$ such that the $m$-th component $F_m$ of $F$ is given by

\[
F_m(w) = k \sum_{j=2}^n w_j e^{i(\theta_m - \theta_j)} \log w_1/2\log \hat{g}_{jm}(\pi_j(w_1, \ldots, w_j)).
\]

Proof. First of all we prove that $F_m(w) = k \sum_{j=2}^n w_j g_{jm}(w_1, \ldots, w_j)$, where $g_{jm}$ are suitable holomorphic functions on $H_j$. 

Then, for any $2 \leq s$, simplify notation we will write $g$ of the functions $p$ and $w$,
then the difference $g$ becomes in the same way, taking
by a recursive procedure, we obtain the existence of the functions $g_j$ such that
$$F_m(w) = k \sum_{j=2}^n w_j g_j(w_1, \ldots, w_j).$$

Now we prove that such $g_j$’s are unique. In fact, if there are two families of holomorphic functions, say $g_j$ and $p_j$, from $H_j$ to $C$ such that
$$F_m(w) = k \sum_{j=2}^n w_j g_j(w_1, \ldots, w_j) = k \sum_{j=2}^n w_j p_j(w_1, \ldots, w_j),$$
then the difference $g_j - p_j$ satisfies the equation
$$\sum_{j=2}^n w_j (g_j - p_j)(w_1, \ldots, w_j) = 0 \text{ for all } w \in H_n.$$ 
Let $w_1 = \cdots = w_n = 0$. Then $g_2(w_1, w_2) = p_2(w_1, w_2)$ for all $(w_1, w_2) \in H_2$ (if $w_2 \neq 0$ the assertion is obvious, otherwise we use a continuity argument). We proceed in the same way, taking $w_1 = \cdots = w_n = 0$ and we obtain $g_3(w_1, w_2, w_3) = p_3(w_1, w_2, w_3)$ for all $(w_1, w_2, w_3) \in H_3$. In conclusion we obtain the uniqueness of the functions $g_j$ recursively.

Now we will get information about the behaviour of the functions $g_j$. In order to simplify notation we will write $\mu(w_1, \ldots, w_j)$ to denote $(\lambda^2 w_1, \lambda e^{i\theta_2} w_2, \ldots, \lambda e^{i\theta_j} w_j)$. The fact that $F \circ \mu = \mu \circ F$ implies that (for any $2 \leq m \leq n$)
$$\sum_{j=2}^n w_j (e^{i\theta_m} g_j(w_1, \ldots, w_j) - e^{i\theta_j} g_j(\mu(w_1, \ldots, w_j))) = 0.$$
Then, for any $2 \leq j, m \leq n$,
$g_j(w_1, \ldots, w_j) = e^{i(\theta_j - \theta_m)} g_j(\mu(w_1, \ldots, w_j))$
(for all $(w_1, \ldots, w_j) \in H_j$). Having defined
$$\hat{g}_j(w_1, \ldots, w_j) = e^{i(\theta_j - \theta_m)} \log w_1 / 2 \log \lambda g_j(w_1, \ldots, w_j),$$
it is easy to see that $\hat{g}_j$ is automorphic under the action of the “restriction” of $\mu$ to $H_j$. Therefore there exist holomorphic functions $\hat{g}_j : X_j \to C$ such that $\hat{g}_j(w_1, \ldots, w_j) = \hat{g}_j(\pi_1(w_1, \ldots, w_j))$, and this concludes the proof.

We turn our attention now to the investigation of the behaviour of the first component of $F$. We already proved that there exists a positive $k$ such that $F_1(w_1, 0, \ldots, 0) = k^2 w_1$. Then we can write
$$F_1(w) = k^2 (w_1 + \sum_{j=2}^n w_j \alpha_j(w_1) + P(w)), \quad (2.3)$$
where $P(w_1,0,\ldots,0) = 0$ and $\frac{\partial P}{\partial w_j}(w_1,0,\ldots,0) = 0$ for all $w_1 \in H_1$ and for $j = 2,\ldots,n$. First of all we want to prove that the $\alpha_j$’s vanish identically.

**Proposition 2.3.** Let $F$ be a holomorphic map from $H_n$ into itself which commutes with $\mu$ given by (1.3). Write $F_1$ as in (2.3). Then $\alpha_j \equiv 0$ for $j = 2,\ldots,n$.

**Proof.** Since $\text{Im} F_1(w) > |F_2(w)|^2 + \cdots + |F_n(w)|^2$ for all $w \in H_n$, it follows that $\text{Im} F_1(w) > 0$ for all $w \in H_n$. Take $j \in \{2,\ldots,n\}$ and consider $F_1$ on $(w_1,0,\ldots,0,w_j,0,\ldots,0)$. Writing $w_2$ instead of $w_j$, it is enough to prove that $\alpha_j = 0$ when $j = 2$. Therefore it is enough to prove the statement when $n = 2$. From now on we will denote $\alpha_2$ by $\alpha$. The fact that $\text{P}(w_1,0) = 0$ and $\frac{\partial P}{\partial w_2}(w_1,0) = 0$ for all $w_1 \in H_1$ implies that there exists a function $\eta$, holomorphic on $H_2$, such that $P(w_1,w_2) = \alpha^2 \eta(w_1,w_2)$. Since $\text{Im} F_1(w_1,w_2) > 0$ for all $(w_1,w_2) \in H_2$, we have

$$\text{Im} w_1 > -\text{Im}(\alpha(w_1)w_2 + \eta(w)w_2^2)$$

for all $(w_1,w_2) \in H_2$. Take $w_1^0 \in \mathbb{C}$ such that $\text{Im} w_1^0 > 0$. Then, for any $w_2 \in \mathbb{C}$ such that $|w_2|^2 < \text{Im} w_1^0$, the point $(w_1^0,w_2)$ belongs to $H_2$. Let $\varepsilon \in \mathbb{R}$ be such that $0 < \varepsilon < \text{Im} w_1^0$ and set $R = \sqrt{\text{Im} w_1^0 - \varepsilon}, \ r = R/2$. The Borel-Carathéodory theorem and inequality (2.4) imply now that

$$\max_{w_2 \in \Delta_r} |\alpha(w_1^0)w_2 + \alpha^2 \eta(w_1^0,w_2)| \leq \frac{2r}{R - r} \max_{w_2 \in \partial \Delta_r} \text{Im}( - (\alpha(w_1^0)w_2 + \eta(w_1^0,w_2)w_2^2))$$

$$\leq \frac{2r}{R - r} \max_{w_2 \in \partial \Delta_r} \text{Im} w_1^0 = 2 \text{Im} w_1^0.$$

We pass to the evaluation of the maximum modulus of the function $w_2 \mapsto w_2^2 \eta(w_1^0,w_2) + \alpha^2 \eta(w_1^0,w_2)$ on $\Delta_r$, and obtain

$$2 \text{Im} w_1^0 \geq \max_{w_2 \in \Delta_r} |w_2^2 \eta(w_1^0,w_2) + \alpha^2 \eta(w_1^0,w_2)| = \max_{w_2 \in \partial \Delta_r} |w_2^2 \eta(w_1^0,w_2) + \alpha^2 \eta(w_1^0,w_2)|$$

$$= \frac{1}{2} \sqrt{\text{Im} w_1^0 - \varepsilon} \max_{w_2 \in \partial \Delta_r} |w_2 \eta(w_1^0,w_2) + \alpha(w_1^0)|$$

$$= \frac{1}{2} \sqrt{\text{Im} w_1^0 - \varepsilon} \max_{w_2 \in \Delta_r} |w_2 \eta(w_1^0,w_2) + \alpha(w_1^0)|.$$

Taking the limit for $\varepsilon \to 0^+$ we get

$$|w_2 \eta(w_1^0,w_2) + \alpha(w_1^0)| \leq 4(\text{Im} w_1^0)^{1/2}$$

for all $w_2 \in \mathbb{C}$ such that $|w_2|^2 < \text{Im} w_1^0$. The number $\alpha(w_1^0)$ is the value at 0 of the holomorphic function $w_2 \mapsto w_2\eta(w_1^0,w_2) + \alpha(w_1^0)$ and therefore inequality (2.6) implies that

$$|\alpha(w_1^0)| \leq 4(\text{Im} w_1^0)^{1/2}$$

for all $w_1^0 \in \mathbb{C}$ such that $\text{Im} w_1^0 > 0$.

Inequality (2.7) implies now that, for any $\tau \in \mathbb{R}$,

$$\limsup_{w_1 \to \tau} \lim_{\text{Im} w_1 > 0} |\alpha(w_1)| = 0,$$

and the reflection principle yields that there exists an entire function $\hat{\alpha}$ which extends $\alpha$. Inequality (2.7) entails that $\hat{\alpha}(w) = 0$ for all $w \in \mathbb{R}$. Therefore $\hat{\alpha} \equiv 0$ and hence $\alpha \equiv 0$, which proves the proposition. \qed
Thus far we have proved that the function \( F_1 \) has the form
\[
F_1(w) = k^2(w_1 + P(w)),
\]
where \( P(w_1,0,\ldots,0) = 0 \) and \( \frac{\partial P}{\partial w_j}(w_1,0,\ldots,0) = 0 \) for all \( w_1 \in H_1 \) and for \( j = 2,\ldots,n \). Now we want to give a “standard” form to the function \( P \) and use this form to give a complete classification of the holomorphic self-maps of \( H_n \), which commute with \( \mu \) given by (1.3).

**Proposition 2.4.** Let \( P \) be a holomorphic function on \( H_n \) such that \( P(w_1,0,\ldots,0) = 0 \) and \( \frac{\partial P}{\partial w_j}(w_1,0,\ldots,0) = 0 \) for all \( w_1 \in H_1 \) and for \( j = 2,\ldots,n \). Then there exist \( n(n - 1)/2 \) holomorphic functions \( \beta_{jl} : H_j \to \mathbb{C} \) for \( 2 \leq j \leq l \leq n \) such that \( P(w) = \sum_{2 \leq j \leq l \leq n} w_j w_l \beta_{jl}(w_1,\ldots,w_j) \). Moreover the family \( \{\beta_{jl}\} \) \((2 \leq j \leq l \leq n)\) is unique.

**Proof.** First of all we prove the assertion on uniqueness: if there are two families of functions with the required properties, then there exists a family \( \{d_{jl}\} \) such that
\[
\sum_{2 \leq j \leq l \leq n} w_j w_l d_{jl}(w_1,\ldots,w_j) = 0
\]
on \( H_n \). Let \( w_3 = \cdots = w_n = 0 \). Then \( w_2^2 d_{22}(w_1,w_2) = 0 \) for all \( (w_1,w_2) \in H_2 \) and therefore \( d_{22} \equiv 0 \). Let \( w_3 = \cdots = w_n = 0 \); then \( w_2 w_j d_{2j}(w_1,w_2) + w_2^2 d_{jj}(w_1,w_2,0,w_j,0,0) = 0 \) and therefore \( d_{2j} \equiv 0 \) on \( H_2 \). By iterating the above procedure, we obtain the uniqueness of the family \( \{\beta_{jl}\} \).

We pass now to the proof of the existence of such a family of functions \( \{\beta_{jl}\} \) as in the statement.

Since \( P(w_1,0,\ldots,0) = 0 \) and \( \frac{\partial P}{\partial w_j}(w_1,0,\ldots,0) = 0 \) for \( j = 2,\ldots,n \) and for all \( w_1 \in H_1 \), then there exist \( \beta_{2j} : H_2 \to \mathbb{C} \), holomorphic, such that \( w_2^2 \beta_{22}(w_1,w_2) = P(w_1,w_2,0,\ldots,0) \) and that \( w_2 \beta_{2j}(w_1,w_2) = \frac{\partial P}{\partial w_j}(w_1,w_2,0,\ldots,0) \) for \( j = 3,\ldots,n \) and for all \( (w_1,w_2) \in H_2 \).

Set \( P_i(w) = P(w) - w_2^2 \beta_{22}(w_1,w_2) - w_2 w_3 \beta_{23}(w_1,w_2) - \cdots - w_2 w_n \beta_{2n}(w_1,w_2) \).

It is easy to see that \( P_i(w_1,w_2,0,\ldots,0) = 0 \) and \( \frac{\partial P_i}{\partial w_j}(w_1,w_2,0,\ldots,0) = 0 \) for \( j = 3,\ldots,n \) and for all \( (w_1,w_2) \in H_2 \). Then there exist \( \beta_{3j} : H_3 \to \mathbb{C} \), holomorphic, such that \( w_3^2 \beta_{33}(w_1,w_2,w_3) = P_i(w_1,w_2,w_3,0,\ldots,0) \) and \( w_3 \beta_{3j}(w_1,w_2,w_3) = \frac{\partial P_i}{\partial w_j}(w_1,w_2,w_3,0,\ldots,0) \) for \( j = 4,\ldots,n \) and for all \( (w_1,w_2,w_3) \in H_3 \). By iterating this procedure, we end with a holomorphic function \( \beta_{nn} : H_n \to \mathbb{C} \) such that \( P_{n-2}(w) = w_n^2 \beta_{nn}(w) \). This proves the existence of the required family. \( \square \)

By gathering together all the results, we have proved that, if \( F : H_n \to H_n \) is a holomorphic map which commutes with \( \mu \) given by (1.3), then there exist: \( k > 0 \), \( \beta_{jl} \in Hol(H_j,\mathbb{C}) \) \((2 \leq j \leq l \leq n)\) and \( \beta_{jm} \in Hol(X_j,\mathbb{C}) \) \((j,m = 2,\ldots,n)\) such that
\[
F_1(w) = k^2(w_1 + \sum_{2 \leq j \leq l \leq n} w_j w_l \beta_{jl}(w_1,\ldots,w_j))
\]
and

\[ F_m(w) = k \sum_{j=2}^{n} w_j e^{i(\theta_j - \theta_m) \log w_1 / 2 \log \lambda} \tilde{g}_{jm}(\pi_j(w_1, \ldots, w_j)) \]

The fact that \( F \) commutes with \( \mu \) and the uniqueness of the family \( \beta_{jm} \) imply that

\[ e^{i(\theta_j + \theta_l)} \beta_{jl}(\mu(w)) = \beta_{jl}(w) \quad \forall w \in H_j. \]

Let us define

\[ \tilde{\beta}_{jl}(w) = e^{i(\theta_j + \theta_l) \log w_1 / 2 \log \lambda} \beta_{jl}(w); \]

then equation (2.8) entails that \( \tilde{\beta}_{jl} \) is automorphic under the action of \( \Gamma_j \). Therefore, if \( F : H_n \to H_n \) is a holomorphic map which commutes with \( \mu \) given by (1.3), we can find \( k > 0 \), \( \tilde{g}_{jl}, \tilde{\beta}_{jl} \in \text{Hol}(X_j, \mathbb{C}) \) such that

\[ F_1(w) = k^2(w_1 + \sum_{2 \leq j \leq l \leq n} w_j w_l e^{i(\theta_j + \theta_l) \log w_1 / 2 \log \lambda} \tilde{\beta}_{jl}(\pi_j(w_1, \ldots, w_j))) \quad \text{and} \]

\[ F_m(w) = k \sum_{j=2}^{n} w_j e^{i(\theta_m - \theta_j) \log w_1 / 2 \log \lambda} \tilde{g}_{jm}(\pi_j(w_1, \ldots, w_j)), \]

for \( m = 2, \ldots, n \).

Moreover it is easy to see that, if \( F \) has the above form, then \( F : H_n \to \mathbb{C}^n \) commutes with \( \mu \) given by (1.3) (to be more precise, with the holomorphic extension of \( \mu \) to \( \mathbb{C}^n \)).

Then we need only a “restriction of the image” to obtain a complete classification of the holomorphic maps from \( H_n \) into itself which commute with \( \mu \). Now we can state:

**Theorem 2.5.** Let \( F : H_n \to H_n \) be a holomorphic map which commutes with \( \mu \), the hyperbolic automorphism of \( H_n \) given by (1.3). Then there exist \( k > 0 \), \( \tilde{g}_{jl} \in \text{Hol}(X_j, \mathbb{C}) \) (for \( j,l = 2, \ldots, n \)) and \( \tilde{\beta}_{jl} \in \text{Hol}(X_j, \mathbb{C}) \) (for \( 2 \leq j \leq l \leq n \)) such that

\[ F_1(w) = k^2(w_1 + \sum_{2 \leq j \leq l \leq n} w_j w_l e^{i(\theta_j + \theta_l) \log w_1 / 2 \log \lambda} \tilde{\beta}_{jl}(\pi_j(w_1, \ldots, w_j))), \]

(2.9)

\[ F_m(w) = k \sum_{j=2}^{n} w_j e^{i(\theta_m - \theta_j) \log w_1 / 2 \log \lambda} \tilde{g}_{jm}(\pi_j(w_1, \ldots, w_j)), \]

(2.10)

for \( m = 2, \ldots, n \) and that

\[ \text{Im} \left( w_1 + \sum_{2 \leq j \leq l \leq n} w_j w_l e^{i(\theta_j + \theta_l) \log w_1 / 2 \log \lambda} \tilde{\beta}_{jl}(\pi_j(w_1, \ldots, w_j)) \right) \]

(2.11)

\[ > \sum_{m=2}^{n} \left| \sum_{j=2}^{n} w_j e^{i(\theta_m - \theta_j) \log w_1 / 2 \log \lambda} \tilde{g}_{jm}(\pi_j(w_1, \ldots, w_j)) \right|^2 \quad \forall w \in H_n. \]

Vice versa, let \( F \) be as in (2.9) and (2.10), where \( \tilde{g}_{jm}, \tilde{\beta}_{jm} \in \text{Hol}(X_j, \mathbb{C}) \) satisfy (2.11). Then \( F : H_n \to H_n \) commutes with \( \mu \). Moreover, the map which associates \( F \) to \( (k, \tilde{g}_{jm}, \tilde{\beta}_{jm}) \) is one-to-one.
The above theorem gives a complete answer to the problem of finding all holomorphic self-maps of $H_n$ which commute with the hyperbolic automorphism $\mu$ given by (1.3). By conjugation—as remarked at the beginning of this paper—it also gives an answer to the problem of finding all holomorphic self-maps of $H_n$ which commute with a given hyperbolic automorphism of $H_n$. Notice that condition (2.11) is fulfilled if the modulus of the $\hat{g}_{jm}$ and $\hat{\beta}_{jm}$ is very small. Therefore an open neighborhood of 0 in $\text{Hol}(X_2, \mathbb{C}) \times \cdots \times \text{Hol}(X_n, \mathbb{C})$ satisfies condition (2.11). This shows how deep the difference between the one-dimensional case and the multidimensional case is (for the one-dimensional case see, e.g., [1] or [6]).

Now we will consider, with particular attention, the case $n = 2$, in which some other results can be given. First of all we restate Theorem 2.5 for $n = 2$.

**Theorem 2.6.** Let $F : H_2 \rightarrow H_2$ be a holomorphic map which commutes with the hyperbolic automorphism $\mu$ given by (1.3). Then there exist $k > 0$ and $\hat{g}, \hat{\beta} \in \text{Hol}(X_2, \mathbb{C})$ such that
\[ F(w) = \left( k^2 w_1 + w_2^2 e^{-i\theta_1 \log w_1 / \log \lambda} \hat{\beta}(\pi(w)), k w_2 \hat{g}(\pi(w)) \right) \quad \text{and} \]
\[ \text{Im} \left( w_1 + w_2^2 e^{-i\theta_1 \log w_1 / \log \lambda} \hat{\beta}(\pi(w)) \right) > |w_2 \hat{g}(\pi(w))|^2 \quad \forall w \in H_2. \]
Vice versa, let $F$ be as in a), where $\hat{g}, \hat{\beta} \in \text{Hol}(X_2, \mathbb{C})$ satisfy b). Then $F : H_2 \rightarrow H_2$ commutes with $\mu$. Moreover, the map which associates $F$ to $(k, \hat{g}, \hat{\beta})$ is one-to-one.

Notice that condition b) is satisfied if $e^{\theta_2 \pi / |\log \lambda|} |\hat{\beta}(\pi(w))| + |\hat{g}(\pi(w))|^2 \leq 1$. Therefore an open neighborhood of 0 in $\text{Hol}(X_2, \mathbb{C}) \times \text{Hol}(X_2, \mathbb{C})$ consists of maps satisfying condition b).

There are “many” functions $f$ which commute with $\gamma$: we will study in detail the “richness” of the class of functions given in Theorem 2.6, when $\beta = 0$.

**Corollary 2.7.** Let $F : H_2 \rightarrow H_2$ be a holomorphic map which commutes with the hyperbolic automorphism $\mu$ given by (1.3). If $F_1$ does not depend on $w_2$, then there exist $k > 0$ and $\hat{g} \in \text{Hol}(X_2, \mathbb{C})$ such that
\[ F(w) = \left( k^2 w_1, k w_2 \hat{g}(\pi(w)) \right) \quad \text{and} \]
\[ |\hat{g}(\pi(w))| \leq 1 \quad \forall w \in H_2. \]
Vice versa, let $F$ be as in a), where $\hat{g} \in \text{Hol}(X_2, \mathbb{C})$ satisfies b). Then $F : H_2 \rightarrow H_2$ commutes with $\mu$.

**Proof.** By Theorem 2.6, if $F$ is a holomorphic self-map of $H_2$ which commutes with $\mu$, then $F_1(w_1, 0) = k^2 w_1$ for a suitable $k > 0$.

The fact that $F_1$ does not depend on $w_2$ implies that
\[ F(w) = \left( k^2 w_1, k w_2 \hat{g}(\pi(w)) \right). \]

Moreover, that fact that $F$ maps $H_2$ into itself entails that
\[ \text{Im} k^2 w_1 > k^2 |w_2|^2 |\hat{g}(\pi(w))|^2 \quad \forall w \in H_2. \]
Consider \( w_1^0 \in \mathbb{C} \) such that \( \text{Im} w_1^0 > 0 \), and let \( \varepsilon \in \mathbb{R} \) be such that \( 0 < \varepsilon < \text{Im} w_1^0 \).

Define \( r(\varepsilon) = \sqrt{\text{Im} w_1^0 - \varepsilon} \). Inequality (2.12) now implies that

\[
\text{Im} w_1^0 > \max_{w_2 \in \Delta(r(\varepsilon))} |w_2|^2 \left| \hat{g}(\pi(w_1^0, w_2)) \right|^2 = \max_{w_2 \in \partial \Delta(r(\varepsilon))} |w_2|^2 \left| \hat{g}(\pi(w_1^0, w_2)) \right|^2 = \left( \text{Im} w_1^0 - \varepsilon \right) \max_{w_2 \in \partial \Delta(r(\varepsilon))} \left| \hat{g}(\pi(w_1^0, w_2)) \right|^2.
\]

Taking the limit for \( \varepsilon \to 0^+ \) we obtain that \( \left| \hat{g}(\pi(w_1^0, w_2)) \right|^2 \leq 1 \) for all \( w_2 \in \mathbb{C} \) such that \( |w_2|^2 < \text{Im} w_1^0 \). This in turn implies that, for any \( w \in H_2 \), \( |\hat{g}(\pi(w))| \leq 1 \).

The sufficiency of \( a \) and \( b \) is obvious.

The above Corollary shows how large the family of all holomorphic self-maps of \( H_2 \) which commute with a given hyperbolic automorphism of \( H_2 \) is. In fact, if \( \hat{g} \circ \pi \) does not depend on \( w_2 \), then \( \hat{g} \circ \pi \) is a holomorphic map from the annulus \( A(\rho, 1) = \{ z \in \mathbb{C} : \rho < |z| < 1 \} \) (where \( \rho = \exp(-\pi^2/\log \lambda) \)) to the closed unit disk \( \Delta \). It is well known that this space of functions is quite large.

Theorem 2.6 also makes it possible to study the fixed points set of a holomorphic self-map of \( \Delta_2 \) which commutes with a given hyperbolic automorphism \( \gamma \). As we are interested in the structure of \( \text{fixf} = \{ z \in \Delta_2 : f(z) = z \} \), up to the action of \( \text{Aut} \Delta_2 \), we can suppose that \( \gamma \) is given by (1.1). First of all we restate Theorem 2.6 on \( \Delta_2 \) by means of the Cayley transform \( \mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2) : \Delta_2 \to H_2 \).

**Corollary 2.8.** Let \( f : \Delta_2 \to \Delta_2 \) be a holomorphic map which commutes with \( \gamma \) given by (1.1). Then there exist \( t_1 \in \mathbb{R} \) and \( \hat{g}, \hat{\beta} \in \text{Hol}(X, \mathbb{C}) \) such that

\[ f(z) = \frac{\cosh t_1 z_1 + \sinh t_1 + i e^{t_1} z_2 e^{-i \theta_2 \log \mathcal{C}_1(z)/\log \lambda} \hat{\beta}(\pi(\mathcal{C}(z))) / 2(1 - z_1)}{\cosh t_1 + z_1 \sinh t_1 + i e^{t_1} z_2 e^{-i \theta_2 \log \mathcal{C}_1(z)/\log \lambda} \hat{\beta}(\pi(\mathcal{C}(z))) / 2(1 - z_1)} \]

and

\[ \text{Im} \left( \mathcal{C}_1(z) + \mathcal{C}_2(z) e^{-i \theta_2 \log \mathcal{C}_1(z)/\log \lambda} \hat{\beta}(\pi(\mathcal{C}(z))) \right) > |\mathcal{C}_2(z) \hat{\beta}(\pi(\mathcal{C}(z)))|^2 \quad \forall z \in \Delta_2. \]

**Vice versa, let \( f \) be as in \( a \), where \( \hat{g}, \hat{\beta} \in \text{Hol}(X, \mathbb{C}) \) satisfy \( b \). Then \( f : \Delta_2 \to \Delta_2 \) commutes with \( \gamma \).**

Now we study the possible fixed points sets of \( f \): in sharp contrast with the one-dimensional case we prove that a map which commutes with a hyperbolic automorphism of \( \Delta_2 \) can have fixed points in \( \Delta_2 \) and we find an explicit form for the possible fixed points sets of \( f \), giving also a necessary and sufficient condition for \( f \) to have fixed points.

**Proposition 2.9.** Let \( f : \Delta_2 \to \Delta_2 \) be a holomorphic map which commutes with the hyperbolic automorphism \( \gamma \) given by (1.1). If \( f \) has fixed points in \( \Delta_2 \), then either \( f = \text{id}_{\Delta_2} \) or \( \text{fixf} = \Delta \times \{0\} \). Moreover, \( f \) has fixed points in \( \Delta_2 \) iff \( f_1(z_1, 0) = z_1 \) for all \( z_1 \in \Delta \).

**Proof.** We recall that the fixed points set of a holomorphic self-map of \( \Delta_n \) is given by the intersection of a complex affine space with \( \Delta_n \); therefore, if \( n = 2 \), \( \text{fixf} \) is always contained in a complex affine line, unless \( f = \text{id}_{\Delta_2} \).
Let $a, b, c \in \mathbb{C}$ be such that the affine line defined by $az_1 + bz_2 + c = 0$ contains the fixed points set of $f$, and let $z^0 = (z_1^0, z_2^0)$ be a fixed point of $f$. Since $\gamma$ and $f$ commute, it is easy to see that $\gamma^m(z^0)$ belongs to $fix f$ for all $m \in \mathbb{Z}$. Calling in the form of $\gamma$ given by (1.1) we obtain that
\[
\frac{\cosh mt_0 z_1^0 + \sinh mt_0}{\sinh mt_0 z_1^0 + \cosh mt_0} + \frac{e^{im\theta} z_2^0}{\sinh mt_0 z_1^0 + \cosh mt_0} + c = 0 \quad \forall m \in \mathbb{Z}.
\]
This is equivalent to
\[
(a z_1^0 + c) \cosh mt_0 + (c z_1^0 + a) \sinh mt_0 + be^{im\theta} z_2^0 = 0 \quad \forall m \in \mathbb{Z}.
\]
Divide both members of the last equation by $\cosh mt_0$ and take the limit both for $m \to +\infty$ and for $m \to -\infty$. Since $\lim_{m \to \pm \infty} \cosh mt_0 = +\infty$, $\lim_{m \to \pm \infty} \tanh mt_0 = \pm 1$ and since the modulus of $be^{im\theta} z_2^0$ is bounded when $m$ diverges, we obtain
\[
(a z_1^0 + c) + (c z_1^0 + a) = 0 \quad \text{and} \quad (a z_1^0 + c) - (c z_1^0 + a) = 0.
\]
Since $|z_2^0| < 1$, the last equations imply $a = c = 0$. Then the complex affine line containing $fix f$ is the complex line $z_2 = 0$, and hence $z_2^0 = 0$. Propositions 1.3 and 1.4 show that, on the complex line $z_2 = 0$, the function $f$ is given by
\[
f(z_1, 0) = \left( \frac{\cosh t_1 z_1 + \sinh t_1}{\sinh t_1 z_1 + \cosh t_1}, 0 \right).
\]
Therefore, if $(z_1^0, 0)$ is a fixed point for $f$, we obtain that $t_1 = 0$ and hence the set $\Delta \times \{0\}$ is contained in $fix f$. The assertion follows.

\[\square\]

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