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ON THE DIMENSION OF HILBERT SPACE REMAINDERS

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Every space is assumed to be separable and metric. A space is called (strongly) countably dimensional if it can be written as a countable union of (closed) finite dimensional subspaces. A space X is called strongly infinite dimensional if the space admits an essential system $(F_n, G_n)_{n=1}^{\infty}$, i.e. F_n and G_n are disjoint closed subsets of X such that if S_n is a closed separator of F_n and G_n for each n, then $\bigcap_{n=1}^{\infty} S_n$ is nonempty. The sequence of left and right endfaces of the Hilbert cube is the standard example of an essential system.

A well-known theorem of Engelking [E] states that every autohomeomorphism h of an n-dimensional space X can be extended to a homeomorphism $\tilde{h} : C \to C$, where C is an n-dimensional compactification of X (and hence we have a $\leq n$-dimensional remainder). We consider the question of whether similar results can be obtained for infinite dimensional spaces, i.e. is it possible to put a bound on the dimension of the remainder? The following example shows that the answer is no if we allow incomplete spaces. Consider the Hilbert cube $Q = [0,1]^{\mathbf{N}}$ and the strongly countably dimensional pseudoboundary $\sigma = \{x \in Q : x_i = 0 \text{ from some index on}\}$. It was shown by R. D. Anderson that $Q \setminus \sigma$ is homeomorphic to Hilbert space (see [BP, Theorem V.5.1]). The following proposition is a slight improvement of the known result that the remainder of every compactification of σ contains a copy of Q.

Proposition 1. *The remainder of every completion of σ contains a dense copy of Hilbert space.*

Proof. Let C be a completion of σ. According to [La] there exist a G_δ-set A in C, a G_δ-set B in Q, and a homeomorphism $h : A \to B$ such that $\sigma \subset A$, $\sigma \subset B$, and $h|\sigma$ is the identity. Since $Q \setminus B$ is σ-compact, it is negligible in the Hilbert space $Q \setminus \sigma$ (see [A]). So $B \setminus \sigma$ and $A \setminus \sigma$ are Hilbert spaces. □

We turn to complete spaces. According to [Le] every complete space can be compactified by adding a strongly countably dimensional remainder. This fact also follows from the aforementioned result that Hilbert space can be compactified to a Hilbert cube by using σ as remainder. So the question naturally arises of whether every autohomeomorphism of a complete space can be "compactified" by adding a strongly countably dimensional remainder. Let us have a closer look at Hilbert

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space which we now represent by $s = \mathbf{R}^{\mathbf{Z}} = \prod_{i=-\infty}^{\infty} \mathbf{R}$. Let α stand for the "left shift" on s, i.e. $\alpha(x)_i = x_{i+1}$ for $i \in \mathbf{Z}$.

Proposition 2. *If α extends over a compactification to a continuous $\tilde{\alpha} : C \to C$, then $C \setminus s$ contains strongly infinite dimensional continua.*

Proof. Let $\{A_1, A_2, \dots\}$ be a partition of $\mathbf{N}$ into infinitely many infinite subsets. We define the following sequence of disjoint pairs of closed subsets of s: for $n \in \mathbf{N}$ and $\varepsilon \in \{0, 1\}$,

$$F_n^\varepsilon = \{(x_i) \in s : x_i = \varepsilon \text{ if for some } k \in A_n \text{ we have } k^2 \le i < (k+1)^2\}.$$

Let $\tilde{F}_n^\varepsilon$ be the closure in C of F_n^ε. We first show that $\tilde{F}_n^0$ and $\tilde{F}_n^1$ are disjoint. Let U_0 and U_1 be two disjoint closed neighbourhoods of $(\dots, 0, 0, 0, \dots)$ and $(\dots, 1, 1, 1, \dots)$ in C. Then there is an $N \in \mathbf{N}$ such that $\bigcap_{i=-N}^{N} \pi_i^{-1}(0) \subset U_0$ and $\bigcap_{i=-N}^{N} \pi_i^{-1}(1) \subset U_1$, where $\pi_i : s \to \mathbf{R}$ stands for the projection on the ith coordinate. Select a $k \in A_n$ such that $k \ge N$. Put $m = k^2 + k$. If $x \in F_n^\varepsilon$, then $x_i = \varepsilon$ for $k^2 \le i \le k^2 + 2k$. Since α^m is a shift to the left over $k^2 + k$ positions we have $\alpha^m(x)_i = \varepsilon$ for $-k \le i \le k$. So $\alpha^m(F_n^0) \subset U_0$ and $\alpha^m(F_n^1) \subset U_1$ and since U_0 and U_1 are compact and disjoint we have that $\tilde{\alpha}^n(\tilde{F}_n^0)$ and $\tilde{\alpha}^n(\tilde{F}_n^1)$ are disjoint. Hence $\tilde{F}_n^0$ and $\tilde{F}_n^1$ are disjoint.

We define the imbedding β of the space $X = [0, \infty) \times Q$ into s as follows: for $a \ge 0$, $x = (x_j) \in Q$, and $i \in \mathbf{Z}$,

$$\beta(a, x)_i = \begin{cases} a, & \text{if } i \le 0, \\ x_j, & \text{if } k^2 \le i < (k+1)^2 \text{ for some } k \text{ and } j \text{ with } k \in A_j. \end{cases}$$

Observe that β is a closed imbedding of a locally compact space in s and hence $K = \mathrm{cl}_C(\beta(X)) \setminus \beta(X)$ is a compactum in $C \setminus s$. Since $K = \bigcap_{i=1}^{\infty} \mathrm{cl}_C(\beta([i, \infty) \times Q))$, it is a continuum. Let $\beta_a : Q \to s$ be defined by $\beta_a(x) = \beta(a, x)$ for $(a, x) \in X$.

Now we prove that K is strongly infinite dimensional. Assume that S_n is a closed separator in K of $\tilde{F}_n^0 \cap K$ and $\tilde{F}_n^1 \cap K$. Since K is compact, we can find for each n a closed separator $\tilde{S}_n$ of $\tilde{F}_n^0$ and $\tilde{F}_n^1$ in C such that $\tilde{S}_n \cap K = S_n$. Put $\tilde{S}_\infty = \bigcap_{n=1}^{\infty} \tilde{S}_n$. Observe that for each $a \ge 0$ the sets $\beta_a^{-1}(F_n^0)$ and $\beta_a^{-1}(F_n^1)$ are precisely the n-endfaces of the Hilbert cube Q and hence they form an essential system for $n \in \mathbf{N}$. So we may conclude that $\bigcap_{n=1}^{\infty} \beta_a^{-1}(\tilde{S}_n)$ and hence $\beta_a(Q) \cap \tilde{S}_\infty$ are nonempty. Since $\pi_0(\beta(a, x)) = a$ we have $\pi_0(\beta(X) \cap \tilde{S}_\infty) = [0, \infty)$. So $\beta(X) \cap \tilde{S}_\infty$ is not compact. Since $\mathrm{cl}_C(\beta(X)) \cap \tilde{S}_\infty$ is compact, we may conclude that $\bigcap_{n=1}^{\infty} S_n = K \cap \tilde{S}_\infty$ is nonempty. □

Propositions 1 and 2 suggest the following questions. If α extends over a compactification to a homeomorphism $\tilde{\alpha} : C \to C$, does $C \setminus s$ contain a Hilbert cube? And if h is an autohomeomorphism of a (strongly) countably dimensional complete space X, can h be extended to a homeomorphism $\tilde{h} : C \to C$, where C is a compactification of X with (strongly) countably dimensional remainder?

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