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ON WEAK COMPACTNESS AND COUNTABLE WEAK COMPACTNESS IN FIXED POINT THEORY

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Abstract. We prove that weak compactness and countable weak compactness in metric spaces are not equivalent. However, if the metric space has normal structure, they are equivalent. It follows that some fixed point theorems proved recently are consequences of a classical theorem of Kirk.

Let (X, d) be a metric space. Let $\mathcal{F}$ be the family of subsets of X consisting of X and sets which are complements of closed balls of X. The weak topology (also called ball topology) on X is the topology whose open sets are generated by $\mathcal{F}$, i.e. $\mathcal{F}$ forms a subbase for the open sets of X. If X is a subset of a Banach space, the weak topology defined in this sense is generally weaker than the usual weak topology on X. For details on this topology in a Banach space setting, see [1] and [2].

We say X is weakly compact if it is compact in the weak topology. A subset of X is called a ball-intersection if it is an intersection of closed balls. X is said to have normal structure if every ball-intersection D of X containing more than one point contains a non-diametral point, i.e. a point $r \in D$ such that

$$\sup\{d(r, x) : x \in D\} < \operatorname{diam}(D).$$

It is clear that X is weakly compact if and only if every nonempty family of ball-intersections with the finite intersection property has a nonempty intersection. And by the Alexander subbase theorem, one can replace 'ball-intersections' with 'closed balls'. X is countably weakly compact if every decreasing sequence of nonempty ball-intersections in X has a nonempty intersection. Obviously every weakly compact metric space is countably weakly compact. In this paper we show that the converse is false. However, if the metric space has normal structure, the converse is true. This shows that the assumption of countable weak compactness in some theorems in the literature is not really weaker than that of weak compactness. It also shows that every complete metric space with uniform normal structure is weakly compact, thereby a fixed point theorem of Khamsi [3] actually follows from that of Kirk [4]. Throughout this paper, $B_r(p)$ will denote the closed ball centered at p with radius r.

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Example. This is an example of a countably weakly compact metric space that is not weakly compact. Let Ω be the first uncountable ordinal. Let P be the set of ordinals $< \Omega$ and Q be the set of countable subsets of P. Let $X = P \cup Q$ with the metric d defined as follows:

$$d(p_1, p_2) = 1.5 \quad \text{if } p_1, p_2 \in P \text{ and } p_1 \neq p_2,$$
$$d(q_1, q_2) = 1 \quad \text{if } q_1, q_2 \in Q \text{ and } q_1 \neq q_2,$$

$$d(p, q) = d(q, p) = \begin{cases} 1 & \text{if } p \in q \\ 1.5 & \text{if } p \notin q \end{cases} \quad \text{for } p \in P \text{ and } q \in Q.$$

(We define without saying that $d(x, y) = 0$ if $x = y$.) One checks that d is indeed a metric.

We have the following six types of balls for $p \in P$ and $q \in Q$:

$$B_r(p) = X \quad \text{if } r \geq 1.5, \tag{1a}$$

$$B_r(p) = \{p\} \cup \{q : p \in q \in Q\} \quad \text{if } 1 \leq r < 1.5, \tag{1b}$$

$$B_r(p) = \{p\} \quad \text{if } 0 \leq r < 1, \tag{1c}$$

$$B_r(q) = X \quad \text{if } r \geq 1.5, \tag{2a}$$

$$B_r(q) = \{p : p \in q, p \in P\} \cup Q \quad \text{if } 1 \leq r < 1.5, \tag{2b}$$

$$B_r(q) = \{q\} \quad \text{if } 0 \leq r < 1. \tag{2c}$$

To show that X is not weakly compact, consider

$$\mathcal{F} = \{B_1(p) : p \in P\}.$$

If $\mathcal{F}_1 = \{B_1(p_n) \colon n = 1, 2, \dots\} \subseteq \mathcal{F}$, then $q = \{p_1, p_2, \dots\} \in \bigcap \mathcal{F}_1$ by (1b) above. So $\mathcal{F}$ has countable (and hence finite) intersection property. Let

$$\mathcal{G} = \{B_1(p_\alpha) \colon \alpha \in D\}, \qquad D \text{ uncountable},$$

be an uncountable subfamily of $\mathcal{F}$. Since every $q \in Q$ is countable, there exists $\alpha \in D$ such that $p_\alpha \notin q$ so $q \notin B_1(p_\alpha)$. Also, since $|D| > 1$, for every $p \in P$ there exists $p_\alpha \neq p$ so $p \notin B_1(p_\alpha)$. This shows $\bigcap \mathcal{G} = \varnothing$ for every uncountable subfamily of $\mathcal{F}$. Thus, in particular, $\bigcap \mathcal{F} = \varnothing$. This proves that X is not weakly compact.

To show that X is countably weakly compact, first note from the six types of balls above that every ball-intersection containing more than one point must be either the whole X or an intersection of countably many (otherwise it would be $\varnothing$ by the paragraph above) balls of type (1b) and some balls of type (2b). Thus such a ball-intersection I is associated with a countable subset C of P ($C = \varnothing$ if $I = X$) such that $q \in I$ whenever $q \supseteq C$ and $q \in Q$. Now suppose $\mathcal{I} = \{I_i \colon i = 1, 2, \dots\}$ is a decreasing sequence of nonempty ball-intersections in X. If some of I_i is a singleton, then obviously the intersection will be the singleton. Otherwise, let C_i be the countable set associated with I_i and let $C = \bigcup C_i$. Then C is countable and belongs to the intersection $\bigcap I_i$. This shows that X is countably weakly compact.

We now proceed to prove the following result.

Theorem 1. *Every countably weakly compact metric space with normal structure is weakly compact.*

Lemma 1. *If X is countably weakly compact but not weakly compact, then there exists an uncountable cardinal K and a decreasing transfinite sequence $\{I_\alpha : \alpha < K\}$ of nonempty ball-intersections such that $\bigcap\{I_\alpha : \alpha < K\} = \varnothing$.*

Proof. Since X is not weakly compact, there exists a family $\mathcal{F}$ of nonempty ball-intersections of X with the finite intersection property and $\bigcap \mathcal{F} = \varnothing$. Let K be the smallest cardinal such that there exists a subfamily $\mathcal{G}$ of $\mathcal{F}$ with cardinality K and $\bigcap \mathcal{G} = \varnothing$. Since X is countably weakly compact, K must be uncountable. Well order $\mathcal{G}$ so $\mathcal{G} = \{B_\gamma : \gamma < K\}$. Now for every $\alpha < K$ define $I_\alpha = \bigcap\{B_\gamma : \gamma \leq \alpha\}$. By the minimality of K, each $I_\alpha \neq \varnothing$. Also $\bigcap\{I_\alpha : \alpha < K\} = \bigcap \mathcal{G} = \varnothing$.

We shall call a decreasing transfinite sequence as in Lemma 1 a $\varnothing$-chain. For each I_α in a $\varnothing$-chain I, let $d_\alpha = \operatorname{diam}(I_\alpha)$. Since K is uncountable and d_α is nonincreasing, there exists $\beta < K$ such that $d_\alpha = d$ for all α with $\beta < \alpha < K$. We have $d > 0$ for otherwise $\bigcap\{I_\alpha : \alpha < K\}$ would be a singleton and hence nonempty—a contradiction. We shall call the number d the diameter of the $\varnothing$-chain and denote it by $\operatorname{diam}(I)$. We shall call a $\varnothing$-chain J a refinement of another $\varnothing$-chain I if $J_\alpha \subseteq I_\alpha$ for every $\alpha < K$.

Lemma 2. *Under the hypothesis of Lemma 1, there exists a $\varnothing$-chain I^* with a minimal diameter, i.e. if J is a refinement of I^*, then* $\operatorname{diam}(J) = \operatorname{diam}(I^*)$.

Proof. Let I be a $\varnothing$-chain as in Lemma 1. Define a sequence of $\varnothing$-chains I_j, $j = 0, 1, 2, \dots$, inductively as follows. Let $I_0 = I$. Suppose I_j has been defined. Let $\hat{d}_j = \inf\{\delta : \text{there exists a refinement of } I_j \text{ with diameter } \delta\}$. Then define I_{j+1} to be a refinement of I_j with $0 \leq \operatorname{diam}(I_{j+1}) - \hat{d}_j < \frac{1}{j+1}$. For each $\alpha < K$ define $I^*_\alpha = \bigcap\{I_{j\alpha} : j = 0, 1, \dots\}$. By countable weak compactness, each I^*_α is nonempty. Obviously I^* is a $\varnothing$-chain. Suppose J is a refinement of I^*. For every j, since both J and I^* are refinements of I_{j+1}, we have $|\operatorname{diam}(J) - \operatorname{diam}(I^*)| \leq \operatorname{diam}(I_{j+1}) - \hat{d}_j < \frac{1}{j+1}$. This proves that $\operatorname{diam}(J) = \operatorname{diam}(I^*)$.

Proof of Theorem 1. Suppose X is countably weakly compact and has normal structure. Suppose on the contrary X is not weakly compact. Then X contains a $\varnothing$-chain $I = \{I_\alpha : \alpha < K\}$ with a minimal diameter as in Lemma 2. Since K is an uncountable cardinal, there exists $\beta < K$ such that $\operatorname{diam}(I_\alpha) = \operatorname{diam}(I) = d$ for all $K > \alpha > \beta$. We have $d > 0$ for otherwise the intersection of I would be a singleton. By normal structure, for each $\beta < \alpha < K$ there exists $0 < r_\alpha < \operatorname{diam}(I_\alpha) = d$ and $p_\alpha \in I_\alpha$ such that $I_\alpha \subseteq B_{r_\alpha}(p_\alpha)$. Let $S_n = [d(1 - \frac{1}{n}), d(1 - \frac{1}{n+1}))$, $n = 1, 2, \dots$. Let $F_n = \{\alpha : \beta < \alpha < K, r_\alpha \in S_n\}$. Since K is an uncountable cardinal, F_m must be cofinal in K for some m. Let $r = d(1 - \frac{1}{m+1})$. Let $J = \bigcap\{B_r(p_\alpha) : \alpha \in F_m\}$. One easily checks that $\{p_\alpha : \alpha \in F_m\} \subseteq J$. Let $M = \bigcap\{B_r(x) : x \in J\}$. Since $\{p_\alpha : \alpha \in F_m\} \subseteq J$, we have $M \subseteq J$. Suppose y, $z \in M$. Then $z \in J$ and $y \in B_r(z)$, so $d(y, z) \leq r$. This implies $\operatorname{diam}(M) \leq r$. Also, $x \in J$ and $\alpha \in F_m \Rightarrow x \in B_r(p_\alpha) \Rightarrow p_\alpha \in B_r(x)$. So $\{p_\alpha : \alpha \in F_m\} \subseteq M$. Now define I^* by setting $I^*_\alpha = I_\alpha$ for $\alpha \leq \beta$ and $I^*_\alpha = I_\alpha \cap M$ for $\beta < \alpha < K$. Note that $\{p_\gamma : \gamma \in F_m, \gamma \geq \alpha\} \subseteq I^*_\alpha$ for $\beta < \alpha < K$ and so $I^*_\alpha \neq \varnothing$. Since $\operatorname{diam}(I^*) \leq \operatorname{diam}(M) < d$, we have a contradiction to the minimality of I. $\square$

The metric version of Kirk's fixed point theorem [4] is the following:

Theorem 2. *Let X be a bounded weakly compact metric space with normal structure. Then every nonexpansive mapping $T : X \to X$ has a fixed point.*

It follows from Theorem 1 that a countable version of a fixed point theorem of Kirk [5], [6] actually follows from Theorem 2 above.

A metric space is said to have uniform normal structure if there exists a number $0 < c < 1$ such that for every ball-intersection B of X with $\operatorname{diam}(B) > 0$, there is a point z in B with

$$\sup\{d(z,x) : x \in B\} \leq c \operatorname{diam}(B).$$

Khamski [3] proved that every complete metric space with uniform normal structure is countably weakly compact. His proof made use of a fixed point theorem which we state as a corollary below. We give here an alternative proof which enables us to conclude the fixed point theorem as a corollary.

Let $\mathcal{A} = \{A_n \colon n = 1, 2, \dots\}$ and $\mathcal{B} = \{B_n \colon n = 1, 2, \dots\}$ be two sequences of sets. We say that $\mathcal{A} \leq \mathcal{B}$ if $A_n \subseteq B_n$ for every n. If $\mathcal{K} = \{K_n \colon n = 1, 2, \dots\}$ is a decreasing sequence of ball-intersections in a metric space and $\mathcal{C} = \{C_n \colon n = 1, 2, \dots\}$ a sequence $\leq \mathcal{K}$, then by taking intersection, one can easily see that there is a smallest (relative to $\leq$) decreasing sequence of ball-intersections $\geq \mathcal{C}$. For a subset B of a metric space, we shall denote by $\operatorname{Ad}(B)$ the smallest ball-intersection containing B.

Theorem 3. *Every complete metric space with uniform normal structure is weakly compact.*

Proof. Suppose X is a complete metric space with uniform normal structure and let c be the constant as in the definition. By Theorem 1, it suffices to prove that X is countably weakly compact. Let $K_n, n = 1, 2, \dots$, be a decreasing sequence of nonempty ball-intersections. Let $d = c \operatorname{diam}(K_1)$. For each n, let $F_n = \{x \in K_n \colon \sup_{y \in K_n} d(x,y) \leq d\}$. Each F_n is a nonempty ball-intersection and has a diameter $\leq d$. Let B_n, $n = 1, 2, \dots$, be the smallest (relative to the order $\leq$) decreasing sequence of ball-intersections containing F_n, $n = 1, 2, \dots$. By the minimality, we have $B_1 = \operatorname{Ad}(B_2 \cup F_1)$, $B_2 = \operatorname{Ad}(B_3 \cup F_2), \dots$. Let $C_n = \{x \in B_n : \sup_{y \in B_n} d(x,y) \leq d\}$. If $x \in C_2$, then $B_d(x) \supseteq B_2$. Since $x \in C_2 \subseteq B_2 \subseteq K_2 \subseteq K_1$, we also have $B_d(x) \supseteq F_1$. It follows that $B_d(x) \supseteq \operatorname{Ad}(B_2 \cup F_1) = B_1$ and hence $x \in C_1$. This shows $C_2 \subseteq C_1$. Similarly, one shows that C_n, $n = 1, 2, \dots$, is decreasing. It is easy to see that $\operatorname{diam}(C_n) \leq d = c \operatorname{diam}(K_1)$. Repeating the process to $C_2, C_3, \dots$, we obtain a decreasing sequence $D_2, D_3, \dots$ such that $D_n \subseteq C_n$ and $\operatorname{diam}(D_n) \leq c^2 \operatorname{diam}(K_1)$. Continuing in this manner, we see that there exists a decreasing sequence $\{L_n \colon n = 1, 2, \dots\}$ of ball-intersections $\leq \mathcal{K}$ with $\operatorname{diam}(L_n) \leq c^n \operatorname{diam}(K_1)$. By the completeness of X, $\bigcap L_n$ is a singleton contained in $\bigcap K_n$ and hence $\bigcap K_n \neq \varnothing$. $\square$

The following corollary follows immediately from Theorems 2 and 3.

Corollary 1 (Khamsi [3])**.** *Every bounded complete metric space with uniform normal structure has the fixed point property for nonexpansive mappings.*

References

1. H. H. Corson and J. Lindenstrauss, *On weakly compact subsets of Banach spaces*, Proc. Amer. Math. Soc. **17** (1966), 407–412. MR **33:**7812
2. G. Godefroy and N. Kalton, *The ball topology and its applications*, Banach Space Theory (B. L. Lin, ed.), Contemp. Math., vol. 85, Amer. Math. Soc., Providence, RI, 1989, pp. 195–237. MR **90c:**46022

3. M. A. Khamsi, *On metric spaces with uniform normal structure*, Proc. Amer. Math. Soc. **106** (1989), 723–726. MR **90c:**54028
4. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004–1006. MR **32:**6436
5. ———, *Nonexpansive mappings in metric and Banach spaces*, Estratto Dai, Rend. Sem. Mat. Fis. Milano, **51** (1981), 133–144. MR **84j:**47091
6. ———, *An abstract fixed point theorem for nonexpansive mappings*, Proc. Amer. Math. Soc. **82** (1981), 640–642. MR **82f:**54075

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