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# ON THE FINITENESS OF BASS NUMBERS OF LOCAL COHOMOLOGY MODULES

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Abstract. In this note we show that, if $I$ is an ideal of dimension 1 of an analytically irreducible local ring, then the Bass numbers of local cohomology modules with support in $V(I)$ are finite.

## 1. Introduction

We assume that all rings are noetherian with identity throughout this note. Our aim in this note is to prove

**Theorem.** *Let $R$ be an analytically irreducible local ring (see [19, chap. VIII]) with maximal ideal $\mathfrak{m}$, $I$ an ideal of dimension 1 and $M$ a finitely generated $R$-module. Then the Bass numbers of local cohomology modules $H^j_I(M)$ are finite for all $j \geq 0$.*

Let $R$ be a ring, $I$ an ideal of $R$, and $N$ an $R$-module. We say $N$ is *$I$-cofinite* if the support of $N$ is contained in $V(I)$ and $\mathrm{Ext}^i_R(R/I, N)$ is finitely generated for all $i \geq 0$ (cf. [5, EXPOSÉ XIII, 1·1] or [6, §2]).

If $M$ is a finitely generated $R$-module, then the local cohomology modules $H^j_{\mathfrak{m}}(M)$ are well known to be Artinian ([13, Proposition 2.1]). The relationship between $\mathfrak{m}$-cofiniteness and Artinian condition is given by the following remark which is a consequence of Matlis duality (cf. [14]) and the fundamental work of A. Grothendieck (cf. [4]):

*Remark* 1 (cf. [6]). Let $R$ be a complete local ring with maximal ideal $\mathfrak{m}$ and residue field $k$ and let $N$ be an $R$-module. Then the following conditions are equivalent:

(i) $N$ satisfies the descending chain condition (dcc );
(ii) $N$ is isomorphic to a submodule of the finite direct sum of copies of the injective hull $E$ of the residue field $k$ of $R$;
(iii) there is a finitely generated $R$-module $M$ such that $\mathrm{Hom}_R(M, E) \simeq N$, and $\mathrm{Hom}_R(N, E) \simeq M$;
(iv) $\mathrm{Supp}_R(N) = V(\mathfrak{m})$, and $\mathrm{Hom}_R(k, N)$ is finitely generated;
(v) $\mathrm{Supp}_R(N) = V(\mathfrak{m})$, and $\mathrm{Ext}^i_R(k, N)$ is finitely generated for all $i \geq 0$;
(vi) $M$ is $\mathfrak{m}$-cofinite.

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There are some questions (cf. [11, Conjecture 4.3, Conjecture 4.4 and Problem 4, etc]). Conjecture 4.4 has been solved affirmatively by C. Huneke and R. Y. Sharp (cf. [10, Theorem 2.1]) for regular local rings of characteristic $p$ and by G. Lyubeznik (cf. [12, Corollary 3.6 (d)]) for those of characteristic zero containing a field. Hartshorne's counter-example [6, §3, An example] says that the above conjecture is not true for an ideal of dimension 2 without the regularity assumption.

## 2. Proof of the lemmas

**Lemma 1.** *Let $I$ be an ideal of $R$, and $p$ a non-negative integer. Then, for any $R$-module $T$ (not necessarily finitely generated), the following conditions are equivalent*:

(i) $\mathrm{Ext}^i_R(R/I,T)$ *is finitely generated for all* $i \le p$;
(ii) *for any* $P \in \mathrm{Min}(R/I)$, $\mathrm{Ext}^i_R(R/P,T)$ *is finitely generated for all* $i \le p$;
(iii) *for any finitely generated $R$-module $N$ with support in $V(I)$*, $\mathrm{Ext}^i_R(N,T)$ *is finitely generated for all* $i \le p$,

*where* $\mathrm{Min}(R/I)$ *is the set of all minimal prime ideals of* $I$.

*Proof.* The equivalence (i) $\Leftrightarrow$ (iii) has been already known (cf. [9, Lemma 4·2] or [3, Lemma 2]).

We show (i) $\Rightarrow$ (ii). Suppose that $\mathrm{Ext}^i_R(R/I,T)$ is finitely generated for all $i \le p$. Let $I'$ be an arbitrary ideal of $R$ containing $I$. Since the support of $R/I'$ is contained in $V(I)$, $\mathrm{Ext}^i_R(R/I',T)$ is finitely generated for all $i \le p$ by the implication (i) $\Rightarrow$ (iii). In particular, for $P \in \mathrm{Min}(R/I)$, $\mathrm{Ext}^i_R(R/P,T)$ is finitely generated for all $i \le p$.

Next we prove (ii) $\Rightarrow$ (i). We may assume that $I$ is a radical ideal by the equivalence (i) $\Leftrightarrow$ (iii). Put $\mathrm{Min}(R/I) = \{P_1, P_2, \cdots, P_n\}$, and $I = P_1 \cap P_2 \cap \cdots \cap P_n$. We prove (ii) $\Rightarrow$ (i) using induction on $n$. Suppose that $\mathrm{Ext}^i_R(R/P_j,T)$ is finitely generated for all $i \le p$ and all $j$ such that $1 \le j \le n$, which is the case $n = 1$. Set $I_j = P_1 \cap P_2 \cap \cdots \cap P_j$ and suppose that $\mathrm{Ext}^i_R(R/I_j,T)$ is finitely generated for all $i \le p$ and all $j$ such that $1 \le j < n$. The exact sequence

$$0 \longrightarrow I_{n-1}/I_{n-1} \cap P_n \longrightarrow R/I_{n-1} \cap P_n \longrightarrow R/I_{n-1} \longrightarrow 0$$

induces a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^i_R(R/I_{n-1},T) \longrightarrow \mathrm{Ext}^i_R(R/I_{n-1} \cap P_n,T) \longrightarrow \mathrm{Ext}^i_R(I_{n-1}/I_{n-1} \cap P_n,T) \longrightarrow \cdots .$$

Since $I_{n-1}/I_{n-1} \cap P_n \simeq (I_{n-1} + P_n)/P_n$, this is a finitely generated $R$-module with support in $V(P_n)$. Hence $\mathrm{Ext}^i_R(R/I_{n-1},T)$ and $\mathrm{Ext}^i_R(I_{n-1}/I_{n-1} \cap P_n,T)$ are finitely generated for all $i \le p$, by inductive hypothesis and the implication (i) $\Rightarrow$ (iii). Consequently $\mathrm{Ext}^i_R(R/I,T)$ is finitely generated for all $i \le p$. □

*Remark* 2. We remark that if $(R,\mathfrak{m})$ is local and $T$ is $I$-cofinite, then $\mathrm{Ext}^i_R(R/\mathfrak{m},T)$ is finitely generated for all $i \ge 0$ by Lemma 1. We use this fact in the sequel.

*Remark* 3. We remark the following facts since we need these to prove Lemma 2:

Let $R \to R'$ be a flat extension of rings, $M$ and $T$ $R$-modules, and $I$ an ideal of $R$. Then

(i) If $M$ is finitely generated, then $\mathrm{Ext}^i_R(M,T) \otimes_R R' \simeq \mathrm{Ext}^i_{R'}(M \otimes_R R', T \otimes_R R')$ (see [15, (3.E)]).

(ii) The local cohomology functor $H^j_I(\,-\,)$ commutes with the tensor functor $(\,-\,)\otimes_R R'$, that is $H^j_I(M)\otimes_R R' \simeq H^j_{IR'}(M\otimes_R R')$ for any $R$-module $M$ (cf. [18, Proposition 4.1.5]).

(iii) Suppose furthermore that $R \to R'$ is faithfully flat. Then $T\otimes_R R'$ is finitely generated as an $R'$-module if and only if $T$ is finitely generated as an $R$-module (cf. [2, chap. I, §3, no. 6, Proposition 11]).

(iv) If $R$ is local, then $\dim R/I = \dim \widehat{R}/I\widehat{R}$, where $\widehat{R}$ is the completion with respect to the maximal ideal of $R$ (cf. [16, §8]).

**Lemma 2.** *Let $R$, $\mathfrak{m}$, $I$ and $M$ be as in the theorem. Let $N$ be a finitely generated $R$-module whose support is contained in $V(\mathfrak{m})$. Then $\operatorname{Ext}^i_R(N, H^j_I(M))$ is a finitely generated $R$-module for all $i \geq 0$ and all $j \geq 0$.*

*Proof.* Since $R$ is an analytically irreducible local ring, we may assume that $R$ is a local domain complete with the topology defined by $\mathfrak{m}$ by Remark 3. Further, we may assume that $I$ is a radical ideal. We only prove the lemma for the case $N = R/\mathfrak{m}$ by Lemma 1. If $R$ is equi-characteristic, then $H^j_I(M)$ is $I$-cofinite for all $j \geq 0$ by [3, Theorem 3]. Thus the lemma holds by Remark 2. Assume that $R$ is not equi-characteristic. Let $q$ be a uniformizing parameter of $R$. Put $\operatorname{Min}(R/I) = \{P_1, P_2, \cdots, P_n\}$, which is the set of minimal prime ideals of $I$, and we have $I = P_1 \cap P_2 \cap \cdots \cap P_n$. We denote $\operatorname{Min}(R/I)$ by $S$. Divide this set $S$ into two parts: one set $S_1$ consisting of prime ideals of $S$ containing $q$ and the other $S_2$ not containing $q$. We set $S_1 = \{P_1, P_2, \cdots, P_t\}$ and $S_2 = S \setminus S_1$, where $t$ is an integer such that $0 \leq t \leq n$. Since the ideal $I$ satisfies the condition of [3, Theorem 3] in the case $t = 0$ or $n$, $H^j_I(M)$ is $I$-cofinite for all $j \geq 0$. Thus the assertion holds by Remark 2. So we may assume $0 < t < n$. Put $I_1 = P_1 \cap \cdots \cap P_t$ and $I_2 = P_{t+1} \cap \cdots \cap P_n$, and we see that $V(I_1) \cup V(I_2) = V(I)$ and $V(I_1) \cap V(I_2) = V(\mathfrak{m})$. We have a Mayer-Vietoris exact sequence (cf. [8, Chap. III, Ex. 2.4]):

$$\cdots \longrightarrow H^j_{\mathfrak{m}}(M) \longrightarrow H^j_{I_1}(M) \oplus H^j_{I_2}(M) \longrightarrow H^j_I(M) \longrightarrow \cdots .$$

We decompose this sequence into short exact sequences of kernels and cokernels as follows:

$$(\alpha)^j \qquad 0 \longrightarrow X^j \longrightarrow H^j_{\mathfrak{m}}(M) \longrightarrow Y^j \longrightarrow 0,$$

$$(\beta)^j \qquad 0 \longrightarrow Y^j \longrightarrow H^j_{I_1}(M) \oplus H^j_{I_2}(M) \longrightarrow Z^j \longrightarrow 0,$$

$$(\gamma)^j \qquad 0 \longrightarrow Z^j \longrightarrow H^j_I(M) \longrightarrow X^{j+1} \longrightarrow 0.$$

We note that $X^q$ and $Y^q$ are Artinian for all $q \geq 0$, since $H^q_{\mathfrak{m}}(M)$ is Artinian for all $q \geq 0$.

Fix $j \geq 0$ arbitrarily. By the sequence $(\beta)^j$, we get an exact sequence:

$$\begin{aligned}\cdots \longrightarrow\ & \operatorname{Ext}^i_R(R/\mathfrak{m}, H^j_{I_1}(M)) \oplus \operatorname{Ext}^i_R(R/\mathfrak{m}, H^j_{I_2}(M)) \\ \longrightarrow\ & \operatorname{Ext}^i_R(R/\mathfrak{m}, Z^j) \longrightarrow \operatorname{Ext}^{i+1}_R(R/\mathfrak{m}, Y^j) \longrightarrow \cdots .\end{aligned}$$

Again by virtue of [3, Theorem 3] and Remark 2, $\operatorname{Ext}^i_R(R/\mathfrak{m}, H^j_{I_1}(M))$ and $\operatorname{Ext}^i_R(R/\mathfrak{m}, H^j_{I_2}(M))$ are finitely generated for all $i \geq 0$. As $\operatorname{Ext}^i_R(R/\mathfrak{m}, Y^j)$ is finitely generated, $\operatorname{Ext}^i_R(R/\mathfrak{m}, Z^j)$ is finitely generated for all $i \geq 0$. Furthermore we get an exact sequence by $(\gamma)^j$:

$$\cdots \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{m}, Z^j) \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{m}, H^j_I(M)) \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{m}, X^{j+1}) \longrightarrow \cdots .$$

As $X^{j+1}$ is also Artinian, $\mathrm{Ext}^i_R(R/\mathfrak{m}, H^j_I(M))$ is finite for all $i \geq 0$ by the above sequence. We have completed the proof. □

## 3. Proof of the main theorem and corollaries

*Remark* 4. We need the following facts for the proof of the main theorem and its corollaries.

(i) Let $(R, \mathfrak{m})$ be local, $\widehat{R}$ its completion with respect to $\mathfrak{m}$, and $T$ an $R$-module. If $T$ has support only at $\mathfrak{m}$, then $T \otimes_R \widehat{R}$ has support only at $\mathfrak{m}\widehat{R}$.

(ii) Let $R \to R'$ be a faithfully flat extension of rings, and $T$ as above. If $T \otimes_R R'$ satisfies the descending chain condition as an $R'$-module, then $T$ satisfies the descending chain condition as an $R$-module.

*Proof of the main theorem.* Let $P$ be any prime ideal of $R$. We show that the Bass numbers (cf. [1]):

$$\mu^i(P, H^j_I(M)) = \dim_{\kappa(P)} \mathrm{Ext}^i_{R_P}(\kappa(P), H^j_I(M)_P)$$

are finite for each $P$, where $\kappa(P) = R_P/PR_P$. If $P$ does not contain $I$, then the local cohomology module $H^j_I(M)$ vanishes at $P$. Thus we may suppose that $P$ contains $I$. Since $I$ is an ideal of dimension 1, $P$ is either a minimal prime ideal of $I$ or the maximal ideal of $R$. If $P$ is a minimal prime ideal of $I$, then $I$ is $PR_P$-primary in $R_P$. Making $R_P$ complete with respect to $PR_P$, $\mathrm{Ext}^i_{R_P}(\kappa(P), H^j_I(M)_P) \otimes_{R_P} \widehat{R}_P$ is finitely generated as an $\widehat{R}_P$-module by Remark 1, where $\widehat{R}_P$ is the completion of $R_P$ defined by the $PR_P$-adic topology. Hence we have the theorem by Remark 3. If $P$ is the maximal ideal, then the assertion follows from Lemma 2. □

We have the following corollaries of the theorem (cf. [11]):

**Corollary 1.** *Let $R$, $\mathfrak{m}$, $I$ and $M$ be as in the theorem. Then $\mathrm{Soc}(H^j_I(M))$ is finitely generated for all $j \geq 0$.*

*Proof.* The assertion immediately follows from the theorem. □

**Corollary 2.** *Let $R$, $\mathfrak{m}$, $I$ and $M$ be as above. Let $j$ be a non-negative integer. If the local cohomology module $H^j_I(M)$ has support only at $\mathfrak{m}$, then $H^j_I(M)$ is Artinian.*

*Proof.* By Remark 4, $H^j_I(M) \otimes_R \widehat{R}$ has support only at $\mathfrak{m}\widehat{R}$, where $\widehat{R}$ is the completion of $R$ with respect to $\mathfrak{m}$. By Lemma 2, $H^j_I(M) \otimes_R \widehat{R}$ is $\mathfrak{m}\widehat{R}$-cofinite. It follows from Remark 1 that $H^j_I(M) \otimes_R \widehat{R}$ is an Artinian $\widehat{R}$-module. Since $\widehat{R}$ is a faithfully flat $R$-module, $H^j_I(M)$ satisfies the descending chain condition as an $R$-module by Remark 4, that is to say, $H^j_I(M)$ is Artinian. □

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## References

1. H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28. MR **27:**3669
2. N. Bourbaki, *Algèbre commutative*, Chapters 1 and 2, Hermann, Paris, 1961. MR **36:**146
3. D. Delfino, *On cofiniteness of local cohomology modules*, Math. Proc. Cambridge Philos. Soc. **115** (1994), 421–429. MR **94m:**13023

4. A. Grothendieck, *Local cohomology, noted by R. Hartshorne*, Lecture Notes in Math., vol. 41, Springer-Verlag, Berlin and New York, 1967. MR **37:**219
5. ———, *Cohomologie locale des faisceaux cohérents et Théorèmes de Lefschtz locaux et globaux (SGA2)*, North-Holland, Amsterdam, 1968. MR **57:**16294
6. R. Hartshorne, *Affine duality and cofiniteness*, Invent. Math. **9** (1970), 145–164. MR **41:**1750
7. ———, *Cohomological dimension of algebraic varieties*, Ann. of Math. **88** (1968), 403–450. MR **38:**1103
8. ———, *Algebraic geometry*, Graduate Texts in Math., vol. 52, Springer-Verlag, Berlin and New York, 1977. MR **57:**3116
9. C. Huneke and J. Koh, *Cofiniteness and vanishing of local cohomology modules*, Math. Proc. Cambridge Philos. Soc. **110** (1991), 421–429. MR **92g:**13021
10. C. Huneke and R. Y. Sharp, *Bass numbers of local cohomology module*, Trans. Amer. Math. Soc. **339** (1993), 765–779. MR **93m:**13008
11. C. Huneke, *Problems on Local Cohomology, in* "Free Resolutions in Commutative Algebra and Algebraic Geometry", Research Notes In Math., ed. by D. Eisenbud and C. Huneke, (1992), 93–108. MR **93f:**13010
12. G. Lyubeznik, *Finiteness properties of local cohomology modules (an application of $D$-modules to commutative algebra)*, Invent. Math. **113** (1993), 41–55. MR **94e:**13032
13. I. G. Macdonald and R. Y. Sharp, *An elementary proof of the non-vanishing of certain local cohomology modules*, Quart. J. Math. Oxford (2), **23** (1972), 197–204. MR **45:**8646
14. E. Matlis, *Injective modules over noetherian rings*, Pacific J. Math., **8** (1958), 511–552. MR **20:**5800
15. H. Matsumura, *Commutative Algebra*, Benjamin/Cummings, Reading, MA, 2nd ed., 1980. MR **82i:**13003
16. ———, *Commutative Ring Theory*, Cambridge University Press, Cambridge, 1986. MR **88h:**13001
17. A. Ogus, *Local cohomological dimension of algebraic varieties*, Ann. of Math. **98** (1973), 1–34. MR **48:**3964
18. J. Strooker, *Homological questions in local algebra*, London Math. Soc. Lecture Note Ser., vol. 145, Cambridge Univ. Press, Cambridge, 1990. MR **91m:**13013
19. O. Zariski and P. Samuel, *Commutative Algebra* Vol. II, Grad. Texts in Math., vol. 29, Springer-Verlag, Berlin and New York, 1960. MR **52:**10706

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