A NEW PROOF OF MASSER’S VANISHING THEOREM

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(Communicated by William W. Adams)

ABSTRACT. We give a simple proof of Masser’s vanishing theorem, which is important in investigating the algebraic independence of the values of Mahler functions.

1. INTRODUCTION

Masser’s vanishing theorem [4] is a complete solution to Problem 1 of Mahler’s article [3], and is important in investigating the algebraic independence of values of Mahler functions. Masser’s proof is divided into two parts. The first part is reproved in Nishioka and Nishioka [5] by using $p$-adic analysis. Their method derives more consequences (Propositions 1 and 2 below), which offer a rather straightforward proof of the vanishing theorem.

Let $\Omega$ be a matrix of $n$ rows and $n$ columns with entries $\omega_{ij}$ ($1 \leq i, j \leq n$) being non-negative integers. For a vector $z = (z_1, \ldots, z_n)$ define $\Omega z$ as the vector $(\prod_{i=1}^{n} z_1^{\omega_{1i}}, \ldots, \prod_{i=1}^{n} z_n^{\omega_{ni}})$. We assume that the matrix $\Omega$ is non-singular and none of its eigenvalues is a root of unity. Let $\alpha$ be an algebraic point of $\mathbb{C}^n$ whose components $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers such that $\Omega^k \alpha \to (0, \ldots, 0)$ as $k \to \infty$.

Theorem (Masser [4]). The following two conditions are equivalent:

(i) There exist a non-zero power series $f(z) \in \mathbb{C}[[z_1, \ldots, z_n]]$ and an integer $k_0$ such that $f(\Omega^k \alpha) = 0$ for all $k \geq k_0$.

(ii) There exist a non-zero difference of monomials $D(z)$ and positive integers $a$ and $m$ such that $D(\Omega^k \alpha) = 0$ for all $r = m + ak$ ($k \geq 0$).

2. PROOF OF THE THEOREM

Proposition 1. Let $C$ be an algebraically closed field of characteristic 0. Let $P(z)$ be a non-zero polynomial in $C[z_1, \ldots, z_n]$, and $x$ a vector in $C^n$ with non-zero components $x_1, \ldots, x_n$. We put $P(z) = \sum_{I=(i_1, \ldots, i_n) \in \Lambda} c_I z^I$ ($c_I \in C^\times$, $I \in \Lambda$). If $P(\Omega^k x) = 0$ for infinitely many $k$, then there exist distinct elements $I, J \in \Lambda$ and positive integers $a$ and $m$ such that

$$x^{(I-J) \Omega^m (\Omega^{ak} - E)} = 1, \quad P(\Omega^{m+ak} x) = 0$$

for any $k \geq 0$, where $E$ is the identity matrix.

Received by the editors June 6, 1994 and, in revised form, March 27, 1995.
1991 Mathematics Subject Classification. Primary 11J81; Secondary 11J91.
Key words and phrases. Mahler function.

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Proof. This is included in the proof of [5, Theorem 3] by setting $DM^2 = a$ therein.

**Proposition 2.** Let $C$ be an algebraically closed field of characteristic 0. Let $P(z)$ be the same as in Proposition 1, and $S$ an infinite subset of the positive integers $\mathbb{N}$. Then there exist polynomials $B_1(z), \ldots, B_s(z) \in C[z_1, \ldots, z_n]$, $b_1, \ldots, b_s \in S$, $u \in \mathbb{N}$ and monomials $U_1(z), \ldots, U_t(z)$, $V_1(z), \ldots, V_t(z)$ such that:

(i) For any $j$ ($1 \leq j \leq t$) there exist distinct $I, J \in \Lambda$ and $a, m \in \mathbb{N}$ such that

$$U_j(z)/V_j(z) = z^{(I-J)\Omega^m}(\Omega^a - E).$$

(ii) $\sum_{i=1}^s B_i(z)P(\Omega^{b_i}z) = (z_1 \cdots z_n)^a \prod_{j=1}^t (U_j(z) - V_j(z)).$

Proof. Let $A$ be the affine algebraic set defined by $\{P(\Omega^k z)\}_{k \in S}$, and $W_1, \ldots, W_r$ the irreducible components of $A$. Let $(x_1, \ldots, x_n)$ be a generic point of $W_j$ over $C$ with components in some field extension of $C$. If $x_i = 0$ for a certain $i$, then $W_j$ is included in the algebraic set defined by $z_i = 0$. If $x_i \neq 0$ for every $i$, then by Proposition 1 there exist distinct $I, J \in \Lambda$ and $a, m \in \mathbb{N}$ such that

$$z^{(I-J)\Omega^m}(\Omega^a - E) = 1.$$

There exist monomials $U_j, V_j$ such that

$$U_j(z)/V_j(z) = z^{(I-J)\Omega^m}(\Omega^a - E).$$

Since $U_j(x) - V_j(x) = 0$, $W_j$ is included in the algebraic set defined by $U_j(z) - V_j(z) = 0$. Therefore we may assume that $A$ is included in the algebraic set defined by

$$(z_1 \cdots z_n)^a \prod_{j=1}^r (U_j(z) - V_j(z)),$$

where $r'$ is the number of $W$'s which are not included in the algebraic set defined by $z_i = 0$ for some $i$. By the Hilbert Nullstellensatz, we obtain the proposition.

For a subset $R$ of $\mathbb{N}$, we define the density $\delta(R)$ of $R$ by

$$\delta(R) = \limsup_{x \to \infty} x^{-1} |\{ k \in R | 1 \leq k \leq x \}|.$$

**Lemma** (Masser [4], Lemma 3). Let $R$ be a subset of $\mathbb{N}$ of positive density. Then there exists a strictly increasing sequence $d_1, d_2, \ldots$ of non-negative integers with the following property. For each $s \geq 1$ let $R_s$ be the set of non-negative integers $r$ such that $r + d_1, \ldots, r + d_s$ are all in $R$. Then $R_s$ has positive density for all $s \geq 1$.

Proof of the Theorem. Replacing $\Omega$ by a suitable positive power of itself, we may assume the multiplicative group generated by the eigenvalues of $\Omega$ is torsion free. Let $f(z)$ be as in (i) and put $f(z) = \sum_{|I| = i_1 + \cdots + i_n} c_I z^I$ ($c_I \in \mathbb{C}$). Then $|c_I| \leq c_1 c_2 |I|$, where $|I| = i_1 + \cdots + i_n$. In what follows $c_1, c_2, \ldots$ denote positive constants independent of $k$. Let $S = \{ I = (i_1, \ldots, i_n) | c_I \neq 0 \}$. By Lemma 3 in Kubota [2], $S$ has a finite subset $T$ such that every element of $S$ majorizes some element of $T$. We define an equivalence relation $\sim$ in $T$ as follows: $I \sim J$ if and only if $\Omega^k |\alpha|^I = \Omega^k |\alpha|^J$ for any $k \geq 0$, where $|\alpha| = (|\alpha_1|, \ldots, |\alpha_n|)$. Let $\{ I_1, \ldots, I_d \}$ be a complete system of representatives and put

$$f_{\sigma \tau}(k) = \log(\Omega^k |\alpha|)_{\tau} - \log(\Omega^k |\alpha|)_{\sigma}, \quad 1 \leq \sigma < \tau \leq q,$$

$$f_i(k) = - \log |\alpha_i^{(k)}|, \quad 1 \leq i \leq n,$$
where \( \Omega^k \alpha = (\alpha^{(k)}_1, \ldots, \alpha^{(k)}_n) \). Then \( f_{\sigma \tau}(k) \) and \( f_i(k) \) are non-zero linear recurrences of which characteristic roots are all eigenvalues of \( \Omega \). Since unity is not an eigenvalue of \( \Omega \), none of \( f_{\sigma \tau}(k), f_i(k) \) is a polynomial. By Lemma 2 in Kubota [2], \( f_{\sigma \tau}(k) \) and \( f_i(k) \) respectively have the degree \((\delta_{\sigma \tau}, \theta_{\sigma \tau})\) and \((\delta_i, \theta_i)\) with \( \theta_{\sigma \tau}, \theta_i > 1 \). Putting \( \theta = \min\{\theta_{\sigma \tau}, \theta_i\}_{1 \leq \sigma < \tau \leq q, 1 \leq i \leq n} \), we have \( \theta > 1 \). By appealing to Corollary 2 of Lemma 1 in [2], there exists a subset \( R_0 \) of \( \mathbb{N} \) such that \( \delta(R_0) > 0 \) and for all \( k \in R_0 \),

\[
|f_{\sigma \tau}(k)|, |f_i(k)| \geq c_2 \theta^k, \quad 1 \leq \sigma < \tau \leq q, \quad 1 \leq i \leq n.
\]

By renumbering the indices of \( I_1, \ldots, I_q \), there exists a subset \( R \) of \( R_0 \) such that \( \delta(R) > 0 \) and for all \( k \in R \),

\[
\log(\Omega^k |\alpha|)^{1_j} \geq c_2 \theta^k, \quad 1 \leq j \leq q - 1,
\]

\[
- \log |\alpha^{(k)}_{i_j}| \geq c_2 \theta^k, \quad 1 \leq i \leq n.
\]

Let

\[
P(z) = \sum_{I \sim I_0} c_I z^I.
\]

For \( I \sim I_0 \), we put \( I^i = I + (0, \ldots, 0, 1, 0) \). Then for all large \( k \in R \),

\[
|P(\Omega^k \alpha)| = |f(\Omega^k \alpha) - P(\Omega^k \alpha)|
\]

\[
\leq \sum_{I \sim I_0} \sum_{J \sim J_0} c_{I,J} |(\Omega^k |\alpha|)^I| + \sum_{I \sim I_0} \sum_{J \sim J_0} c_{J,I} |(\Omega^k |\alpha|)^J|
\]

\[
\leq \sum_{I \sim I_0} \sum_{I \sim I_0} c_{I,J} |(\Omega^k |\alpha|)^I| + \sum_{I \sim I_0} \sum_{I \sim I_0} c_{I,J} |(\Omega^k |\alpha|)^J|.
\]

By the inequality (1), we have

\[
|P(\Omega^k \alpha)| \leq |T| c_4 e^{-c_2 \theta^k} (\Omega^k |\alpha|)^{I_1} + n |T| c_4 e^{-c_2 \theta^k} (\Omega^k |\alpha|)^{J_1}
\]

\[
\leq c_4 e^{-c_2 \theta^k} (\Omega^k |\alpha|)^{I_1},
\]

for all large \( k \in R \). Setting \( \beta_i = \alpha_i/|\alpha| \) \((1 \leq i \leq n)\), we have

\[
P(\Omega^k \beta) = P(\Omega^k \alpha)/(\Omega^k |\alpha|)^I_1
\]

and so

\[
|P(\Omega^k \beta)| \leq c_4 e^{-c_2 \theta^k}
\]

for all large \( k \in R \). Applying the lemma to the set \( R \), we obtain a strictly increasing sequence \( d_1, d_2, \ldots \) as in the lemma. We apply Proposition 2 to the set \( S = \{d_1, d_2, \ldots\} \) and \( P(z) \). Then \( r + b_1, \ldots, r + b_s \in R \) for infinitely many positive integers \( r \). Therefore

\[
\left| \sum_{i=1}^s B_i(\Omega^r \beta) P(\Omega^{r+b_i} \beta) \right| \leq c_5 e^{-c_2 \theta^r}
\]

for infinitely many \( r \). Then by the equality (ii) in Proposition 2, for some \( j \) \((1 \leq j \leq t)\) and infinitely many \( r \) we have

\[
|U_j(\Omega^r \beta) - V_j(\Omega^r \beta)| \leq c_7 e^{-c_2 \theta^r/t}.
\]
By Baker’s theorem [1, Theorem 3.1] for linear forms in logarithms of algebraic numbers,

\[ |U_j(\Omega^r \beta)/V_j(\Omega^r \beta) - 1| \geq c_8 e^{-c_9 r} \]  

if \( U_j(\Omega^r \beta) \neq V_j(\Omega^r \beta) \). By (2) and (3) we conclude that \( U_j(\Omega^r \beta) = V_j(\Omega^r \beta) \) for infinitely many \( r \). On the other hand, since \( I \sim J \sim I_1 \),

\[
U_j(\Omega^r |\alpha|)/V_j(\Omega^r |\alpha|) = (\Omega^r |\alpha|)^{(I-J)\Omega^m(\Omega^s - E)}  \\
= (\Omega^{r+m+e}|\alpha|)^{I-J(\Omega^{r+m}|\alpha|)}^{J-I} = 1.
\]

Therefore

\[ U_j(\Omega^r \alpha) = V_j(\Omega^r \alpha) \]

for infinitely many \( r \). Applying Proposition 1 to the polynomial \( U_j(z) - V_j(z) \), we complete the proof.

REFERENCES


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