

**K -INVARIANT KAEHLER STRUCTURES ON $K_{\mathbf{C}}/N$
AND THE ASSOCIATED LINE BUNDLES**

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ABSTRACT. Let K be a compact semi-simple Lie group, and let N be a maximal unipotent subgroup of the complexified group $K_{\mathbf{C}}$. In this paper, we classify all the K -invariant Kaehler structures on $K_{\mathbf{C}}/N$. For each Kaehler structure ω , let \mathbf{L} be the line bundle with connection whose curvature is ω . We then study the holomorphic sections of \mathbf{L} , which constitute a K -representation space.

1. INTRODUCTION

Let K be a compact semi-simple Lie group, let $G = K_{\mathbf{C}}$ be its complexification, and let KAN be an Iwasawa decomposition of G . Since G and N are complex, the space $X = G/N$ is a complex homogeneous space, with left K action. We denote by T the centralizer of A in K ; T is a Cartan subgroup of K here. Since T normalizes N , it acts on X on the right.

Given a suitable K -invariant symplectic structure ω on X , the process of geometric quantization [5] converts it into a K -representation space V . A desired property of V is that every irreducible K -representation occurs with multiplicity one (termed a *model* in [3], if V is in addition unitary). Several years ago, A.S. Schwarz suggested the space $X = G/N$ as a candidate for this process [6], and this was worked out in [1] for $K \times T$ -invariant Kaehler structures on X .

In this paper, we classify all the K -invariant Kaehler structures on X . For each K -invariant Kaehler structure, we study its associated line bundle whose holomorphic sections constitute the K -representation space V .

Let $H = TA = T_{\mathbf{C}}$ (which is a Cartan subgroup of G), with Lie algebra \mathfrak{h} . Let n be the rank of G , and denote by $\lambda_1, \dots, \lambda_n \in \mathfrak{h}^*$ the positive simple roots. For each positive simple root λ_j , let $\chi_j : H \rightarrow \mathbf{C}^*$ be the character that satisfies $\chi_j(\exp v) = \exp \lambda_j(v)$. We say that a differential form β transforms by χ_j if $R_t^* \beta = \chi_j(t) \beta$, where $t \in T$ and R_t is the right T action. We shall prove

Theorem I. *Every K -invariant Kaehler structure on X can be uniquely written as*

$$\omega = \sqrt{-1} \partial \bar{\partial} F + \sum_1^n d\beta_j,$$

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where $\sqrt{-1}\partial\bar{\partial}F$ is a $K \times T$ -invariant Kaehler structure; each β_j is K -invariant and transforms by χ_j under the right T action.

Since the $K \times T$ -invariant component $\sqrt{-1}\partial\bar{\partial}F$ has been described carefully in [1], this theorem completely classifies all the K -invariant Kaehler structures on X . Observe that ω is $K \times T$ -invariant if and only if the component $\sum d\beta_j$ vanishes. We shall see in the next theorem that this is in fact the desired property to perform geometric quantization.

Let ω be a K -invariant Kaehler structure on X . By Theorem I, ω is exact, hence is in particular integral. Therefore, we can consider the complex line bundle \mathbf{L} on X whose Chern class is $[\omega] = 0$. It is equipped with a connection ∇ whose curvature is ω . Let \mathfrak{k} be the Lie algebra of K . For $\xi \in \mathfrak{k}$, we denote by $\xi^\#$ the vector field on X induced by the K action. There is a canonical representation of \mathfrak{k} on the smooth sections of \mathbf{L} , given by the operators

$$\nabla_{\xi^\#} + \sqrt{-1}\phi^\xi, \quad \xi \in \mathfrak{k},$$

where ϕ is the moment map associated to the K action on (X, ω) ([2], [5]). We shall assume that this representation is K -invariant; namely, it lifts to a holomorphic K action on \mathbf{L} . Thus $\mathcal{O}(\mathbf{L})$, the space of holomorphic sections of \mathbf{L} , becomes a K -representation space. The following theorem asserts that $\mathcal{O}(\mathbf{L})$ suits the purpose of geometric quantization best when ω is $K \times T$ -invariant:

Theorem II. *The following are equivalent:*

- (i) ω is $K \times T$ -invariant ;
- (ii) ω has a potential function ;
- (iii) every irreducible K -representation occurs in $\mathcal{O}(\mathbf{L})$ with multiplicity one.

2. K -INVARIANT KAEHLER STRUCTURES ON $K_{\mathbf{C}}/N$

In this section, we prove Theorem I, which classifies all the K -invariant Kaehler structures on $X = K_{\mathbf{C}}/N$. Let $\partial, \bar{\partial}$ be the Dolbeault operators on X , and $Z_K^{0,1}(X, \mathbf{C})$ be the space of K -invariant $\bar{\partial}$ -closed $(0,1)$ -forms on X . We shall see that every K -invariant Kaehler structure ω on X can be written as

$$\omega = \partial\alpha + \bar{\partial}\bar{\alpha},$$

where $\alpha \in Z_K^{0,1}(X, \mathbf{C})$. Therefore, we now develop some machineries to calculate $Z_K^{0,1}(X, \mathbf{C})$.

Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}, \mathfrak{t}$ be the Lie algebras of G, K, A, N, T respectively. Let $n = \text{rank } G = \dim_{\mathbf{C}} H$. Let $\lambda_{\pm 1}, \dots, \lambda_{\pm m} \in \Delta$ be the root system of \mathfrak{g} , where $\lambda_1, \dots, \lambda_n$ are positive simple roots, and $n \leq m$. Let

$$(2.1) \quad \{\xi_j, \xi_{-j}\} \subset \mathfrak{g}/\mathfrak{h} \quad , \quad \xi_{\pm j} \in \mathfrak{g}_{\pm\lambda_j}$$

be a Weyl basis ([4], p. 421) of $\mathfrak{g}/\mathfrak{h}$. Then

$$(2.2) \quad \zeta_j = \xi_j - \xi_{-j} \quad , \quad \gamma_j = \sqrt{-1}(\xi_j + \xi_{-j}) \in \mathfrak{k} \quad ; \quad j = 1, \dots, m.$$

In fact, under the image of $\mathfrak{k} \rightarrow \mathfrak{k}/\mathfrak{t}$, $\{\zeta_j, \gamma_j\}$ form a basis of $\mathfrak{k}/\mathfrak{t}$. By Iwasawa, $\mathfrak{g}/\mathfrak{n} \cong \mathfrak{k} + \mathfrak{a}$, which induces an almost complex structure J on $\mathfrak{k} + \mathfrak{a}$. Then

$$(2.3) \quad J\zeta_j = \gamma_j \quad ; \quad J\gamma_j = -\zeta_j.$$

The Killing form identifies these vectors with $\zeta_j^*, \gamma_j^* \in \mathfrak{k}^*$. Consider

$$(2.4) \quad q_j = \zeta_j + \sqrt{-1}\gamma_j \in \wedge^{0,1}(\mathfrak{k} + \mathfrak{a}) \quad , \quad v_j = \zeta_j^* - \sqrt{-1}\gamma_j^* \in \wedge^{0,1}(\mathfrak{k} + \mathfrak{a})^* \quad ,$$

for $j = 1, \dots, m$. By Iwasawa, $X = G/N = KA$. Therefore, we may identify $\wedge^{0,1}(\mathfrak{k} + \mathfrak{a})$ and $\wedge^{0,1}(\mathfrak{k} + \mathfrak{a})^*$ with the $K \times A$ -invariant anti-holomorphic vector fields and complex $(0,1)$ -forms on X .

Let $\xi \in \mathfrak{t}$, and $ad_\xi^* : \mathfrak{k}^* \rightarrow \mathfrak{k}^*$. Then

$$\begin{aligned}
 (2.5) \quad ad_\xi^* \zeta_j^* &= ad_\xi^*(\xi_j^* - \xi_{-j}^*) \\
 &= \lambda_j(\xi)\xi_j^* + \lambda_j(\xi)\xi_{-j}^* \\
 &= -\sqrt{-1}\lambda_j(\xi)\gamma_j^*,
 \end{aligned}$$

and

$$\begin{aligned}
 (2.6) \quad ad_\xi^* \gamma_j^* &= ad_\xi^*\sqrt{-1}(\xi_j^* + \xi_{-j}^*) \\
 &= \sqrt{-1}(\lambda_j(\xi)\xi_j^* - \lambda_j(\xi)\xi_{-j}^*) \\
 &= \sqrt{-1}\lambda_j(\xi)\zeta_j^*.
 \end{aligned}$$

Note that, in (2.5) and (2.6), the root λ_j satisfies $\sqrt{-1}\lambda_j(\xi) \in \mathbf{R}$ for $\xi \in \mathfrak{t}$.

For $\xi \in \mathfrak{t}$, the action of ad_ξ^* on $\wedge^{0,1}(\mathfrak{g})^*$ preserves $\wedge^{0,1}(\mathfrak{n})^*$. Therefore ad_ξ^* acts on $\wedge^{0,1}(\mathfrak{g}/\mathfrak{n})^* = \wedge^{0,1}(\mathfrak{k} + \mathfrak{a})^*$. Let $v_j \in \wedge^{0,1}(\mathfrak{k} + \mathfrak{a})^*$ be the $(0,1)$ -form given in (2.4). Then (2.5) and (2.6) give

$$(2.7) \quad ad_\xi^* v_j = \lambda_j(\xi)v_j.$$

We now go from Lie algebra representation to group representation; so consider

$$Ad_t^* : \wedge^{0,1}(\mathfrak{k} + \mathfrak{a})^* \rightarrow \wedge^{0,1}(\mathfrak{k} + \mathfrak{a})^*,$$

for $t \in T$. Also, for each root λ_j , we define the character $\chi_j : T \rightarrow \mathbf{C}^*$ which satisfies

$$(2.8) \quad \chi_j(\exp \xi) = \exp(\lambda_j, \xi)$$

for all $\xi \in \mathfrak{t}$. Then (2.7) implies that

$$(2.9) \quad Ad_t^* v_j = \chi_j(t)v_j$$

for all $t \in T$.

Since T normalizes N , there is a right T action on $X = G/N$, which induces T representation on the $K \times A$ -invariant $(0,1)$ -forms $\wedge^{0,1}(\mathfrak{k} + \mathfrak{a})^*$. For $t \in T$, let L_t and R_t denote the left and right T actions on X respectively. Then, by (2.9),

$$\begin{aligned}
 (2.10) \quad R_t^* v_j &= R_t^* L_t^* v_j \\
 &= Ad_t^* v_j \\
 &= \chi_j(t)v_j.
 \end{aligned}$$

Let $\{\zeta_j, \gamma_j\}$ be the vectors in (2.2), and let

$$(2.11) \quad V = \oplus_1^m \mathbf{R}(\zeta_j, \gamma_j) \subset \mathfrak{k}.$$

Then (2.3) says that V is preserved by the almost complex structure on $\mathfrak{k} + \mathfrak{a} = \mathfrak{g}/\mathfrak{n}$. In fact,

$$(2.12) \quad \mathfrak{k} + \mathfrak{a} = V \oplus \mathfrak{h}$$

is a decomposition of $\mathfrak{k} + \mathfrak{a}$ into complex vector subspaces. This decomposition is orthogonal with respect to the Killing form on $\mathfrak{k} + \mathfrak{a} = \mathfrak{g}/\mathfrak{n}$. It induces the inclusions

$$\wedge^{0,k}(V)^*, \wedge^{0,k}(\mathfrak{h})^* \subset \wedge^{0,k}(\mathfrak{k} + \mathfrak{a})^*,$$

where $\wedge^{0,k}(V)^*$ annihilates $\wedge^{0,k}(\mathfrak{h})$ and $\wedge^{0,k}(\mathfrak{h})^*$ annihilates $\wedge^{0,k}(V)$. Note that the $(0,k)$ -forms in $\wedge^{0,k}(\mathfrak{h})^*$ are $K \times T$ -invariant: If $\xi \in \wedge^{0,k}(\mathfrak{h})^*$, then $ad_v^* \xi = 0$ for all $v \in \mathfrak{h}$. Hence $Ad_t^* \xi = \xi$ for all $t \in T$. It follows that

$$(2.13) \quad R_t^* \xi = R_t^* L_t^* \xi = Ad_t^* \xi = \xi$$

for all $t \in T$.

Let v_1, \dots, v_m be the $K \times A$ -invariant $(0,1)$ -forms in (2.4). We want to consider the $K \times A$ -invariant $(0,2)$ -forms $\{\bar{\partial}v_j\} \subset \wedge^{0,2}(\mathfrak{k} + \mathfrak{a})^*$. Fix $j \in \{1, \dots, m\}$, and the general expression for $\bar{\partial}v_j$ is

$$(2.14) \quad \bar{\partial}v_j = w + \sum_k u_k \wedge v_k + \sum_{r < s} b_{rs} v_r \wedge v_s,$$

for some $w \in \wedge^{0,2}(\mathfrak{h})^*$, $u_k \in \wedge^{0,1}(\mathfrak{h})^*$, $b_{rs} \in \mathbf{C}$. The following lemma describes w, u_k and b_{rs} . Recall that $\lambda_1, \dots, \lambda_n$ are simple, among the positive roots $\lambda_1, \dots, \lambda_m$. Then,

Lemma 2.1. *In (2.14), $w = 0$; and $u_k = 0$ if and only if $k \neq j$. Finally, all b_{rs} vanish if and only if $j = 1, \dots, n$.*

Proof. In view of (2.10),

$$R_t^* \bar{\partial}v_j = \bar{\partial}R_t^* v_j = \chi_j(t) \bar{\partial}v_j$$

for all $t \in T$. Therefore, we also need RHS of (2.14) to transform by χ_j under the right T action. But

$$(2.15) \quad \begin{aligned} R_t^*(u_k \wedge v_k) &= R_t^* u_k \wedge R_t^* v_k \\ &= L_{t^{-1}}^* u_k \wedge \chi_k(t) v_k \\ &= \chi_k(t) u_k \wedge v_k, \end{aligned}$$

and

$$(2.16) \quad R_t^*(v_r \wedge v_s) = R_t^* v_r \wedge R_t^* v_s = \chi_r(t) \chi_s(t) v_r \wedge v_s.$$

Since the non-zero elements of $\{w, u_k \wedge v_k, v_r \wedge v_s\} \subset \wedge^{0,2}(\mathfrak{k} + \mathfrak{a})^*$ are linearly independent, the vectors that do not transform by χ_j have to vanish. Therefore, (2.13) and (2.15) imply that

$$w = 0, \quad \text{and } u_k = 0 \text{ if } k \neq j.$$

However, $u_j \neq 0$ in (2.14): Let q_j be the vector in (2.4). By arguments similar to the ones in (2.5) and (2.6), we see that $[\xi, q_j] = \lambda_j(\xi) q_j$ for all $\xi \in \wedge^{0,1}(\mathfrak{h})$. Choose ξ such that $\lambda_j(\xi) \neq 0$. Then

$$\begin{aligned} 0 \neq \lambda_j(\xi)(v_j, q_j) &= (v_j, [\xi, q_j]) \\ &= (\bar{\partial}v_j, \xi \wedge q_j). \end{aligned}$$

Since $\wedge^{0,1}(V)^*$ annihilates $\wedge^{0,1}(\mathfrak{h})$, $(w, \xi \wedge q_j) = (b_{rs} v_r \wedge v_s, \xi \wedge q_j) = 0$. It follows that $(u_j \wedge v_j, \xi \wedge q_j) \neq 0$, i.e. $u_j \neq 0$.

We next compute the b_{rs} , and show that they all vanish if and only if $j = 1, \dots, n$. If $j = 1, \dots, n$, then λ_j is simple so $\chi_r \chi_s \neq \chi_j$ for all $r, s \in \{1, \dots, m\}$. Hence by (2.16), all $b_{rs} = 0$.

On the other hand, consider $j = n + 1, \dots, m$, so that λ_j is not simple. There exist some roots λ_k, λ_l such that $\lambda_k + \lambda_l = \lambda_j$, and ξ_k, ξ_l, ξ_j be the eigenvectors in (2.1) such that

$$(2.17) \quad [\xi_k, \xi_l] = c \xi_j,$$

where $c \in \mathbf{C}$ is non-zero. Let p_k, p_l, v_j be the $K \times A$ -invariant vector fields and differential form given in (2.4). With some computations following (2.4), we can conclude from (2.17) that

$$(v_j, [p_k, p_l]) \neq 0.$$

But $(\bar{\partial}v_j, p_k \wedge p_l) = (v_j, [p_k, p_l])$, which means that $b_{kl} \neq 0$ in (2.14). This completes the proof of the lemma. \square

Let $\Omega_K^{0,1}(X, \mathbf{C})$ be the space of K -invariant $(0,1)$ -forms on X . Since we identify $\wedge^{0,1}(\mathfrak{k} + \mathfrak{a})^*$ with the $K \times A$ -invariant $(0,1)$ -forms on X , it follows that

$$\Omega_K^{0,1}(X, \mathbf{C}) = C_K^\infty(X, \mathbf{C}) \otimes \wedge^{0,1}(\mathfrak{k} + \mathfrak{a})^*.$$

However, by Iwasawa $X = KA$, so a K -invariant function on X is simply a function on A . Therefore,

$$(2.18) \quad \Omega_K^{0,1}(X, \mathbf{C}) = C^\infty(A, \mathbf{C}) \otimes \wedge^{0,1}(\mathfrak{k} + \mathfrak{a})^*.$$

We are interested in

$$Z_K^{0,1}(X, \mathbf{C}) = \{\alpha \in \Omega_K^{0,1}(X, \mathbf{C}) ; \bar{\partial}\alpha = 0\}.$$

For all positive simple roots $\lambda_1, \dots, \lambda_n$ with their characters χ_j defined in (2.8), let

$$Z_{K, \lambda_j}^{0,1}(X, \mathbf{C}) = \{\alpha \in Z_K^{0,1}(X, \mathbf{C}) ; R_i^* \alpha = \chi_j(t)\alpha \text{ for all } t \in T\}.$$

Similarly, let $Z_{KT}^{0,1}(X, \mathbf{C})$ denote the elements in $Z_K^{0,1}(X, \mathbf{C})$ that are invariant under the right T action. Then

Proposition 2.2. (i) For every positive simple root λ_j , $Z_{K, \lambda_j}^{0,1}(X, \mathbf{C})$ is one dimensional ;

$$(ii) \quad Z_{KT}^{0,1}(X, \mathbf{C}) = Z_{KT}^{0,1}(X, \mathbf{C}) \oplus (\oplus_1^n Z_{K, \lambda_j}^{0,1}(X, \mathbf{C})).$$

Proof. Let $\alpha \in \Omega_K^{0,1}(X, \mathbf{C})$. By (2.18), we have

$$(2.19) \quad \alpha = w + \sum_1^m f_j v_j,$$

where $w \in C^\infty(A, \mathbf{C}) \otimes \wedge^{0,1}(\mathfrak{h})^*$, $f_j \in C^\infty(A, \mathbf{C})$, and $v_j \in \wedge^{0,1}(V)^*$ are the $(0,1)$ -forms in (2.4).

Clearly $C^\infty(A, \mathbf{C})$ is $K \times T$ -invariant. It follows from (2.13) that w is $K \times T$ -invariant, and from (2.10) that each $f_j v_j$ transforms by χ_j under the right T action.

Since R_i^* commutes with $\bar{\partial}$,

$$\bar{\partial}w \in \Omega_{KT}^{0,2}(X, \mathbf{C}), \quad \bar{\partial}(f_j v_j) \in \Omega_{K, \lambda_j}^{0,2}(X, \mathbf{C})$$

for all $j = 1, \dots, m$. Therefore, in (2.19), $\bar{\partial}\alpha = 0$ if and only if $\bar{\partial}w = \bar{\partial}(f_1 v_1) = \dots = \bar{\partial}(f_m v_m) = 0$; so we can investigate these components separately. Clearly if $\bar{\partial}\alpha = 0$, then $w \in Z_{KT}^{0,1}(X, \mathbf{C})$.

Suppose that $\bar{\partial}(f_j v_j) = 0$, for $j = 1, \dots, n$. By Lemma 2.1, $\bar{\partial}v_j = u_j \wedge v_j$, for some $u_j \in \wedge^{0,1}(\mathfrak{h})^*$. Then

$$(2.20) \quad \begin{aligned} 0 = \bar{\partial}(f_j v_j) &= (\bar{\partial}f_j) \wedge v_j + f_j \bar{\partial}v_j \\ &= (\bar{\partial}f_j + f_j u_j) \wedge v_j. \end{aligned}$$

If $0 \neq \bar{\partial}f_j + f_j u_j \in \wedge^{0,1}(\mathfrak{h})^*$, then $\bar{\partial}f_j + f_j u_j$ and v_j are linearly independent, which contradicts (2.20). Therefore,

$$(2.21) \quad \bar{\partial}f_j + f_j u_j = 0.$$

We claim that the solutions f_j of (2.21) form a one dimensional vector space: We make the identification

$$f_j \in C_K^\infty(X, \mathbf{C}) = C^\infty(A, \mathbf{C}), \quad u_j \in \wedge^{0,1}(\mathfrak{h})^* = \text{Hom}(\mathfrak{a}, \mathbf{C}) = \Omega_A^1(A, \mathbf{C}),$$

so that f_j and u_j are a complex function and an invariant form on A respectively. However, the Lie group A is isomorphic to its Lie algebra \mathfrak{a} via the exponential map, and by a choice of Euclidean coordinates, $\mathfrak{a} = \mathbf{R}^n$. Let dx_1, \dots, dx_n be the standard 1-forms on \mathbf{R}^n . Then, under these identifications, u_j becomes a complex linear 1-form on \mathbf{R}^n . Namely, $u_j = \sum_k c_{jk} dx_k$ for some $c_{jk} \in \mathbf{C}$. Also, the operator $\bar{\partial}$ on $C_K^\infty(X, \mathbf{C})$ is identified with the operator d on $C^\infty(A, \mathbf{C})$. Therefore, (2.21) becomes

$$0 = df_j + f_j u_j = \sum_k \frac{\partial f_j}{\partial x_k} dx_k + c_{jk} f_j dx_k,$$

which means that

$$\frac{\partial f_j}{\partial x_k} = -c_{jk} f_j$$

for all $k = 1, \dots, n$. This equation can be solved with

$$f_j(x) = a \exp\left(-\sum_k c_{jk} x_k\right),$$

and is unique up to the constant $a \in \mathbf{C}$. Hence the space of solutions of (2.21) is one dimensional, as claimed. This proves part (i) of the proposition.

In order to complete the proof, we need to show that $f_{n+1}, \dots, f_m = 0$ in (2.19). Since $\bar{\partial}\alpha = 0$, $\bar{\partial}(f_j u_j) = 0$ for all j . Let $j \in \{n+1, \dots, m\}$. Then

$$\begin{aligned} 0 = \bar{\partial}(f_j v_j) &= (\bar{\partial}f_j) \wedge v_j + f_j (\bar{\partial}v_j) \\ (2.22) \qquad &= (\bar{\partial}f_j) \wedge v_j + f_j u_j \wedge v_j + f_j x \\ &= (\bar{\partial}f_j + f_j u_j) \wedge v_j + f_j x, \end{aligned}$$

where $u_j \in \wedge^{0,1}(\mathfrak{h})^*$, and $0 \neq x \in \wedge^{0,2}(V)^*$ by Lemma 2.1. But $(\bar{\partial}f_j + f_j u_j) \wedge v_j$ and $f_j x$ are linearly independent if they are both non-zero. So (2.22) implies $f_j x = 0$, and hence $f_j = 0$. This proves the proposition. \square

We have shown that every $\alpha \in Z_K^{0,1}(X, \mathbf{C})$ can be uniquely written as

$$\alpha = \alpha_0 + \alpha_1 + \dots + \alpha_n,$$

where α_0 is $K \times T$ -invariant and $R_t^* \alpha_j = \chi_j(t) \alpha_j$ for all $j = 1, \dots, n$. With this result, we now consider a K -invariant Kaehler structure ω on X . Since K is semi-simple,

$$H^2(X, \mathbf{R}) = H^2(KA, \mathbf{R}) = H^2(K, \mathbf{R}) = 0.$$

Therefore ω , being closed, can be written as

$$\omega = d\beta,$$

for some real 1-form β on X . Let

$$\beta = \alpha + \bar{\alpha}$$

be its Dolbeault decomposition, where α and $\bar{\alpha}$ are $(0,1)$ and $(1,0)$ -forms respectively. Averaging by K if necessary, we may assume that $\beta, \alpha, \bar{\alpha}$ are K -invariant. Since ω is of type $(1,1)$,

$$(2.23) \qquad \omega = \partial\alpha + \bar{\partial}\bar{\alpha}$$

and

$$\bar{\partial}\alpha = \partial\bar{\alpha} = 0.$$

Therefore, $\alpha \in Z_K^{0,1}(X, \mathbf{C})$. We apply Proposition 2.2 and write

$$(2.24) \quad \alpha = \sum_0^n \alpha_j,$$

where $\alpha_0 \in Z_{KT}^{0,1}(X, \mathbf{C})$ and $\alpha_j \in Z_{K,\lambda_j}^{0,1}(X, \mathbf{C})$ for $j = 1, \dots, n$. We claim that α_0 is $\bar{\partial}$ -exact:

Recall from (2.19) that α_0 can be written as an element of $C^\infty(A, \mathbf{C}) \otimes \wedge^{0,1}(\mathfrak{h})^*$. We make the natural identification

$$\begin{aligned} \alpha_0 &\in C^\infty(A, \mathbf{C}) \otimes \wedge^{0,1}(\mathfrak{h})^* \\ &= C^\infty(A, \mathbf{C}) \otimes \text{Hom}(\mathfrak{a}, \mathbf{C}) \\ &= C^\infty(A, \mathbf{C}) \otimes \Omega_A^1(A, \mathbf{C}) \\ &= \Omega^1(A, \mathbf{C}), \end{aligned}$$

so that α_0 is identified with a complex 1-form on A . Then α_0 , being a $\bar{\partial}$ -closed $(0,1)$ -form, is identified with a closed 1-form on A . Since

$$H^1(A, \mathbf{C}) = 0,$$

it means that α_0 is identified with an exact 1-form on A . Hence

$$(2.25) \quad \alpha_0 = \bar{\partial}f \in C^\infty(A, \mathbf{C}) \otimes \wedge^{0,1}(\mathfrak{h})^*,$$

for some $f \in C^\infty(A, \mathbf{C})$, as claimed.

Set $F = \sqrt{-1}(f - \bar{f})$ and $\beta_j = \alpha_j + \bar{\alpha}_j$. Then (2.23), (2.24) and (2.25) imply that

$$\omega = \sqrt{-1}\partial\bar{\partial}F + \sum_1^n d\beta_j,$$

which satisfies the decomposition for ω described in Theorem I.

Let $\iota : H \hookrightarrow X$ be the natural holomorphic imbedding of the Cartan subgroup H into X . Then each $\iota^*\beta_j$ is a T -invariant form that transforms by χ_j under the right T action. Since H and T are abelian, $L_t = R_{t^{-1}}$. Therefore

$$(2.26) \quad \iota^*d\beta_j = dt^*\beta_j = 0,$$

which means that each $d\beta_j$ degenerates along H . Hence if $\omega = \sqrt{-1}\partial\bar{\partial}F + \sum d\beta_j$ is Kaehler, then $\sqrt{-1}\partial\bar{\partial}F$ cannot vanish. We shall show that more is true: $\sqrt{-1}\partial\bar{\partial}F$ has to be Kaehler.

Let $\mathfrak{k} + \mathfrak{a} = V \oplus \mathfrak{h}$ be the decomposition of $\mathfrak{k} + \mathfrak{a}$ into complex subspaces V and \mathfrak{h} , given in (2.12). Note that

$$(2.27) \quad \mathfrak{k} = V + \mathfrak{t}.$$

For each positive simple root λ_j , we let $\chi_j : H \rightarrow \mathbf{C}^*$ be its corresponding character. We then say that a differential form β transforms by χ_j if $R_t^*\beta = \chi_j(t)\beta$. The following proposition completes the proof of Theorem I.

Proposition 2.3. *Let $\omega = \sqrt{-1}\partial\bar{\partial}F + \sum d\beta_j$ be a K -invariant Kaehler structure, where $\sqrt{-1}\partial\bar{\partial}F$ is $K \times T$ -invariant, and each β_j transforms by χ_j under the right T action. Then $\sqrt{-1}\partial\bar{\partial}F$ is necessarily Kaehler.*

Proof. For simplicity, we write $\omega = \omega' + \omega''$, where $\omega' = \sqrt{-1}\partial\bar{\partial}F$ and $\omega'' = \sum d\beta_j$. Since X is diffeomorphic to KA , the points on X can be written as $ka, k \in K, a \in A$.

Suppose that ω' is not Kaehler. Since it is K -invariant, ω'_a is degenerate for some $a \in A$. Given $\xi \in \mathfrak{k}$, let ξ^\sharp be the infinitesimal vector field on X generated by the K action. Let $V \subset \mathfrak{k}$ be the subspace given in (2.11), generated by the basis $\{\zeta_j, \gamma_j\}$ in (2.2). Then

$$(V^\sharp)_a \oplus (\mathfrak{t}^\sharp)_a \oplus J(\mathfrak{t}^\sharp)_a = T_aX.$$

Further, $(V^\sharp)_a$ and $(\mathfrak{t}^\sharp)_a \oplus J(\mathfrak{t}^\sharp)_a$ are complementary with respect to ω'_a (see [1]). Therefore, one of the following two cases is valid:

Case 1. ω'_a is degenerate on $(\mathfrak{t}^\sharp)_a \oplus J(\mathfrak{t}^\sharp)_a$. Then, together with (2.26), we see that ω_a is degenerate.

Case 2. ω'_a is degenerate on V^\sharp . There exists a non-zero vector

$$\eta = \sum_1^m a_j \zeta_j + b_j \gamma_j \in V$$

such that $\omega'(\eta^\sharp, J\eta^\sharp)_a \leq 0$. Let

$$\pi : \mathfrak{k} = V \oplus \mathfrak{t} \longrightarrow \mathfrak{t}$$

be the projection onto the second factor, and let

$$\Phi : X \longrightarrow \mathfrak{k}^*$$

be the moment map associated to the K action on (X, ω') . Then

$$\begin{aligned} 0 &\geq \omega'(\eta^\sharp, J\eta^\sharp)_a \\ &= (\Phi(a), [\eta, J\eta]) \\ &= (\Phi(a), \pi[\eta, J\eta]) \quad \text{as } \Phi(a) \in \mathfrak{t}^* [1], \\ &= \sum_1^m (a_j^2 + b_j^2)(\Phi(a), \lambda_j). \end{aligned}$$

Since η is non-zero, there exists some positive root λ_j such that

$$(2.28) \quad (\Phi(a), \lambda_j) \leq 0.$$

For this λ_j , we see that

$$(2.29) \quad \omega(\zeta_j^\sharp, J\zeta_j^\sharp)_a = \omega(\zeta_j^\sharp, \gamma_j^\sharp)_a = (\Phi(a), \lambda_j) + \left(\sum_i \beta_i, [\zeta_j, \gamma_j]^\sharp\right)_a.$$

But in view of (2.26) and $[\zeta_j, \gamma_j] \in \mathfrak{t}$,

$$(2.30) \quad \left(\sum_i \beta_i, [\zeta_j, \gamma_j]^\sharp\right)_a = 0.$$

Combining equations (2.28), (2.29) and (2.30), we get

$$\omega(\zeta_j^\sharp, J\zeta_j^\sharp)_a \leq 0,$$

i.e. ω is not Kaehler. This solves the situation of Case 2, hence Proposition 2.3. □

We have thus proved Theorem I. The $K \times T$ -invariant component, $\sqrt{-1}\partial\bar{\partial}F$, has been studied carefully in [1], and we briefly state it here: By K -invariance and the exponential map, F becomes a function on \mathfrak{a} . Then $\sqrt{-1}\partial\bar{\partial}F$ is Kaehler if and only if the following conditions hold.

- (i) $F : \mathfrak{a} \longrightarrow \mathbf{R}$ is strictly convex.

(ii) Let $\Phi : X \rightarrow \mathfrak{k}^*$ be the moment map corresponding to the K action on $(X, \sqrt{-1}\partial\bar{\partial}F)$. Then the image of Φ intersects \mathfrak{t}^* inside the positive Weyl chamber.

Hence this result, together with Theorem I, classifies all the K -invariant Kaehler structures on X .

3. LINE BUNDLES ON $K_{\mathbb{C}}/N$

Let ω be a K -invariant Kaehler structure on $X = K_{\mathbb{C}}/N$. We write ω in the canonical form

$$\omega = \sqrt{-1}\partial\bar{\partial}F + \sum_1^n d\beta_j,$$

as expressed in Theorem I.

We claim that ω is $K \times T$ -invariant if and only if it has a potential function:

Since each β_j transforms by the character χ_j , we know that ω is $K \times T$ -invariant if and only if $\sum d\beta_j$ vanishes. This will imply that ω has a potential function F . Conversely, suppose that ω has a potential function F . Then, averaging by K if necessary, we may assume that F is K -invariant. But by Iwasawa, $X = KA$; so the K -invariant function F is just a function on A . Consequently F , and hence ω , are $K \times T$ -invariant. We have shown that the first two properties of Theorem II, the $K \times T$ -invariance and the existence of a potential function, are equivalent.

As we shall see, this is the desired property to perform geometric quantization. The above formula proves that ω is exact, and hence is in particular integral. Therefore, there exists a complex line bundle \mathbf{L} whose Chern class is $[\omega] = 0$; it is equipped with a connection ∇ whose curvature is ω . There is a natural \mathfrak{k} representation on the smooth sections of \mathbf{L} given by the operators

$$\nabla_{\xi^\sharp} + \sqrt{-1}\phi^\xi, \quad \xi \in \mathfrak{k},$$

where ϕ is the moment map corresponding to the K action on X ([2], [5]). We shall assume that this representation is induced from a holomorphic K action on \mathbf{L} . With nice topological conditions, this assumption is always valid. For instance, it is always possible to do this if K is simply-connected [5]. This way, we get a K representation on $\mathcal{O}(\mathbf{L})$, the space of holomorphic sections of \mathbf{L} . In [1], we see that if ω is $K \times T$ -invariant, then $\mathcal{O}(\mathbf{L})$ contains every irreducible K representation with multiplicity one. We shall show that $\mathcal{O}(\mathbf{L})$ is not so nice if ω is not invariant under the right T action. Therefore, the most appropriate setting to perform geometric quantization is a $K \times T$ -invariant Kaehler manifold.

Proposition 3.1. *Suppose ω is not invariant under the right T action. Then there is no non-vanishing holomorphic section on \mathbf{L} .*

Proof. As in (2.23), we write

$$\omega = \partial\alpha + \bar{\partial}\bar{\alpha},$$

where α is a $(0,1)$ -form and $\bar{\partial}\bar{\alpha} = 0$. Since ω is not $K \times T$ -invariant, it has no potential function; hence α is not $\bar{\partial}$ -exact. Write

$$(3.1) \quad \beta = \alpha + \bar{\alpha},$$

so that $d\beta = \omega$.

Suppose s is a non-vanishing holomorphic section of \mathbf{L} . Then

$$(3.2) \quad \gamma = \sqrt{-1} \frac{\nabla s}{s}$$

is a complex 1-form, and, by the definition of curvature, $d\gamma = \omega$. This means that $\gamma - \beta$ is closed. Since K is semi-simple,

$$H^1(X, \mathbf{C}) = H^1(KA, \mathbf{C}) = H^1(K, \mathbf{C}) = 0.$$

Therefore, there exists a complex-valued function h such that

$$\gamma - \beta = dh.$$

From (3.2), we see that

$$(3.3) \quad \sqrt{-1}\nabla s = (\beta + dh)s.$$

Let J be the almost complex structure on X . Since s is holomorphic,

$$(3.4) \quad \nabla_{\sqrt{-1}\xi - J\xi}s = 0$$

for every real vector field ξ . Combining (3.3) and (3.4), we get

$$(\beta + dh, \xi + \sqrt{-1}J\xi)s = 0.$$

But s is non-vanishing, so $(\beta + dh, \xi + \sqrt{-1}J\xi) = 0$. Since $\xi + \sqrt{-1}J\xi$ is anti-holomorphic,

$$\beta + dh \in \Omega^{1,0}(X, \mathbf{C}).$$

From (3.1),

$$\alpha + \bar{\alpha} + \partial h + \bar{\partial} h \in \Omega^{1,0}(X, \mathbf{C}).$$

We end up with

$$\alpha + \bar{\partial} h \in \Omega^{1,0}(X, \mathbf{C}),$$

where α is a $(0,1)$ -form that is not $\bar{\partial}$ -exact. This is a contradiction, and hence the proposition. \square

Using this result, we now show that if ω is not right T invariant, then the trivial (one dimensional) representation does not occur in $\mathcal{O}(\mathbf{L})$ as a subrepresentation. This is because, if the trivial representation occurs in $\mathcal{O}(\mathbf{L})$, then it contains some K -invariant holomorphic sections other than the zero section. However:

Proposition 3.2. *Suppose ω is not right T invariant. Then the only K -invariant holomorphic section of \mathbf{L} is the zero section.*

Proof. Suppose s is a K -invariant holomorphic section. By the previous proposition, $s_p = 0$ for some $p \in X$. Let Kp denote the K orbit through p . Then s , being K -invariant, vanishes on Kp . However, the orbit Kp contains some totally real subspace of X , as can be seen from (2.12) and (2.27). Hence s , being holomorphic, has to be the zero section. \square

This completes the proof of Theorem II. We conclude that $\mathcal{O}(\mathbf{L})$ serves the purpose of geometric quantization best when X is a $K \times T$ -invariant Kaehler manifold.

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