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$C(K, A)$ AND $C(K, H^\infty)$ HAVE THE DUNFORD-PETTIS PROPERTY

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ABSTRACT. Denote by X either the disc algebra A, or the space H^∞ of bounded analytic functions on the disc, or any of their even duals. Then $C(K, X)$ has the Dunford-Pettis property.

An open question in the theory of Banach spaces is when the space $C(K, X)$ of all continuous functions defined over a Hausdorff compact space K with values in a Banach space X has the Dunford-Pettis property. The natural conjecture, that this occurs *if and only if X does*, was disproved by Talagrand [13]. On the other hand, Bourgain [1] showed that this is the case when X is an abstract $L^1(\mu)$-space and this was extended to $\mathcal{L}^1$-spaces by Emmanuele [8]. Moreover, it is easy to see that the conjecture is true for $\mathcal{L}^\infty$-spaces. It is known that some classical Banach spaces of analytic functions have the Dunford-Pettis property (see Bourgain [2], Chaumat [3]). This led E. Saab and P. Saab to pose the following problem [12, Question 14]: if A is the disc algebra, does $C(K, A)$ have the Dunford-Pettis property? In this paper, we give a positive answer to this question and also show that $C(K, H^\infty)$ has this property, where H^∞ is the space of bounded analytic functions on the disc. Note that A and H^∞ are not $\mathcal{L}^p$-spaces for any $1 \le p \le \infty$, since they do not have even local unconditional structure [11, Theorem 4.2].

A Banach space X is said to have the Dunford-Pettis property if every weakly compact operator from X into an arbitrary Banach space is completely continuous, or equivalently, if given sequences (x_n) in X and (x_n^*) in X^*, both weakly convergent to zero, then $\langle x_n^*, x_n\rangle$ tends to zero. A detailed exposition of this property can be found in [5]. For the rest of the notation and terminology used and not explained here, we refer the reader to the monographs of Diestel and Uhl [6], Lindenstrauss and Tzafriri [10], and Wojtaszczyk [14]. We begin by proving a technical result, which is of independent interest.

Theorem 1. *If $\ell^\infty(X)$ has the Dunford-Pettis property, then $C(K, X)$ has the same property for every compact Hausdorff space K.*

Proof. Let Σ be the Borel σ-field of K and $\mathcal{B}(\Sigma, X)$ be the Banach space of all totally Σ-measurable functions with values in X [7, Chapter II. §9]. It is well-known [6, p. 182] that $C(K, X)^*$ is isometrically isomorphic to the Banach space

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$M(K, X^*)$ of all weak*-regular X^*-valued vector measures defined on Σ that are of bounded variation. Then, using the representation of $\mathcal{B}(\Sigma, X)$ as the injective tensor product of $\mathcal{B}(\Sigma)$ and X, and [8, Lemma 1], we have that the canonical injection J_1 of $C(K, X)$ into its bidual can be extended to an isometric operator, which we still denote J_1, defined over $\mathcal{B}(\Sigma, X)$ and with values in $C(K, X)^{**}$. Namely,

$$\langle \mu, J_1(f) \rangle = \int_K f \, d\mu, \quad \mu \in M(K, X^*), \ f \in \mathcal{B}(\Sigma, X).$$

Suppose that (f_n) and (μ_n) are sequences in $C(K, X)$ and $C(K, X)^*$ respectively, both weakly convergent to zero. We have to prove that $\langle \mu_n, f_n \rangle$ tends to zero. For each $n \in \mathbb{N}$, $f_n(K)$ is a norm-compact subset of X. Therefore, there is a simple Σ-measurable X-valued function $g_n(\cdot) = \sum_{i=1}^{p_n} x_n(i) \chi_{A_n(i)}(\cdot)$ such that $\|f_n - g_n\| \leq 1/n$, for the $\mathcal{B}(\Sigma, X)$-norm. Note that

$$|\langle \mu_n, f_n \rangle| = \left| \int_K f_n \, d\mu_n \right| \leq \frac{1}{n} \|\mu_n\| + \left| \int_K g_n \, d\mu_n \right|.$$

Hence, it is enough to prove that $\langle \mu_n, J_1(g_n) \rangle$ tends to zero.

For every $n \in \mathbb{N}$, consider the following family of pairwise disjoint Σ-measurable subsets of K:

$$F_n = \{A_1(m_1) \cap A_2(m_2) \cap \cdots \cap A_n(m_n) : \ m_i = 1, ..., p_i, \ i = 1, ..., n\}.$$

Since F_n has a finite number of elements, by removing those with empty intersections, we can write $F_n = \{B_1^n, B_2^n, \ldots, B_{q_n}^n\}$ for some $q_n \in \mathbb{N}$. Now, define the closed subspace of $\mathcal{B}(\Sigma, X)$

$$Y_n = \left\{ \sum_{i=1}^{q_n} x_i \chi_{B_i^n} \ : \ x_i \in X \right\},$$

which is clearly isometrically isomorphic to $\ell_{q_n}^\infty(X)$. Since (Y_n) is an increasing sequence of subspaces, we can consider the space Y defined as the closure in $\mathcal{B}(\Sigma, X)$ of their union. We observe that g_n belongs to Y, for all $n \in \mathbb{N}$. On the other hand, let J_2 be the canonical injection of $C(K, X)^*$ into $C(K, X)^{***}$ and denote by $J_2^Y(\mu_n)$ the restriction of $J_2(\mu_n)$ to $J_1(Y)$. It is easy to check that $(J_1(g_n))$ and $(J_2^Y(\mu_n))$ are weakly null sequences in $J_1(Y)$ and $J_1(Y)^*$, respectively. Moreover,

$$\langle J_2^Y(\mu_n), J_1(g_n) \rangle = \langle \mu_n, J_1(g_n) \rangle = \int_K g_n \, d\mu_n.$$

Therefore, the theorem will we proved if we show that Y has the Dunford-Pettis property. According to a result due to Bourgain [1, Proposition 2], we only have to prove that $(\oplus_n Y_n)_\infty$ has the Dunford-Pettis property. But, this follows on combining the hyphotesis that $\ell^\infty(X)$ has the Dunford-Pettis property with the following topological isomorphisms:

$$(\oplus_n Y_n)_\infty \cong (\oplus_n \ell_{q_n}^\infty(X))_\infty \cong (\oplus_n X)_\infty = \ell^\infty(X). \quad \square$$

In general, the converse of the above theorem is not true. The example goes as follows: Let M_n be the algebra of $n \times n$ complex matrices. Then, by [4, p. 61], we know that $Z = (\oplus_n M_n)_0$ is a C^*-algebra which has the Dunford-Pettis property; thus $C(K, Z)$ has the Dunford-Pettis property for every compact Hausdorff space K [12, Theorem 48]. On the other hand, $\ell^\infty(Z)$ contains a complemented copy of $(\oplus_n M_n)_\infty$, thus a complemented copy of $(\oplus_n \ell_n^2)_\infty$. Finally, note that this last space has a complemented copy of the reflexive space ℓ^2 [5, p. 22].

Our next theorem is the main result of this paper. As we pointed out, it answers Question 14 of [12].

Theorem 2. *$C(K, A)$ and $C(K, H^\infty)$ have the Dunford-Pettis property for every compact Hausdorff space K.*

Proof. In both cases, we shall apply Theorem 1. Since H^∞ has the Dunford-Pettis property [2, Corollary 5.4], the case of H^∞ follows directly from the topological isomorphism $\ell^\infty(H^\infty) \cong H^\infty$ [14, Part III.E.13].

For the case of the disc algebra A, recall that A can be described as the smallest closed subspace of $C(\mathbb{T})$ containing all complex polynomials. Therefore, it is easy to see that

$$\ell^\infty(A) = \|\cdot\|_\infty\text{-closure}\left\{\ell^\infty\left(\bigcup_{k=1}^{\infty} A_k\right)\right\},$$

where $A_k = \operatorname{span}\{1, z, ..., z^{2k}\}$.

Suppose that (x_n) and (x_n^*) are sequences in $\ell^\infty(A)$ and $\ell^\infty(A)^*$, respectively, that converge weakly to zero. Using the above expresion for $\ell^\infty(A)$, we can find $y_n \in \ell^\infty(\bigcup_{k=1}^\infty A_k)$ such that $\|x_n - y_n\| \leq \frac{1}{n}$, for all $n \in \mathbb{N}$. Moreover, for each $m \in \mathbb{N}$, we can find $k = k(n, m) \in \mathbb{N}$ such that $y_n(m) \in A_{k(n,m)}$ and $k(n+1, m) > k(n, m)$, for $n, m \in \mathbb{N}$. Of course, we only have to show that $\langle x_n^*, y_n\rangle$ tends to zero.

Consider the subspace $Z_n = (\oplus_m A_{k(n,m)})_\infty \subset \ell^\infty(A)$. For every $n \in \mathbb{N}$, Z_n is contained in Z_{n+1}, so we can define Z as the subspace of $\ell^\infty(A)$ given by the closure of $\bigcup_{n=1}^\infty Z_n$. We observe that (y_n) is a weakly null sequence in Z, and the restriction of x_n^* to Z is also a weakly null sequence in Z^*. Hence, to finish the proof, it is enough to show that Z has the Dunford-Pettis property or, according again to [1, proposition 2], that $(\oplus_n Z_n)_\infty$ has the Dunford-Pettis property. It is clear that $(\oplus_n Z_n)_\infty = (\oplus_n(\oplus_m A_{k(n,m)})_\infty)_\infty$ is isomorphic to a complemented subspace of $(\oplus_n(\oplus_m A_m)_\infty)_\infty$. Finally, by [14, Part III.E.18], this last space is isomorphic to $\ell^\infty(H^\infty)$, and we reduce to the former situation. □

In our next result, we extend Theorem 2 to other Banach spaces, namely, the even duals of A and H^∞. To fix the notation, we denote by $X^{(n)}$ the n^{th} dual of the Banach space X and $X^{(0)} = X$.

Theorem 3. *$C(K, A^{(2n)})$ and $C(K, (H^\infty)^{(2n)})$ have the Dunford-Pettis property for every compact Hausdorff space K and for all $n \in \mathbb{N}$.*

Proof. In the same way as in the scalar case, it can be shown that $\ell^\infty(X)^*$ is the topological direct sum of (a subspace isomorphic to) $\ell^1(X^*)$ and the space of those functionals vanishing over $c_0(X)$. Then, $\ell^\infty(X)^{**}$ contains a complemented subspace isomorphic to $\ell^\infty(X^{**})$ and, by induction, we obtain that $\ell^\infty(X)^{(2n)}$ contains a complemented subspace isomorphic to $\ell^\infty(X^{(2n)})$, for all $n \in \mathbb{N}$.

Since any of the duals of H^∞ have the Dunford-Pettis property [2, Corollary 5.4], we have that $\ell^\infty((H^\infty)^{(2n)})$ has the Dunford-Pettis property, and the case of H^∞ follows from Theorem 1.

On the other hand, since A^{**} is isomorphic to the ℓ^∞-sum of H^∞ and a certain $\mathcal{L}^\infty$-space L [11, Chapter 1], we have the following topological isomorphisms:

$$\ell^\infty(A^{(2n)}) \cong \ell^\infty((H^\infty)^{(2n-2)}) \oplus_\infty \ell^\infty(L^{(2n-2)}), \quad n \in \mathbb{N}.$$

Since $L^{(2n-2)}$ are again $\mathcal{L}^\infty$-spaces, in order to prove that $\ell^\infty(A^{(2n)})$ has the Dunford-Pettis property, we only need to show that $\ell^\infty(X)$ has the Dunford-Pettis property whenever X is an arbitrary $\mathcal{L}^\infty$-space. Now we recall the following characterization: a Banach space X is an $\mathcal{L}^\infty$-space if and only if there is a real number λ such that for all Banach spaces $Z \supset Y$, every compact operator $T : Y \to X$ has an extension $\tilde{T} : Z \to X$ with $\|\tilde{T}\| \leq \lambda \|T\|$ [9, Theorem 4.1 and its proof]. Hence, dealing with the projections from $\ell^\infty(X)$ onto its different coordinates, we see that $\ell^\infty(X)$ is an $\mathcal{L}^\infty$-space and, therefore, its bidual is isomorphic to a complemented subspace of some $C(K)$. □

Addendum. After this paper was submitted, N. Randrianantoanina has proved, with completely different techniques, that $C(K, A)^*$ has the Dunford-Pettis property in a paper posted at the Banach bulletin board entitled "Some results on the Dunford-Pettis property". The authors want to thank the referee for pointing out to them Randrianantoanina's result.

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