C(K,A) AND C(K,H∞) HAVE THE DUNFORD-PETTIS PROPERTY

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Abstract. Denote by X either the disc algebra $A$, or the space $H^\infty$ of bounded analytic functions on the disc, or any of their even duals. Then $C(K,X)$ has the Dunford-Pettis property.

An open question in the theory of Banach spaces is when the space $C(K,X)$ of all continuous functions defined over a Hausdorff compact space $K$ with values in a Banach space $X$ has the Dunford-Pettis property. The natural conjecture, that this occurs if and only if $X$ does, was disproved by Talagrand [13]. On the other hand, Bourgain [1] showed that this is the case when $X$ is an abstract $L^1(\mu)$-space and this was extended to $L^1$-spaces by Emmanuele [8]. Moreover, it is easy to see that the conjecture is true for $L^\infty$-spaces. It is known that some classical Banach spaces of analytic functions have the Dunford-Pettis property (see Bourgain [2], Chaumat [3]). This led E. Saab and P. Saab to pose the following problem [12, Question 14]: if $A$ is the disc algebra, does $C(K,A)$ have the Dunford-Pettis property? In this paper, we give a positive answer to this question and also show that $C(K,H^\infty)$ has this property, where $H^\infty$ is the space of bounded analytic functions on the disc. Note that $A$ and $H^\infty$ are not $L^p$-spaces for any $1 \leq p \leq \infty$, since they do not have even local unconditional structure [11, Theorem 4.2].

A Banach space $X$ is said to have the Dunford-Pettis property if every weakly compact operator from $X$ into an arbitrary Banach space is completely continuous, or equivalently, if given sequences $(x_n)$ in $X$ and $(x^*_n)$ in $X^*$, both weakly convergent to zero, then $(x^*_n, x_n)$ tends to zero. A detailed exposition of this property can be found in [5]. For the rest of the notation and terminology used and not explained here, we refer the reader to the monographs of Diestel and Uhl [6], Lindenstrauss and Tzafriri [10], and Wojtaszczyk [14]. We begin by proving a technical result, which is of independent interest.

Theorem 1. If $\ell^\infty(X)$ has the Dunford-Pettis property, then $C(K,X)$ has the same property for every compact Hausdorff space $K$.

Proof. Let $\Sigma$ be the Borel $\sigma$-field of $K$ and $B(\Sigma,X)$ be the Banach space of all totally $\Sigma$-measurable functions with values in $X$ [7, Chapter II. §9]. It is well-known [6, p. 182] that $C(K,X)^*$ is isometrically isomorphic to the Banach space...
\( M(K, X^*) \) of all weak*-regular \( X^* \)-valued vector measures defined on \( \Sigma \) that are of bounded variation. Then, using the representation of \( B(\Sigma, X) \) as the injective tensor product of \( B(\Sigma) \) and \( X \), and [8, Lemma 1], we have that the canonical injection \( J_1 \) of \( C(K, X) \) into its bidual can be extended to an isometric operator, which we still denote \( J_1 \), defined over \( B(\Sigma, X) \) and with values in \( C(K, X)^{**} \). Namely,

\[
\langle \mu, J_1(f) \rangle = \int_K f \, d\mu, \quad \mu \in M(K, X^*), \ f \in B(\Sigma, X).
\]

Suppose that \((f_n)\) and \((\mu_n)\) are sequences in \( C(K, X) \) and \( C(K, X)^* \) respectively, both weakly convergent to zero. We have to prove that \( \langle \mu_n, f_n \rangle \) tends to zero. For each \( n \in \mathbb{N} \), \( f_n(K) \) is a norm-compact subset of \( X \). Therefore, there is a simple \( \Sigma \)-measurable \( X \)-valued function \( g_n(\cdot) = \sum_{i=1}^{p_n} x_n(i) \chi_{A_n(i)}(\cdot) \) such that \( \|f_n - g_n\| \leq 1/n \), for the \( B(\Sigma, X) \)-norm. Note that

\[
|\langle \mu_n, f_n \rangle | = \left| \int_K f_n \, d\mu_n \right| \leq \frac{1}{n} \|\mu_n\| + \left| \int_K g_n \, d\mu_n \right|.
\]

Hence, it is enough to prove that \( \langle \mu_n, J_1(g_n) \rangle \) tends to zero.

For every \( n \in \mathbb{N} \), consider the following family of pairwise disjoint \( \Sigma \)-measurable subsets of \( K \):

\[
F_n = \{ A_1(m_1) \cap A_2(m_2) \cap \cdots \cap A_n(m_n) : m_i = 1, \ldots p_i, \ i = 1, \ldots, n \}.
\]

Since \( F_n \) has a finite number of elements, by removing those with empty intersections, we can write \( F_n = \{ B_1^n, B_2^n, \ldots, B_{q_n}^n \} \) for some \( q_n \in \mathbb{N} \). Now, define the closed subspace of \( B(\Sigma, X) \)

\[
Y_n = \left\{ \sum_{i=1}^{q_n} x_i \chi_{B_i^n} : x_i \in X \right\},
\]

which is clearly isometrically isomorphic to \( \ell_{q_n}^\infty(X) \). Since \( (Y_n) \) is an increasing sequence of subspaces, we can consider the space \( Y \) defined as the closure in \( B(\Sigma, X) \) of their union. We observe that \( g_n \) belongs to \( Y \), for all \( n \in \mathbb{N} \). On the other hand, let \( J_2 \) be the canonical injection of \( C(K, X)^* \) into \( C(K, X)^{**} \) and denote by \( J_2^Y(\mu_n) \) the restriction of \( J_2(\mu_n) \) to \( J_1(Y) \). It is easy to check that \( (J_1(g_n)) \) and \( (J_2^Y(\mu_n)) \) are weakly null sequences in \( J_1(Y) \) and \( J_2(Y)^* \), respectively. Moreover,

\[
\langle J_2^Y(\mu_n), J_1(g_n) \rangle = \langle \mu_n, J_1(g_n) \rangle = \int_K g_n \, d\mu_n.
\]

Therefore, the theorem will we proved if we show that \( Y \) has the Dunford-Pettis property. According to a result due to Bourgain [1, Proposition 2], we only have to prove that \( (\oplus_n Y_n)_\infty \) has the Dunford-Pettis property. But, this follows on combining the hypotheses that \( \ell^\infty(X) \) has the Dunford-Pettis property with the following topological isomorphisms:

\[
(\oplus_n Y_n)_\infty \cong (\oplus_n \ell_{q_n}^\infty(X))_\infty \cong (\oplus_n X)_\infty = \ell^\infty(X).
\]

In general, the converse of the above theorem is not true. The example goes as follows: Let \( M_n \) be the algebra of \( n \times n \) complex matrices. Then, by [4, p. 61], we know that \( Z = (\oplus_n M_n)_0 \) is a \( C^* \)-algebra which has the Dunford-Pettis property; thus \( C(K, Z) \) has the Dunford-Pettis property for every compact Hausdorff space \( K \) [12, Theorem 48]. On the other hand, \( \ell^\infty(Z) \) contains a complemented copy of \( (\oplus_n M_n)_\infty \), thus a complemented copy of \( (\oplus_n \ell_{n}^2)_\infty \). Finally, note that this last space has a complemented copy of the reflexive space \( \ell^2 \) [5, p. 22].
Our next theorem is the main result of this paper. As we pointed out, it answers Question 14 of [12].

**Theorem 2.** $C(K, A)$ and $C(K, H^\infty)$ have the Dunford-Pettis property for every compact Hausdorff space $K$.

**Proof.** In both cases, we shall apply Theorem 1. Since $H^\infty$ has the Dunford-Pettis property [2, Corollary 5.4], the case of $H^\infty$ follows directly from the topological isomorphism $\ell^\infty(H^\infty) \cong H^\infty$ [14, Part III.E.13].

For the case of the disc algebra $A$, recall that $A$ can be described as the smallest closed subspace of $C(\mathbb{T})$ containing all complex polynomials. Therefore, it is easy to see that

$$\ell^\infty(A) = \| \cdot \|_{\infty}\text{-closure} \left\{ \ell^\infty\left( \bigcup_{k=1}^\infty A_k \right) \right\},$$

where $A_k = \text{span}\{1, z, \ldots, z^{2k}\}$.

Suppose that $(x_n)$ and $(x_n^*)$ are sequences in $\ell^\infty(A)$ and $\ell^\infty(A)^*$, respectively, that converge weakly to zero. Using the above expression for $\ell^\infty(A)$, we can find $y_n \in \ell^\infty(\bigcup_{k=1}^\infty A_k)$ such that $\|x_n - y_n\| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Moreover, for each $m \in \mathbb{N}$, we can find $k = k(n, m) \in \mathbb{N}$ such that $y_n(m) \in A_{k(n,m)}$ and $k(n+1, m) > k(n, m)$, for $n, m \in \mathbb{N}$. Of course, we only have to show that $\langle x_n^*, y_n \rangle$ tends to zero.

Consider the subspace $Z_n = (\oplus_m A_{k(n,m)})_\infty \subset \ell^\infty(A)$. For every $n \in \mathbb{N}$, $Z_n$ is contained in $Z_{n+1}$, so we can define $Z$ as the subspace of $\ell^\infty(A)$ given by the closure of $\bigcup_{n=1}^\infty Z_n$. We observe that $(y_n)$ is a weakly null sequence in $Z$, and the restriction of $x_n^*$ to $Z$ is also a weakly null sequence in $Z^*$. Hence, to finish the proof, it is enough to show that $Z$ has the Dunford-Pettis property or, according again to [1, proposition 2], that $(\oplus_n Z_n)_\infty$ has the Dunford-Pettis property. It is clear that $(\oplus_n Z_n)_\infty = (\oplus_n (\oplus_m A_{k(n,m)})_\infty)_\infty$ is isomorphic to a complemented subspace of $(\oplus_n (\oplus_m A_{n})_\infty)_\infty$. Finally, by [14, Part III.E.18], this last space is isomorphic to $\ell^\infty(H^\infty)$, and we reduce to the former situation.

In our next result, we extend Theorem 2 to other Banach spaces, namely, the even duals of $A$ and $H^\infty$. To fix the notation, we denote by $X^{(n)}$ the $n^{th}$ dual of the Banach space $X$ and $X^{(0)} = X$.

**Theorem 3.** $C(K, A^{(2n)})$ and $C(K, (H^\infty)^{(2n)})$ have the Dunford-Pettis property for every compact Hausdorff space $K$ and for all $n \in \mathbb{N}$.

**Proof.** In the same way as in the scalar case, it can be shown that $\ell^\infty(X)^*$ is the topological direct sum of (a subspace isomorphic to) $\ell^1(X^*)$ and the space of those functionals vanishing over $c_0(X)$. Then, $\ell^\infty(X)^{**}$ contains a complemented subspace isomorphic to $\ell^\infty(X^{**})$ and, by induction, we obtain that $\ell^\infty(X)^{(2n)}$ contains a complemented subspace isomorphic to $\ell^\infty(X^{(2n)})$, for all $n \in \mathbb{N}$.

Since any of the duals of $H^\infty$ have the Dunford-Pettis property [2, Corollary 5.4], we have that $\ell^\infty((H^\infty)^{(2n)})$ has the Dunford-Pettis property, and the case of $H^\infty$ follows from Theorem 1.

On the other hand, since $A^{**}$ is isomorphic to the $\ell^\infty$-sum of $H^\infty$ and a certain $\mathcal{L}^\infty$-space $L$ [11, Chapter 1], we have the following topological isomorphisms:

$$\ell^\infty(A^{(2n)}) \cong \ell^\infty((H^\infty)^{(2n-2)}) \oplus_\infty \ell^\infty(L^{(2n-2)}), \quad n \in \mathbb{N}.$$
Since $L^{(2n-2)}$ are again $\mathcal{L}^\infty$-spaces, in order to prove that $\ell^\infty(A(2^n))$ has the Dunford-Pettis property, we only need to show that $\ell^\infty(X)$ has the Dunford-Pettis property whenever $X$ is an arbitrary $\mathcal{L}^\infty$-space. Now we recall the following characterization: a Banach space $X$ is an $\mathcal{L}^\infty$-space if and only if there is a real number $\lambda$ such that for all Banach spaces $Z \supset Y$, every compact operator $T : Y \to X$ has an extension $\tilde{T} : Z \to X$ with $\|\tilde{T}\| \leq \lambda\|T\|$ [9, Theorem 4.1 and its proof]. Hence, dealing with the projections from $\ell^\infty(X)$ onto its different coordinates, we see that $\ell^\infty(X)$ is an $\mathcal{L}^\infty$-space and, therefore, its bidual is isomorphic to a complemented subspace of some $C(K)$.

**Addendum.** After this paper was submitted, N. Randrianantoanina has proved, with completely different techniques, that $C(K,A)^*$ has the Dunford-Pettis property in a paper posted at the Banach bulletin board entitled “Some results on the Dunford-Pettis property”. The authors want to thank the referee for pointing out to them Randrianantoanina’s result.

**References**


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