

COMMUTATIVITY OF AUTOMORPHISMS OF SUBFACTORS MODULO INNER AUTOMORPHISMS

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ABSTRACT. We introduce a new algebraic invariant $\chi_a(M, N)$ of a subfactor $N \subset M$. We show that this is an abelian group and that if the subfactor is strongly amenable, then the group coincides with the relative Connes invariant $\chi(M, N)$ introduced by Y. Kawahigashi. We also show that this group is contained in the center of $\text{Out}(M, N)$ in many interesting examples such as quantum $SU(n)_k$ subfactors with level k ($k \geq n+1$), but not always contained in the center. We also discuss its relation to the most general setting of the orbifold construction for subfactors.

§0. INTRODUCTION

Recently Y. Kawahigashi introduced the relative Connes invariant

$$\chi(M, N) = \frac{\text{Ct}(M, N) \cap \overline{\text{Int}}(M, N)}{\text{Int}(M, N)}$$

for subfactors in [Ka3], where $\text{Ct}(M, N)$ and $\overline{\text{Int}}(M, N)$ denote classes of centrally trivial and approximately inner automorphisms of the subfactor respectively. S. Popa [P1], [P2] has shown that the central triviality of automorphisms of strongly amenable subfactors of type II_1 in the sense of [P3] is equivalent to their non-strong outerness. (Popa uses the terminology “proper outerness” instead of “strong outerness”.) It has been known by P. H. Loi [L] that the approximate innerness of an automorphism of a strongly amenable subfactor of type II_1 is characterized by triviality of his invariant [L]. These facts show that we can generalize this relative Connes invariant to properly infinite subfactors. In this paper we introduce a new algebraic invariant of a subfactor $N \subset M$. It is the intersection of the sets of non-strongly outer automorphisms and automorphisms with trivial Loi invariant modulo inner automorphisms arising from the normalizers. We define

$$\chi_a(M, N) = \frac{\Psi(M, N) \cap \text{Ker } \Phi(M, N)}{\text{Ad } \mathcal{N}(M, N)},$$

where we denote the set of non-strongly outer automorphisms on the subfactor by $\Psi(M, N)$, the map assigning Loi’s invariant by Φ and the set of normalizers of N in M , i.e., $\{u \in \mathcal{U}(M) \mid uNu^* = N\}$ by $\mathcal{N}(M, N)$. Note that we define

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this invariant modulo inner automorphisms arising from the normalizers instead of inner automorphisms in N . This is because we think that it is more natural when we think of the orbifold construction. For example, in the case of subfactors of an AFD II_1 factor M arising from an outer action of \mathbf{Z}_n on this factor, $\chi(M, M^{\mathbf{Z}_n})$ becomes $\mathbf{Z}_n \oplus \mathbf{Z}_n$ as in [Ka3], that is, the double of the acting group because of the normalizers. We think this is not natural when we consider the orbifold actions on subfactors. So we use normalizers instead of $\mathcal{U}(N)$.

We show that this invariant $\chi_\alpha(M, N)$ is always an abelian group and that if the subfactor is strongly amenable, then the group coincides with the relative Connes invariant $\chi(M, N)$ introduced in [Ka3]. We also show that this group is contained in the center of $\text{Out}(M, N)$ in many interesting examples, but not always contained in the center. We also discuss its relation to the most general setting of the orbifold construction for subfactors.

In this paper we mainly deal with type II_1 subfactors for simplicity. But the same method also works for properly infinite subfactors, if we use the general bimodule theory as in [Y1] and [Y2].

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§1. TRIVIALITY OF LOI'S INVARIANT

In this section we give a necessary condition for the triviality of Loi's invariant in terms of bimodules. We refer the readers to [L] for more details on Loi's invariant of automorphisms on subfactors.

Let $N \subset M$ be a pair of factors of type II_1 with finite index. And let E be the trace preserving conditional expectation from M onto N . (We can also work on properly infinite subfactors. In such a case, we take this conditional expectation E to be minimal.) Let β be an automorphism in $\text{Aut}(M, N)$, then the condition $\beta \circ E = E \circ \beta$ is automatically satisfied because of the uniqueness of the conditional expectation E with respect to the trace. In this case we can extend β uniquely to the k th Jones extension algebra M_k of the subfactor $N \subset M$ so that $\beta(e_k) = e_k$. Here e_k is the k th Jones projection. (See [L],[Ka1].) We write the extension of β to the M_k with the same symbol β .

Furthermore β induces a unitary operator on the Hilbert space $L^2(M_k)$ which is the completion of M_k with respect to the trace. We denote this operator by $\tilde{\beta}$. In this paper we use the following notation. Let ${}_A X_B$ be an A - B bimodule, $\alpha \in \text{Aut}A$ and $\beta \in \text{Aut}B$, then we define another A - B bimodule ${}_\alpha X = {}_A(\alpha X)_B$ (resp. $X_\beta = {}_A(X_\beta)_B$) by

$$\begin{aligned} a \cdot x' \cdot b &\equiv \alpha(a)xb, & \text{for } a \in A, b \in B, x' \in {}_\alpha X, x \in X \\ (\text{resp. } a \cdot x' \cdot b &\equiv ax\beta(b), & \text{for } a \in A, b \in B, x' \in X_\beta, x \in X), \end{aligned}$$

where $x=x'$ as an element of the Hilbert space X .

Now we have the following lemma which is easily obtained.

Lemma 1.1. *Let $N \subset M$ be a pair of factors of type II_1 with finite index, β an automorphism of the subfactor $N \subset M$, and ${}_A X_B$ an irreducible bimodule contained in ${}_A(M_k)_B$. Then an irreducible bimodule ${}_A(\tilde{\beta}(X))_B$ appears in the irreducible decomposition of ${}_A(M_k)_B$. Moreover, let ξ be an intertwiner between the two bimodules ${}_A(M_k)_B$ and ${}_A X_B$, then we get $\beta \circ \xi' \circ \beta^{-1} \in \text{Hom}({}_A(M_k)_B, {}_A(\tilde{\beta}(X))_B)$ induced by*

the action of an automorphism β , where ξ' is the intertwiner obtained from ξ by changing the right and left actions via β^{-1} . Here A and B are factors M or N .

Proof. We have only to prove in the case of $A = B = M$, because the other cases are proved similarly. From the assumption that ${}_M X_M$ be an irreducible bimodule with ${}_M X_M \subset {}_M(M_k)_M$, there exists a non-zero intertwiner $\xi \in \text{Hom}({}_M(M_k)_M, {}_M X_M)$. If we change the right and the left actions of these bimodules via β^{-1} , we get another intertwiner $\xi' \in \text{Hom}({}_M(\beta^{-1}(M_k)_{\beta^{-1}})_M, {}_M(\beta^{-1}X_{\beta^{-1}})_M)$. The automorphism β induces a unitary operator between the two bimodules ${}_M(\beta^{-1}(M_k)_{\beta^{-1}})_M$ and ${}_M(M_k)_M$ and also between the two bimodules ${}_M(\beta^{-1}X_{\beta^{-1}})_M$ and ${}_M(\tilde{\beta}(X))_M$. Thus we get the following three intertwiners:

$$\begin{aligned} \xi' &\in \text{Hom}({}_M(\beta^{-1}M_{k\beta^{-1}})_M, {}_M(\beta^{-1}X_{\beta^{-1}})_M), \\ \beta &\in \text{Hom}({}_M(\beta^{-1}M_{k\beta^{-1}})_M, {}_M(M_k)_M), \\ \beta &\in \text{Hom}({}_M(\beta^{-1}X_{\beta^{-1}})_M, {}_M(\tilde{\beta}(X))_M). \end{aligned}$$

Here we use the same notation β for the above two intertwiners induced by the automorphism β . We compose these intertwiners and obtain the following non-zero intertwiner:

$$\beta \circ \xi' \circ \beta^{-1} \in \text{Hom}({}_M(M_k)_M, {}_M(\tilde{\beta}(X))_M).$$

So the bimodule ${}_M(\tilde{\beta}(X))_M$ appears in the irreducible decomposition of ${}_M(M_k)_M$. And the irreducibility of ${}_M(\tilde{\beta}(X))_M$ is easy. \square

The next proposition gives a necessary condition for the triviality of Loi's invariant of automorphisms on subfactors in terms of bimodules.

Proposition 1.2. *Let $N \subset M$ be a pair of factors of type II_1 with finite index, and β an automorphism of the subfactor. Let ${}_A X_B$ be an irreducible bimodule appearing in the irreducible decomposition of a bimodule ${}_A(M_k)_B$ for some k . If Loi's invariant of β is trivial, then ${}_A(\tilde{\beta}(X))_B \cong {}_A X_B$. Here A and B are factors M or N .*

Proof. We show a proof in the case of $A = B = M$ and the other cases are proved similarly. Since we have the isomorphism $\text{End}({}_M(M_k)_M) \cong M' \cap M_{2k}$, any intertwiner $\xi \in \text{End}({}_M(M_k)_M)$ has the corresponding operator $m_\xi \in M' \cap M_{2k}$ on $L^2(M_k)$ with $\xi(x) = m_\xi x$ for all $x \in L^2(M_k)$. Suppose that Loi's invariant of β is trivial. Then we have $\beta(m_\xi) = m_\xi \in M' \cap M_{2k}$. Since

$$\begin{aligned} \xi(mxm') &= m_\xi(mxm') \\ &= \beta(m_\xi)(mxm') \\ &= \beta(m_\xi\beta^{-1}(mxm')) \\ &= m\beta m_\xi\beta^{-1}(x)m' \\ &= m\beta \circ \xi \circ \beta^{-1}(x)m' \\ &= \beta \circ \xi \circ \beta^{-1}(mxm') \end{aligned}$$

for $m, m' \in M$, $x \in L^2(M_k) \cap M_k$, we have $\xi = \beta \circ \xi \circ \beta^{-1}$ on $L^2(M_k)$. Here $\xi \in \text{Hom}({}_M(M_k)_M, {}_M X_M)$ and $\beta \circ \xi \circ \beta^{-1} \in \text{Hom}({}_M(M_k)_M, {}_M(\tilde{\beta}(X))_M)$. So there exists a non-zero intertwiner $(\beta \circ \xi \circ \beta^{-1}) \circ \xi^* \in \text{Hom}({}_M X_M, {}_M(\tilde{\beta}(X))_M)$ and we

have the isomorphism ${}_M X_M \cong {}_M(\tilde{\beta}(X))_M$ because of the irreducibility of these bimodules ${}_M X_M$ and ${}_M(\tilde{\beta}(X))_M$. \square

§2. THE ALGEBRAIC RELATIVE CONNES INVARIANT

In this section we define an algebraic relative Connes invariant

$$\chi_a(M, N) = \frac{\Psi(M, N) \cap \text{Ker } \Phi(M, N)}{\text{Ad } \mathcal{N}(M, N)}$$

for arbitrary subfactors, where we denote the set of non-strongly outer automorphisms on the subfactor by $\Psi(M, N)$, the map assigning Loi's invariant by Φ and the set of normalizers of N in M , i.e., $\{ u \in \mathcal{U}(M) \mid uNu^* = N \}$ by $\mathcal{N}(M, N)$. We will show that it is always an abelian group. We refer readers to [CK] and [Ko] for a notion of strong outerness of automorphisms. We also refer readers to [P1] and [P2] for a notion of proper outerness of automorphisms which is equivalent to the strong outerness and has been defined independently.

Theorem 2.1. *Let $N \subset M$ be a pair of factors of type II_1 with finite index. Then a non-strongly outer automorphism and an automorphism with trivial Loi invariant on the subfactor $N \subset M$ commute up to inner perturbation. That is, if α is a non-strongly outer automorphism and β is an automorphism with trivial Loi invariant on the subfactor $N \subset M$, then $\alpha\beta = (\text{Ad } u) \circ \beta\alpha$ for some $u \in \mathcal{U}(M)$.*

Proof. Since α is non-strongly outer, there exists a bimodule ${}_M X_M$ such that ${}_M(M_\alpha)_M \cong {}_M X_M \subset {}_M(M_k)_M$ by [CK]. So we have the isomorphism ${}_M(\beta^{-1}M_{\alpha\beta^{-1}})_M \cong {}_M(\beta^{-1}X_{\beta^{-1}})_M$. And the automorphism β induces the two isomorphisms ${}_M(\beta^{-1}M_{\alpha\beta^{-1}})_M \cong {}_M(M_{\beta\alpha\beta^{-1}})_M$ and ${}_M(\beta^{-1}X_{\beta^{-1}})_M \cong {}_M(\tilde{\beta}(X))_M$. Hence by composing these three isomorphisms we have ${}_M(M_{\beta\alpha\beta^{-1}})_M \cong {}_M(\tilde{\beta}(X))_M$. Since the Loi invariant of β is trivial, we have ${}_M(\tilde{\beta}(X))_M \cong {}_M X_M$ by Proposition 1.2. So we get the isomorphism ${}_M(M_{\beta\alpha\beta^{-1}})_M \cong {}_M(M_\alpha)_M$, and there exists a unitary $u \in \text{Hom}({}_M(M_{\beta\alpha\beta^{-1}})_M, {}_M(M_\alpha)_M)$. Because $u(mx\beta\alpha\beta^{-1}(m')) = u(m \cdot x \cdot m') = m \cdot u(x) \cdot m' = mu(x)\alpha(m')$ for $m, m', x \in M$, the operator u is in $\text{End}({}_M M) \cong M$, i.e., u is the right multiplication of a unitary in M . From this, we have $x\beta\alpha\beta^{-1}(m)u = (x \cdot m)u = (x)u \cdot m = xu\alpha(m)$. Therefore, we get $\beta\alpha\beta^{-1}(m) = \text{Ad } u \circ \alpha(m)$ by setting $x = 1$ in this equality. \square

The next corollary is an immediate conclusion from the theorem.

Corollary 2.2. *The group $\chi_a(M, N)$ is abelian.*

Corollary 2.3. *If a subfactor $N \subset M$ has finite depth, then $\chi_a(M, N)$ is a finite abelian group.*

Proof. We have only to show the finiteness of $\chi_a(M, N)$. This follows immediately from the fact that the set of non-strongly outer automorphisms on $N \subset M$ is finite modulo inner automorphisms, since it corresponds to a subset of vertices of the dual principal graph and the graph is finite by the finite depth condition by [CK]. \square

We remark that without the finite depth assumption, we do not get finiteness of $\chi_a(M, N)$ in general. See [Ka3, Remark 4.2].

In the next two propositions, we give the most fundamental examples.

Proposition 2.4. *Let G be a finite group, and α an outer action on a factor M . Then $\chi_a(M, M^\alpha) = \mathcal{Z}(G)$, the center of G .*

Proof. The inclusion $\chi_a(M, M^\alpha) \subset \mathcal{Z}(G)$ can be easily obtained by using the result in [I] and Theorem 2.1.

We show the inverse inclusion $\mathcal{Z}(G) \subset \chi_a(M, M^\alpha)$. Suppose g is in $\mathcal{Z}(G)$, then $\alpha_g(u_h) = u_h$ for all $h \in G$, where u_h is the implementing unitary of the automorphism α_h . Since the higher relative commutants $N' \cap M_k$ are generated by u_h 's and Jones projections and the automorphism α_g fixes these elements, the Loi invariant of the automorphism α_g is trivial. It is obvious that the automorphism α_g is non-strongly outer. \square

Proposition 2.5. *Let G be a finite group, and α an outer action on a factor M . Then $\chi_a(M \rtimes_\alpha G, M)$ is the group of one-dimensional representations of G .*

Proof. Let M_2 be a Jones' basic extension algebra of the subfactor $M = M_0 \subset M \rtimes G = M_1$, and $e_2 = e_M^{M_1}$ a Jones projection in M_2 . We denote the implementing unitary of the automorphism α_g by u_g . A one-dimensional representation π of G appears as an automorphism of M in the dual principal graph of this subfactor, so it is non-strongly outer. Since we have $\pi(u_g) = \langle \pi, g \rangle u_g$, this automorphism π acts trivially on the higher relative commutants $M' \cap M_k$ because they are generated by $\{u_g e_2 u_g^* \mid g \in G\}$ and Jones projections. Hence, Loi's invariant of π is trivial. Now the inverse inclusion is easy. \square

The next proposition gives examples of $\chi_a(M, N)$, which are obtained by the definition of $\chi_a(M, N)$ and Proposition 4.4 in [Ka3].

Proposition 2.6 (Kawahigashi, [Ka3, Proposition 4.4]). *In the case of AFD II_1 subfactor with index less than 4, we get*

$$\chi_a(M, N) = \begin{cases} 0, & \text{for } A_{2n}, D_{2n+4}, E_8, \text{ with } n \geq 1, \\ \mathbf{Z}_2, & \text{for } A_{2n+1}, E_6, \text{ with } n \geq 1, \\ \mathbf{Z}_3, & \text{for } D_4. \end{cases}$$

The next proposition is obtained from [EK2] and [G2], which generalizes the papers [EK1], [X1] and [X2].

Proposition 2.7. *Let $N \subset M$ be a (not necessarily AFD) quantum $SU(n)$ subfactor with level k which has the same paragroup as the AFD quantum $SU(n)_k$ subfactor, and let d be the greatest common measure of n and k . Then we get $\chi_a(M, N) = \mathbf{Z}_d$.*

As pointed out in [Ka3], several people noticed the similarity between modular automorphisms of a type III factor and $\chi_a(M, N)$. It is well-known that modular automorphisms of a type III factor are included in the center of $\text{Out}(M)$ (see [C1]). And $\chi_a(M, N)$ is also included in the center of $\text{Out}(M, N)$ in many interesting cases.

Example 2.8. Let $N \subset M$ be a subfactor of an AFD II_1 factor with index less than 4, then an equality $\chi_a(M, N) = \mathcal{Z}(\text{Out}(M, N))$ holds except the case with principal graph D_4 . Actually, an argument similar to the proof of Corollary 2.3.2 in [C2] also works for the case of automorphisms of AFD II_1 subfactor. So in the AFD II_1 subfactor case, we have $\varepsilon(\text{Ct}(M, N)) = \varepsilon(\overline{\text{Int}}(M, N))'$. Here ε is a canonical quotient map from $\text{Aut}(M, N)$ to $\text{Out}(M, N)$. In the case of subfactors with principal graphs A_n , E_6 and E_8 , we have $\text{Aut}(M, N) = \overline{\text{Int}}(M, N)$ by [L], so

$\mathcal{Z}(\text{Out}(M, N)) = \varepsilon(\overline{\text{Int}}(M, N)) \cap \varepsilon(\overline{\text{Int}}(M, N))' = \varepsilon(\overline{\text{Int}}(M, N)) \cap \varepsilon(\text{Ct}(M, N)) = \chi_a(M, N)$. In the case of subfactors with principal graph D_{2n} ($n \geq 3$),

$$\mathcal{Z}(\text{Out}(M, N)) \subset \varepsilon(\overline{\text{Int}}(M, N))' = \varepsilon(\text{Ct}(M, N)) = 0 = \chi_a(M, N).$$

Here is another example.

Example 2.9. Let $N \subset M$ be an $SU(3)$ subfactor of an AFD type II_1 factor with level $k = 3m$ for $m \geq 2$, then they cannot have a \mathbf{Z}_2 paragroup symmetry which flips the two tails of the (dual) principal graph and fixes the distinguished vertex $*$ by looking at the principal graphs. (See for example Fig 3.1 in [EK1].) (We refer the readers to [OI] for paragroup.) So all automorphisms β in $\text{Aut}(M, N)$ fix α up to inner conjugacy and we have $\beta\alpha\beta^{-1} = \text{Ad } u \circ \alpha$ by Proposition 1.2. We remark that we don't need the triviality of Loi's invariant in this case. This implies $\chi_a(M, N) \subset \mathcal{Z}(\text{Out}(M, N))$.

The same kind of argument works for the quantum $SU(n)$ subfactors with $n \geq 4$.

But note that the set of automorphisms in $\chi_a(M, N)$ is not always included with $\mathcal{Z}(\text{Out}(M, N))$. That is, even if an automorphism β is non-strongly outer and with trivial Loi invariant, it may not commute with an outer automorphism on the subfactor. The following gives such an example.

Example 2.10. Let $N \subset M$ be an $SU(3)$ subfactor of an AFD type II_1 factor with level 3, that is, an AFD type II_1 subfactor with principal graph $E_6^{(1)}$ with index 4. Then the graph automorphism flipping the two tails of this graph and fixing the distinguished vertex $*$ preserves the connection on this subfactor. (See [Ka2], page 78 for the connection on this subfactor.) So it induces an outer action of \mathbf{Z}_2 on this subfactor. This subfactor has \mathbf{Z}_3 as $\chi_a(M, N)$ by Proposition 2.7, and if we denote a non-trivial automorphism in $\chi_a(M, N)$ by $\alpha (\neq id)$, then we can denote the two bimodules on the tail of the dual principal graph by ${}_M(M_\alpha)_M$ and ${}_M(M_{\alpha^2})_M$. The automorphism β exchanges these bimodules. This means $\beta\alpha\beta^{-1} = \text{Ad } u \circ \alpha^2$ for some $u \in \mathcal{N}(M, N)$ by a similar computation to that in Theorem 2.1. So α and β do not commute. This means $\chi_a(M, N) \not\subset \mathcal{Z}(\text{Out}(M, N))$ in this case.

We remark that the above two examples show that an automorphism α in $\chi_a(M, N)$ ($\alpha \neq id$) is outer conjugate to α^2 in the case of subfactor with principal graph $E_6^{(1)}$ (i.e., $SU(3)_3$ subfactor case), but such outer conjugacy does not happen in the case of $SU(3)_k$ subfactors with $k \geq 4$. A similar argument also works for general $SU(n)_k$ subfactors.

Finally, we discuss the orbifold construction. The orbifold construction has originally arisen from the technique in conformal field theory and in solvable lattice model theory. This technique was first used in subfactor theory by Y. Kawahigashi [Ka2]. At first he used it to show the existence of subfactors of the AFD factor of type II_1 with principal graph D_{2n} and non-existence of those with principal graph D_{2n+1} . But this construction has turned out to be a more general one, and it has been extended to the case of AFD quantum $SU(n)_k$ subfactors arising from the solvable lattice models, which is the same as Wenzl's Hecke algebra subfactor of type A in [W], by D. E. Evans, Y. Kawahigashi and F. Xu. (See [EK1], [X1], [X2].) Moreover it has been generalized to the case of arbitrary (not necessarily AFD) quantum $SU(n)_k$ subfactors by the author in [G1] and [G2].

Y. Kawahigashi introduced the relative Connes invariant $\chi(M, N)$ for type II_1 subfactors and applied it to a classification of $\text{Aut}(M, N)$. He noticed that this

invariant is deeply related to the orbifold construction. The orbifold construction in subfactor theory is a procedure to take a quotient of a subfactor by a certain symmetry. It is considered the procedure to take simultaneous fixed point algebras by a certain action on a subfactor. But a general definition of orbifold construction is still a little obscure because it has been applied only to concrete examples. We will give a general definition here to show what should be called the orbifold construction.

Suppose α is an outer action of a group G on a subfactor $N \subset M$. If we take simultaneous fixed point algebras of this subfactor, we get another subfactor $N^\alpha \subset M^\alpha$. Then in what case should we call it an orbifold subfactor? If the above automorphism α has trivial Loi invariant, then the higher relative commutants do not decrease by this procedure. And if it is strongly outer, then the higher relative commutants do not increase. So if α is strongly outer and has trivial Loi invariant, the new subfactor $N^\alpha \subset M^\alpha$ has the same higher relative commutants as the original subfactor $N \subset M$. This means that if the subfactor is strongly amenable, it does not change by this construction. Actually, such actions are completely classified by [P2]. Such a case is not interesting. But if α has a trivial Loi invariant and the strong outerity of α breaks at some level, then the relative commutants $N' \cap \tilde{M}_k$ strictly increase at that level; here \tilde{M}_k is the k th extension algebra of the subfactor $N \rtimes_\alpha G \subset M \rtimes_\alpha G$. And in this case the (dual) principal graph may change. So we would like to call the procedure to take simultaneous fixed point algebras (or simultaneous crossed products) of a subfactor by an outer action of a group *the orbifold construction* only when the action is non-strongly outer and has trivial Loi invariant.

But note that $\chi_\alpha(M, N)$ does not always induce an action on the subfactor $N \subset M$ because the Connes obstruction of the group may be non-trivial. Now we get the following definition.

Definition 2.11. If the Connes obstruction of $\chi_\alpha(M, N)$ is trivial, then it induces an action on this subfactor $N \subset M$. We call the procedure of taking the simultaneous fixed point algebras (or taking the simultaneous crossed products) by this action *the orbifold construction*. And we call the new subfactor $N^\alpha \subset M^\alpha$ *an orbifold subfactor* of a subfactor $N \subset M$.

We remark that according to our definition, an orbifold action on a subfactor is always abelian by Corollary 2.2.

REFERENCES

- [CK] M. Choda & H. Kosaki, *Strongly outer actions for an inclusion of factors*, J. Funct. Anal. **122** (1994), 315–332. MR **96b**:46085
- [C1] A. Connes, *Sur le theoreme de Radon-Nikodym pour les poids normaux-fideles semi-finis*, Bull. Sci. Math. **97** (1973), 253–258. MR **50**:10841
- [C2] A. Connes, *Outer conjugacy classes of automorphisms of factors*, Ann. Sci. Ec. Norm. Sup. 4me serie, t.8 (1975), 383–420. MR **52**:15031
- [EK1] D. E. Evans & Y. Kawahigashi, *Orbifold subfactors from Hecke algebras*, Comm. Math. Phys. **165** (1994), 445–484. CMP 95:03
- [EK2] D. E. Evans & Y. Kawahigashi, *Subfactors and conformal field theory*, Quantum and non-commutative analysis, Kluwer Academic, 1993, pp. 341–369. MR **95j**:46074
- [G1] S. Goto, *Orbifold construction for non-AFD subfactors*, Int. J. of Math **5** (1994), 725–746. MR **95h**:46094
- [G2] S. Goto, *Symmetric flat connections triviality of Loi's invariant and orbifold subfactors*, to appear in Publ. RIMS Kyoto Univ.

- [I] M. Izumi, *Application of fusion rules to classification of subfactors*, Publ. RIMS Kyoto Univ **27** (1991), 953–994. MR **93b**:46121
- [Ka1] Y. Kawahigashi, *Automorphism commuting with a conditional expectation onto a subfactor with finite index*, J. Operator theory **28** (1992), 127–145. MR **95b**:46086
- [Ka2] Y. Kawahigashi, *On flatness of Ocneanu’s connections on the Dynkin diagrams and classification of subfactors*, J. Funct. Anal. **127** (1995), 63–107. MR **95j**:46075
- [Ka3] Y. Kawahigashi, *Centrally trivial automorphisms and an analogue of Connes’s $\chi(M)$ for subfactors*, Duke Math. J. **71** (1993), 93–118. MR **94k**:46131
- [Ko] H. Kosaki, *Automorphisms in the irreducible decompositions of sectors*, “Quantum and non-commutative analysis”, Kluwer Academic, 1993, pp. 305–316. MR **95e**:46073
- [L] P. H. Loi, *On automorphisms of subfactors*, preprint, 1990.
- [O1] A. Ocneanu, *Quantized group string algebras and Galois theory for algebras*, “Operator algebras and applications, Vol. 2 (Warwick, 1987)”, London Math. Soc. Lect. Note Series Vol. 136, Cambridge University Press, 1988, pp. 119–172. MR **91k**:46068
- [P1] S. Popa, *On the classification of actions of amenable groups on subfactors*, C. R. Acad. Sc. Paris. **315** (1992), 295–299. MR **93i**:46110
- [P2] S. Popa, *Classification of actions of discrete amenable groups on amenable subfactors of type II*, preprint, 1992.
- [P3] S. Popa, *Classification of amenable subfactors of type II* Acta Math., vol. 172, 1994, pp. 352–445. MR **95f**:46105
- [W] H. Wenzl, *Hecke algebras of type A_n and subfactors*, Invent. Math. **92** (1988), 345–383. MR **90b**:46118
- [X1] F. Xu, *Orbifold construction in subfactors*, Comm. Math. Phys. **166** (1994), 237–254. CMP 95:06
- [X2] F. Xu, *The flat part of non-flat orbifolds*, 1993, to appear in Pac. J. of Math.
- [Y1] S. Yamagami, *A note on Ocneanu’s approach to Jones’ index theory*, Internat. J. of Math. **4** (1993), 859–871. MR **95f**:46114
- [Y2] S. Yamagami, *Modular theory for bimodules*, J. Funct. Anal. **125** (1994), 327–357. CMP 95:02

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