ON THE SET OF TOPOLOGICALLY INVARIANT MEANS ON
AN ALGEBRA OF CONVOLUTION OPERATORS ON $L^p(G)$

EDMOND E. GRANIRER

(Communicated by Dale E. Alspach)

Abstract. Let $G$ be a locally compact group, $A_p = A_p(G)$ the Banach algebra defined by Herz; thus $A_2(G) = A(G)$ is the Fourier algebra of $G$. Let $PM_p = A_p^*$ the dual, $J \subset A_p$ a closed ideal, with zero set $F = Z(J)$, and $F = (A_p/J)^*$. We consider the set $TIM_p(x) \subset F^*$ of topologically invariant means on $F$ at $x \in F$, where $F$ is “thin.” We show that in certain cases $TIM_p(x) \geq 2^c$ and $TIM_p(x)$ does not have the WRNP, i.e. is far from being weakly compact in $F^*$. This implies the non-Arens regularity of the algebra $A_p/J$.

Introduction

Let $G$ be a locally compact group with unit $e$ and left Haar measure $dx$. Let $A_p(G) = A_p$, the Figa-Talamanca-Gaudry-Herz algebra of $G$ (see [Hz] or [Gr3], [Gr4]), thus $A_2(G) = A(G)$ is the Fourier algebra of $G$ as in [Ey1].

Let $PM_p = PM_p(G) = A_p^*$, $PM_2 = PM_2(G) = A(G)^*$ their Banach space duals (thus algebras of left convolution operators on $L^p(G)$). $PM_p$ is an $A_p$ module via $(u \cdot \phi, v) = (\phi, uv)$ for $u,v \in A_p$.

If $\phi \in PM_p$ let supp $\phi \subset G$ denote its support (see sequel). If $F \subset PM_p$ is a closed subspace, let $\sigma(F) = \{ x \in G; \lambda \delta_x \in F \}$, where for any bounded Borel measure on $G$ let $\lambda \mu \in PM_p$ be given by $(\lambda \mu, v) = \int v d\mu$ for $v \in A_p$, and $\delta_x$ be the point mass at $x$. Denote $E_\phi(x) = ncl \{ \phi \in F; x \notin \text{supp} \phi \}$ (ncl=norm closure) and $F_e = ncl \{ \phi \in F; \text{supp} \phi \text{ is compact} \}$.

If $x \in \sigma(F)$ let $TIM_p(x) = \{ \psi \in F^*; 1 = (\psi, \lambda \delta_x) = ||\psi||, \psi = 0 \text{ on } E_\phi(x) \}$. Prop. 1 of [Gr3], p. 42 shows that this definition is consistent with [Gr3], p. 39.

Our main interest in this paper is in this set. Ching Chou has proved in [Ch2] that if $G$ is second countable nondiscrete and $F = PM_2(G)$, then $TIM_p(e) = 2^c$, where $c$ is the cardinality of $R$, the real line. This result has been definitively improved by Zhiguo Hu [Hu] who found the exact cardinality of $TIM_p(e)$ for an arbitrary $G$ and $F = PM_2(G)$. The method of proof in both results is $C^*$ algebraic and does not apply if $p \neq 2$.

We have proved in [Gr2], Theorem 2.7 that for any $1 < p < \infty$, if $G$ is second countable nondiscrete and $F = PM_p(G)$, then $TIM_p(e) = 2^c$ and $TIM_p(e)$
Theorem. Let $\mathbb{P} \subset PM_p(G)$ be a $w^*$ closed $A_p$ submodule, $\sigma(\mathbb{P}) = F$, $a, b \in G$. Assume that $F$ is metrisable.

(a) If $x \in \text{int}_{aHb} F$ for some closed nondiscrete subgroup $H$, then $\text{card} \ TIM_{\mathbb{Q}}(x) \geq 2^c$, where $\mathbb{Q} = \mathbb{P}$ or $\mathbb{P}_c$.

(b) If $p = 2$, $G$ contains $R$ (or $T$) as a closed subgroup and $S \subset R$ is a symmetric set such that $x \in aSb \subset F$, then $\text{card} \ TIM_{\mathbb{Q}}(x) \geq 2^c$, where $\mathbb{Q} = \mathbb{P}_c$ or $\mathbb{P}$.

What happens if $F$ is not metrisable? In particular what happens if $G$ is non-metrisable, $F = G = H$, thus $\mathbb{P} = PM_p(G)$? Does, in this case, $TIM_F(e)$ contain at least two elements?

We answer this question, and much more, by improving the above results to nonmetrisable $F$, and showing that $TIM_F(x)$ does not have the WRNP, if $x$ is as above.

The method of proof requires an impressive result of I. Zelmanov [Ze] which guarantees an adequate supply of nondiscrete metrisable subgroups in any nondiscrete $G$, and the above mentioned results of [Gr4]. Many thanks are due to Wistar Comfort for pointing out [Ze] to us. We will prove the

Theorem. Let $\mathbb{P}$ be a $w^*$ closed $A_p$ submodule of $PM_p(G)$, $\sigma(\mathbb{P}) = F$, $a, b \in G$. Let $\mathbb{P}_c \subset \mathbb{Q} \subset PM_p(G)$ be any norm closed $A_p$ submodule.

(a) If $1 < p < \infty$, $H$ is a nondiscrete closed subgroup and $x \in \text{int}_{aHb} F$, then $\text{card} \ TIM_{\mathbb{Q}}(x) \geq 2^c$ and $TIM_{\mathbb{Q}}(x)$ does not have the WRNP.

(b) If $p = 2$, $G$ contains $R$ (or $T$) as a closed subgroup, $S \subset R$ (or $T$) is a symmetric set and $x \in aSb \subset F$, then $\text{card} \ TIM_{\mathbb{Q}}(x) \geq 2^c$ and $TIM_{\mathbb{Q}}(x)$ does not have the WRNP.

In particular, if $F = H = G$, $1 < p < \infty$ (thus $\mathbb{P} = PM_p(G)$), we get that $\text{card} \ TIM_F(x) \geq 2^c$ and $TIM_F(x)$ does not have the WRNP, for any nondiscrete $G$.

We also note that for any nondiscrete $G$ there exists $\mathbb{P} \subset PM_p(G)$ as in the above theorem such that $\text{card} \ TIM_F(x) = 2^c$, hence the above estimate for $\text{card} \ TIM_F(x)$ cannot be improved.

Furthermore we show that if $J \subset A_p(G)$ is a closed ideal such that $Z(J) = F$ satisfies (a) or (b) of the above theorem, then the Banach algebra $A_p/J$ is not Arens regular.

Some more notation: If $\mathbb{P} \subset PM_p(G)$, let $W_p(x) = C(\lambda\delta_x) + E_p(x)$, $x \in G$. If $\phi \in PM_p = A_p^*$, then $\text{supp} \phi$ is the set of $x \in G$ such that for any open $V \subset G$ with $x \in V$ there is some $v \in A_p$ such that $\text{supp} v \subset V$ and $(\phi, v) \neq 0$, [Hz]. For the definition of WRNP, see [Sa1] or [Gr2], p. 156. These, as well as other notations can be found in any of [Gr3], [Gr4] or [Hz].
The main results

Proposition 1. Every nonmetrisable locally compact group $G$ contains an infinite compact abelian metrisable subgroup.

Proof. $G$ contains a compact normal infinite subgroup $N$ (see [HR] (8.7)). By E.I. Zelmanov’s impressive result ([Ze], Theorem 2) $N$ contains an infinite abelian subgroup whose closure $A$ is a compact abelian infinite subgroup. Now by Rudin’s Theorem 7 ([Ru3], p. 203) $A$ contains an infinite compact metrisable subgroup. □

Corollary 2. Let $G$ be a nondiscrete locally compact group. Then $G$ contains a separable metric nondiscrete closed subgroup.

Proof. If $G$ is metrisable, let $x_n \in G$ be distinct and such that $x_n \to e$. Let $H$ be the closed subgroup generated by $\{x_n\}$. □

Proposition 3. Let $\mathbb{P} \subset \mathbb{R}$ be norm closed $A_p(G)$ submodules of $PM_p(G)$ such that $a \in \sigma(\mathbb{P})$. Then any $\psi \in TIM_p(a)$ has an extension $\psi_0 \in TIM_R(a)$.

Proof. Let $\psi_1 \in \mathbb{R}^*$ be a norm preserving extension of $\psi$, by Hahn-Banach. Then $1 = \|\psi_1\| = \|\psi\| = (\psi, \Lambda_\alpha) = (\psi_1, \Lambda_\alpha)$ since $\Lambda_\alpha \in \mathbb{P} \subset \mathbb{R}$. Let now $S = S(a) = \{u \in A_p; 1 = \|u\| = u(a)\}$. Then pointwise multiplication in $A_p$ renders $S(a)$ into an abelian semigroup as readily checked. Let $M \in L^\infty(S)^*$ be a translation invariant mean. Define $\psi_0 \in \mathbb{R}^*$ by $\psi_0 = M(\psi_1, u \cdot \phi)$, where $(\psi_1, u \cdot \phi)$ is considered as a bounded function on $S$ (i.e. in $L^\infty(S)$). It is now routine to check that $(\psi_0, v \cdot \phi) = (\psi_0, \phi)$ for all $v \in S(a)$ and $\phi \in \mathbb{R}$. Now, by [Gr3], Proposition 1, $\psi_0 \in TIM_R(a)$ is the required extension. □

Corollary 4. $\text{card} TIM_R(a) \geq \text{card} TIM_p(a)$. Furthermore, if $TIM_p(a)$ fails to have the WRNP, then so does $TIM_R(a)$.

Proof. Let $i : \mathbb{P} \to \mathbb{R}$ be the canonical imbedding. If $TIM_R(a)$ has the WRNP, then by E. Saab’s Corollary 2 ([Sa1], p. 310), $i^*TIM_R(a) = TIM_p(a)$ also has the WRNP, which cannot be. □

The following is folklore.

Proposition 5. Let $X, Y$ be Banach spaces and $W_1 \subset W_2 \subset X$ closed subspaces and $Y_0 \subset Y$ a finite dimensional subspace. Let $q : Y \to Y/Y_0$, $p_j : X \to X/W_j$ be the canonical maps. (a) If $p_2$ is not weakly compact, then $p_1$ is not weakly compact; (b) If $t : X \to Y$ is a nonweakly compact operator, then $qt : X \to Y/Y_0$ is not weakly compact.

Proof. (a) Let $p_3 : X/W_1 \to X/W_2$ be the canonical map. If $p_1$ is weakly compact, then so is $p_2 = p_3p_1$. (b) There is a closed subspace $Y_1$, such that $Y = Y_0 \oplus Y_1$, thus $Y/Y_0 \cong Y_1$. Thus $q[I - q]$ is the projection on $Y_1/Y_0$ resp. and $I - q$ is weakly compact. If $qt$ is weakly compact, so is $t = qt + (I - q)t$ which cannot be. □

Theorem 6. Let $G$ be any locally compact group and $\mathbb{P} \subset PM_p(G)$ a $w^*$ closed $A_p$ submodule with $F = \sigma(\mathbb{P})$. Assume that for some nondiscrete closed subgroup $H$ and $a, b \in G$, $\text{int}_{aHb}F \neq \emptyset$. If $x \in \text{int}_{aHb}F$, then $\text{card} TIM_Q(x) \geq 2^c$ and $TIM_Q(x)$ does not have the WRNP, for any norm closed $A_p$ module $Q$ such that $\mathbb{P}_c \subset Q \subset PM_p(G)$.
Remarks. (1) If \( F \) is metrisable, then our Corollary 7 of [Gr4] implies part of the above theorem. The above is an improvement in that \( F \) need not be metrisable.

In particular one can take for an arbitrary nondiscrete locally compact group \( G = H = F \) and get that \( \mathbb{P} = PM_p(G) \) satisfies \( TIM_p(x) \geq 2^c \) and \( TIM_p(x) \) does not have the WRNP (a fortiori is not weakly compact in \( PM_p(G)^* \)) for any \( x \in G \).

If \( p \neq 2 \) this is a new result. The \( C^* \) algebra methods of Z. Hu in [Hu] do not seem to work in this case.

(2) One cannot improve the cardinality estimate of Theorem 6. For assume that \( 1 < p < \infty \), and \( G \) is an arbitrary nondiscrete locally compact group. Then by Corollary 2 \( G \) contains a closed separable metrisable nondiscrete subgroup \( H \). By Herz’s theorem [Hz], p. 92 the restriction map \( r : A_p(G) \to A_p(H) \) is onto and \( |r| \leq 1 \). Hence \( r*PM_p(H) = \mathbb{P} \) is norm closed and by Theorem 4.14 in [Ru1], \( \mathbb{P} \) is \( w^* \)-closed. \( \mathbb{P} \) is an \( A_p \) module, since \( (\cdot v)*\phi, u) = (r*[(rv)*\phi], u) \) and \( \sigma(\mathbb{P}) \subset H \), see [Hz]. But \( \sigma(\mathbb{P}) = H \) since if \( x \in H \) and \( \phi = \lambda \delta_x \in PM_p(H) \), then \( r*\phi = \delta_x \in \mathbb{P} \).

Now \( A_p(H) \) is norm separable since \( H \) is separable metric. Since \( r* \) is an isometry into (see [Hz], p. 91), \( PM_p(H) = \mathbb{P} \), \( \mathbb{P} \subset H \). Hence \( \mathbb{P} = \mathbb{P}_1 \subset \beta \mathbb{P} \). We can apply now our Theorem 6 to \( \mathbb{P} \subset PM_p(G) \), \( \sigma(\mathbb{P}) = H = F \) and \( a = b = c \) and get that \( 2^c \geq \mathbb{P} = \mathbb{P}_1 = card TIM_p(x) \geq 2^c \) for all \( x \in H \). Thus \( card TIM_p(x) = 2^c \).

Proof. Let \( V \subset G \) be open such that \( V \) is compact and \( x \in V \cap ahHb \subset F \). Let, by Corollary 2, \( H_0 \) be a nondiscrete separable closed metrisable subgroup of \( G \).

Then \( x = abh \) for some \( h \in H \). Thus \( x \in V \cap ahHb \subset V \cap ahHb \subset F \). Let \( F_0 = cl(V \cap ahHb) \), a compact metrisable subset of \( F \). If \( c = abh \), then \( x \in \int_{ahHb} F_0 \) since \( x \in V \cap chHb \subset F_0 \). Let \( F_0 = w^* \text{cl} \{\lambda \delta_x; x \in F_0 \}, F_1 = w^* \text{cl} \{\lambda \delta_x; x \in F \} \) where \( cl \) denotes closure and \( \text{lin}, \text{linear span} \). Then \( F_0 \subset \subset F_0 \subset F_1 \subset F \), since \( F_0 \) is compact and since \( F_1 \) is the smallest \( w^* \)-closed \( A_p \) module with \( \sigma(F_1) = F \), [Hz].

Apply now our Theorem 3 of [Gr4] to \( F_0 \), the metrisable set \( F_0 \) and the closed nondiscrete group \( H_0 \). Then \( \sigma(F_0) = F_0 \), \( x \in \int_{chHb} F_0 \), thus \( x \in D_1(J_0) \) (see [Gr4] where \( J_0 = \{u \in A_p; (\phi, u) = 0, \text{for } \phi \in F_0\} \), a closed ideal such that \( F_0 = (A_p/J_0)^* \). By Theorem 4 of [Gr4] we get that there is some onto operator \( t : F_0 \to \ell^\infty \) (thus \( t* \) is \( w^*-w^* \)-norm isomorphism) such that \( t*F \subset TIM_p(x) \) where \( F = \{\eta \in \ell^\infty*; 1 = (\eta, 1) = ||\eta|| \text{ and } \eta = 0 \text{ on } c_0\} \).

Let now \( \beta N \) be the Stone-Čech compactification of the positive integers \( N \). Then \( \beta N \sim N \subset F \) is a \( w^* \)-perfect set of cardinality \( 2^c \) (see [Ru1], p. 204). Thus \( \Gamma = t*(\beta N \sim N) \) is a \( w^* \)-perfect subset of \( TIM_p(x) \) and \( card TIM_p(x) \geq 2^c \).

But \( \Gamma \) is isomorphic to a canonical \( \ell^1 \) basis, i.e. there is some \( d > 0 \) such that \( \sum |a/i| \geq \sum |a_1^i \delta_i \phi_1| \geq d \sum |a_1| \) for all \( n \geq 1, a_1 \in C \) and distinct \( \phi_1, \ldots, \phi_n \in \beta N \sim N \). To prove this it is enough (since \( ||t|| \leq 1 \) and \( t^* \) is a norm isomorphism into) to show that \( \sum |a_1| = ||\sum a_1 \delta_1 \phi_1|| \). Now \( \ell^\infty = C(\beta N) \), hence there is some \( f \in C(\beta N) \) such that \( ||f|| = 1 \) and \( f(\delta_1) = 1/|a_1| \). Thus \( \sum a_1 \delta_1 \phi_1 = \sum |a_1| \leq ||\sum |a_1| \delta_1 | \). It follows that the \( w^* \) compact set \( TIM_p(x) \) contains a \( w^* \)-perfect set which is isomorphic to a canonical \( \ell^1 \) basis. Hence by our Lemma 1.2 on p. 157 in [Gr2], \( TIM_p(x) \) does not have the WRNP. To get the result about \( TIM_Q(x) \) apply Corollary 4.

With a view to future applications, we have under the assumptions of Theorem 6:

**Corollary 6.** Let \( W \subset W_0(x) \) be any closed subspace. Then the canonical map \( q : Q \to Q/W \) is not a weakly compact operator, for any \( x \in \int_{ahHb} F \).
Proof. By Proposition 5(a) we can assume that $W = W_0(x)$. Since $\mathbb{Q}/W_0(x) = (\mathbb{Q}/E_0(x))/\mathbb{C}(\lambda_\delta_x)$ and by Proposition 5(b) we need only show that the canonical map $q : \mathbb{Q} \to \mathbb{Q}/E_0(x)$ is not weakly compact. If $q$ is weakly compact, then so is $q^* : (\mathbb{Q}/E_0(x))^* \to \mathbb{Q}^*$. But then \( \{ \psi \in \mathbb{Q}^* : ||\psi|| \leq 1 \text{ and } \psi = 0 \text{ on } E_0(x) \} \) and a fortiori $TIM_0(x)$ is a weakly relatively compact subset of $\mathbb{Q}^*$. But $TIM_0(x)$ is $w^*$, hence weakly, closed. Thus $TIM_0(x)$ is weakly compact and a fortiori has the WRNP. This cannot be by Theorem 6.

We do not know if the next theorem holds for $p \neq 2$.

**Theorem 7.** Let $G$ be a locally compact group, and $\mathbb{P} \subset PM_2(G)$ a $w^*$ closed $A_2(G)$ module with $F = \sigma(\mathbb{P})$, $a,b \in G$. Assume that $R$ (or $T$) is a closed subgroup of $G$ and $S \subset R$ (or $T$) is a symmetric set such that $aSb \subset F$. If $x \in aSb$, then $card TIM_0(x) \geq 2^c$ and $TIM_0(x)$ does not have the WRNP for any norm closed $A_2(G)$ module $\mathbb{Q}$ such that $\mathbb{P} \subset \mathbb{Q} \subset PM_2(G)$.

**Remarks.** If $F$ is metrisable, then Corollary 6 of [Gr4] implies the fact that $card TIM_0(x) \geq 2^c$.

Proof. $S$ is a compact subset of $R$, since the map $t : \prod_{i=1}^{\infty} D_i \to S$, $D_i = \{ 0,1 \}$, $tc = \sum_{i=1}^{\infty} \varepsilon_i t_i$, is continuous. Let $F_0 = aSb$, a compact metrisable subset of $aRb$, hence of $G$. Let $\mathbb{P}_0 = w^* \text{ cl lin } \{ \lambda \delta_x : x \in F_0 \}$. Then since $\sigma(\mathbb{P}_0) = F_0$, $\mathbb{P}_0 \subset \mathbb{P}$. We can apply Corollary 2' of [Gr4] with $F_0$ instead of $F$ and get that $x \in D_1(J_0)$, where $J_0 = \{ u \in \mathbb{A}(G) : \phi(u) = 0 \text{ for } \phi \in \mathbb{P}_0 \}$. Hence by [Gr4], Theorem 4, there is an onto operator $t : \mathbb{P}_0/\mathbb{W}_0(x) \to F^\infty$ such that $t^*F \subset TIM_0(x)$. The proof of Theorem 6 shows that card $TIM_0(x) \geq 2^c$ and $TIM_0(x)$ does not have the WRNP. Apply now Corollary 4 to $Q$.

**Remarks.** (1) Theorem 7 holds true if $S = \bigcup_{\alpha \in I}(x_\alpha + S_\alpha)$, where $x_\alpha \in R$, $S_\alpha$ or $-S_\alpha$ are ultrathin symmetric and $I$ is any index set. Symmetric sets are such. This holds since Corollary 2' of [Gr4] holds for such sets $S$.

(2) If $F = G$ thus $\mathbb{P} = PM_2(G)$ and $x = e$, a much better and definitive result on card $TIM_0(e)$ has been obtained by Zhiguo Hu [Hu].

(3) One cannot improve the cardinality estimate of Theorem 7. Indeed, $R$ (or $T$) is a closed subgroup of the otherwise arbitrary group $G$. Thus $R$ (or $T$) is a set of synthesis for $G$ (see [Hz]). Thus $PM_2(R)$ (or $PM_2(T)$) can be identified with $\mathbb{P}_1 = \{ \phi \in PM_2(G) : \text{ supp } \phi \subset R \}$, a $w^*$ closed $A_2(G)$ submodule of $PM_2(G)$ with $F = \sigma(\mathbb{P}_1) = R$ (or $T$), see [Hz]. Thus $\mathbb{P}_1 = L_\infty(R)$ (or $\mathbb{P}_1 \approx L_\infty$). Now let $F = aSb$ and $\mathbb{P} = w^* \text{ cl lin } \{ \lambda \delta_x : x \in aSb \}$. Then $\mathbb{P} \subset \mathbb{P}_1$ and card $\mathbb{P} \leq \text{ card } \mathbb{P}_1 = c$. But Theorem 7 implies $2^c \leq \text{ card } TIM_0(x) \leq \text{ card } \mathbb{P}^* \leq \text{ card } \mathbb{P}_1^* = 2^c$ for all $x \in aSb$.

**Corollary 7'.** With assumptions as in Theorem 7 let $W \subset W_0(x)$ be any closed subspace. Then the canonical map $q : \mathbb{Q} \to \mathbb{Q}/W$ is not a weakly compact operator, for any $x \in aSb$.

**Proof.** See the proof of Corollary 6'.

**Corollary 8.** Let $J \subset A_p(G)$ be a closed ideal, $F = Z(J) = \{ x \in G : v(x) = 0 \text{ for } v \in J \}$, $a,b \in G$. Assume one of the following:

(a) For some nondiscrete closed subgroup $H \subset G$, $\text{ int}_aHbF \neq \emptyset$ or

(b) $p = 2$, $G$ contains $R$ (or $T$) as a closed subgroup, $S \subset R$ (or $T$) is a symmetric set such that $aSb \subset F$.

Then $A_p/J$ is a Banach algebra which is not Arens regular.
Proof. Let $\mathcal{P} = (A/J)^*$. Then $W = WAP(\mathcal{P}) \subset W_\mathcal{P}(x)$ for $x \in \interact_{\mathcal{A}_\mathcal{H}}(x \in aSb)$, respectively, by [Gr4], Proposition 5. But then $q : \mathcal{P} \to \mathcal{P}/W$ is not weakly compact, thus clearly $\mathcal{P}/W \neq \{0\}$. Hence $A/J$ is not Arens regular. 

Remarks. (1) As shown by J.P. Kahane, there exist perfect sets $F \subset G = T$ (and even continuous curves $F$ in $Lip_\beta, \beta < 1$ in $G = R^2$) such that $A(G)/IF = A(F) = C(F)$, an Arens regular Banach algebra, where $IF = \{v \in A(G); v = 0 \text{ on } F\}$. If $\mathcal{P} = A(F)^*$, one has in this case, card $TIM_\mathcal{P}(x) = 1$ for all $x \in F$ (see [Gr3], p. 56–57).

(2) If $J$ satisfies the conditions of Corollary 8 and $G$ is second countable, then $A_p/J$ is even an extremely non-Arens regular (ENAR) Banach algebra, i.e. there is a closed subspace of $\mathcal{P}/WAP(\mathcal{P})$ which has $\mathcal{P}$ as a quotient. We do not know if this is the case if $G$ is not second countable.

The Abelian Case

If $G$ is abelian (l.c.a.), the above results have implications on translation invariant (tr. inv.) subspaces $P \subset L^\infty(\hat{G})$. Denote $L^\infty(\hat{G}) = L^\infty, L^1(\hat{G}) = L^1, UC$ the uniformly continuous functions in $L^\infty, UC_P = UC \cap P, \overline{P} = \{f; f \in P\}$ and $\sigma(P) = \overline{P} \cap G$, where $G \subset L^\infty$ (are the continuous characters on $\hat{G}$). Let $M_P(x) = \{\psi \in P^*; 1 = \|\psi\| = (\psi, \pi)\}$. Let $TIM_P(x) = \{\psi \in M_P(x); (\psi, f) = (\psi, \pi h) * f\}$ if $f \in P, 0 \leq h \leq L^1, \int h d\chi = 1$ and $IM_P(x) = \{\psi \in M_P(x); (\psi, f) = x(\chi)(\psi, f_\chi)\}$ for $\chi \in \hat{G}$, $f \in P\}$, where $f_\chi(\gamma) = f(\chi \gamma)$. Recall that by Proposition 9 in [Gr4], $IM_Q(x) \subset TIM_Q(x)$ with equality if $Q \subset UC$.

Corollary 9. Let $G$ be l.c.a., $P [Q]$ be $w^*$ [norm] closed tr. inv. subspaces of $L^\infty$ such that $UC_P \subset Q \subset L^\infty, F = \sigma(P), a \in G$. Assume that either

(a) for some nondiscrete closed subgroup $H \subset G, int_{\mathcal{A}_H} F \neq \emptyset$ or
(b) $G$ contains $R$ (or $T$) as a closed subgroup, and $S \subset R$ (or $T$) is a symmetric set such that $aS \subset F$.

Then for any $x \in int_{\mathcal{A}_H} F \{x \in aS\} |card IM_Q(x) \geq card TIM_Q(x) \geq 2^c$ and both $IM_Q(x)$ and $TIM_Q(x)$ do not have the WRNP.

Proof. By [Sa1], p. 308 it is enough to prove the result for $TIM_Q(x)$. But by [Gr4] $F^* : L^\infty \to PM(G)^*$ is a norm and $w^*-w^*$ linear homeomorphism such that $F^*TIM_Q(x) = TIM_Q(x)$, where $F^*Q = Q$ and $F : L^1 \to A(G)$ is Fourier transform (see [Gr4]). Then as in [Gr4] and by the above Theorem 6, $7$, card $TIM_Q \geq 2^c$ and $TIM_Q(x)$ does not have the WRNP. Hence again by [Sa1], p. 308 the same holds for $TIM_Q(x)$.

Remarks. (1) The cardinality part of the above result is implied by Corollaries 10, 11 in [Gr4], in case $F$ is metrisable.

(2) Let $F \subset R = G, \chi_t(x) = e^{itx}$ and $P_*\{F\} = w^* \text{ llin } \{\chi_t; t \in F\} \subset L^\infty(\hat{G})$.

(a) Assume that $F$ is the Cantor 1/3 set; thus for $t_n = 2/3^n, F = \{\sum_{i=0}^\infty \varepsilon_i t_i; \varepsilon_i = 0, 1\}$, a symmetric set. Then for $t \in F$ and $P = P_\ast(F), TIM_P(t) = \{\psi \in P^*; 1 = \|\psi\| = (\psi, \pi_\chi), (\psi, f_\chi) = \chi_t(x)(\psi, f)\}$ for all $f \in P, x \in R\}$, where $f_\chi(y) = f(x+y)$, since $F$ is compact and by [Gr4], Proposition 9. Then, since $|\text{card } L^\infty(R)^* = 2^c \geq |\text{card } TIM_P(t)|$, Corollary 9 yields that card $TIM_P(t) = 2^c$ and $TIM_P(t)$ does not have the WRNP for any $t \in F$. Note that if $t = 0$, then $TIM_P(0) = IM_P(0)$, the set of honest to goodness invariant means on $P$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
(b) Assume that $F \subset R$ is a compact perfect Helson $S$ set with $0 \in F$. There exist such by [He] (or see [Ru1]). In this case, with $P = P_r(F)$ we have that $\text{card} TIM_P(t) = 1$, and $TIM_P(t)$ contains one element, hence certainly has the WRNP, for all $t \in F$. In particular for $t = 0$, $\text{card} IM_P(0) = 1$, for the set of honest to goodness invariant means on $P$ (see [Gr3]). The same is the case if $F \subset R$ is any scattered compact set, by Loomis' lemma [Lo].

**Question:** Does there exist a perfect set $F \subset R$ such that for $P = P_r(F)$, there is some $t_0$ in $F$ for which $\text{card} TIM_P(t_0) = c$?

**References**


[Gr3] ________, *On convolution operators with small support which are far from being convolution by a bounded measure*, Colloq. Math. 67 (1994), 33–60. CMP 94:17


Department of Mathematics, The University of British Columbia, Vancouver, British Columbia, Canada V6T 1Z2

E-mail address: granirer@math.ubc.ca