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ON THE SET OF TOPOLOGICALLY INVARIANT MEANS ON AN ALGEBRA OF CONVOLUTION OPERATORS ON $L^p(G)$

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ABSTRACT. Let G be a locally compact group, $A_p = A_p(G)$ the Banach algebra defined by Herz; thus $A_2(G) = A(G)$ is the Fourier algebra of G. Let $PM_p = A_p^*$ the dual, $J \subset A_p$ a closed ideal, with zero set $F = Z(J)$, and $\mathbb{P} = (A_p/J)^*$. We consider the set $TIM_{\mathbb{P}}(x) \subset \mathbb{P}^*$ of topologically invariant means on $\mathbb{P}$ at $x \in F$, where F is "thin." We show that in certain cases card $TIM_{\mathbb{P}}(x) \geq 2^c$ and $TIM_{\mathbb{P}}(x)$ does not have the WRNP, i.e. is far from being weakly compact in $\mathbb{P}^*$. This implies the non-Arens regularity of the algebra A_p/J.

INTRODUCTION

Let G be a locally compact group with unit e and left Haar measure dx. Let $A_p(G) = A_p$, the Figa-Talamanca-Gaudry-Herz algebra of G (see [Hz] or [Gr3], [Gr4]), thus $A_2(G) = A(G)$ is the Fourier algebra of G as in [Ey1].

Let $PM_p = PM_p(G) = A_p^*$, $PM_2 = PM_2(G) = A(G)^*$ their Banach space duals (thus algebras of left convolution operators on $L^p(G)$). PM_p is an A_p module via $(u \cdot \phi, v) = (\phi, uv)$ for $u, v \in A_p$.

If $\phi \in PM_p$ let supp $\phi \subset G$ denote its support (see sequel). If $\mathbb{P} \subset PM_p$ is a closed subspace, let $\sigma(\mathbb{P}) = \{x \in G; \lambda\delta_x \in \mathbb{P}\}$, where for any bounded Borel measure on G let $\lambda\mu \in PM_p$ be given by $(\lambda\mu, v) = \int v\, d\mu$ for $v \in A_p$, and δ_x be the point mass at x. Denote $E_{\mathbb{P}}(x) = \text{ncl}\,\{\phi \in \mathbb{P}; x \notin \text{supp}\,\phi\}$ (ncl=norm closure) and $\mathbb{P}_c = \text{ncl}\{\phi \in \mathbb{P}; \text{supp}\ \phi \text{ is compact}\}$.

If $x \in \sigma(\mathbb{P})$ let $TIM_{\mathbb{P}}(x) = \{\psi \in \mathbb{P}^*; 1 = (\psi, \lambda\delta_x) = \|\psi\|,\ \psi = 0 \text{ on } E_{\mathbb{P}}(x)\}$. Prop. 1 of [Gr3], p. 42 shows that this definition is consistent with [Gr3], p. 39.

Our main interest in this paper is in this set. Ching Chou has proved in [Ch2] that if G is second countable nondiscrete and $\mathbb{P} = PM_2(G)$, then card $TIM_{\mathbb{P}}(e) = 2^c$, where c is the cardinality of R, the real line. This result has been definitively improved by Zhiguo Hu [Hu] who found the exact cardinality of $TIM_{\mathbb{P}}(e)$ for an arbitrary G and $\mathbb{P} = PM_2(G)$. The method of proof in both results is C^* algebraic and does not apply if $p \neq 2$.

We have proved in [Gr2], Theorem 2.7 that for any $1 < p < \infty$, if G is second countable nondiscrete and $\mathbb{P} = PM_p(G)$, then card $TIM_{\mathbb{P}}(e) = 2^c$ and $TIM_{\mathbb{P}}(e)$

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does not have the WRNP, see [Sa1] (which roughly means that $TIM_{\mathbb{P}}(e)$ is far from being a weakly compact set in $\mathbb{P}^*$).

Improving results in [Gr3] we have obtained in [Gr4] results about the set $TIM_{\mathbb{P}}(x)$ for the case that $\sigma(\mathbb{P})$ is "thin" in G. For these we need a few more notations. If F, H are subsets of G, let $\mathrm{int}_H F$ be the interior of $F \cap H$ in H, thus $x \in \mathrm{int}_H F$ iff for some open V in G, $x \in V \cap H \subset F \cap H$. The set $S \subset R$ (the real line) is symmetric if there are $t_n > 0$ such that $t_n > \sum_{n+1}^{\infty} t_i$ for all n and $S = \{\sum_1^{\infty} \varepsilon_i t_i; \varepsilon_i = 0, 1\}$. If in addition $\sum_1^{\infty} (t_{i+1}/t_i)^2 < \infty$, then S is ultrathin symmetric, [GMc], p. 333. Among the results obtained in [Gr4] is the

Theorem. *Let $\mathbb{P} \subset PM_P(G)$ be a w^* closed A_p submodule, $\sigma(\mathbb{P}) = F$, $a, b \in G$.* **Assume that F is metrisable.**

(a) *If $x \in \mathrm{int}_{aHb} F$ for some closed nondiscrete subgroup H, then* $\mathrm{card}\, TIM_{\mathbb{Q}}(x) \geq 2^c$*, where $\mathbb{Q} = \mathbb{P}$ or $\mathbb{P}_c$.*

(b) *If $p = 2$, G contains R (or T) as a closed subgroup and $S \subset R$ is a symmetric set such that $x \in aSb \subset F$, then* $\mathrm{card}\, TIM_{\mathbb{Q}}(x) \geq 2^c$*, where $\mathbb{Q} = \mathbb{P}_c$ or $\mathbb{P}$.*

What happens if F is not metrisable? In particular what happens if G is nonmetrisable, $F = G = H$, thus $\mathbb{P} = PM_p(G)$? Does, in this case, $TIM_{\mathbb{P}}(e)$ contain at least two elements?

We answer this question, and much more, by improving the above results to nonmetrisable F, and showing that $TIM_{\mathbb{P}}(x)$ does not have the WRNP, if x is as above.

The method of proof requires an impressive result of I. Zelmanov [Ze] which guarantees an adequate supply of nondiscrete metrisable subgroups in any nondiscrete G, and the above mentioned results of [Gr4]. Many thanks are due to Wistar Comfort for pointing out [Ze] to us. We will prove the

Theorem. *Let $\mathbb{P}$ be a w^* closed A_p submodule of $PM_p(G)$, $\sigma(\mathbb{P}) = F$, $a, b \in G$. Let $\mathbb{P}_c \subset \mathbb{Q} \subset PM_p(G)$ be any norm closed A_p submodule.*

(a) *If $1 < p < \infty$, H is a nondiscrete closed subgroup and $x \in \mathrm{int}_{aHb} F$, then* $\mathrm{card}\, TIM_{\mathbb{Q}}(x) \geq 2^c$ *and $TIM_{\mathbb{Q}}(x)$ does not have the WRNP.*

(b) *If $p = 2$, G contains R (or T) as a closed subgroup, $S \subset R$ (or T) is a symmetric set and $x \in aSb \subset F$, then* $\mathrm{card}\, TIM_{\mathbb{Q}}(x) \geq 2^c$ *and $TIM_{\mathbb{Q}}(x)$ does not have the WRNP.*

In particular, if $F = H = G$, $1 < p < \infty$ (thus $\mathbb{P} = PM_p(G)$), we get that $\mathrm{card}\, TIM_{\mathbb{P}}(x) \geq 2^c$ and $TIM_{\mathbb{P}}(x)$ does not have the WRNP, for any nondiscrete G.

We also note that for any nondiscrete G there exists $\mathbb{P} \subset PM_p(G)$ as in the above theorem such that $\mathrm{card}\, TIM_{\mathbb{P}}(x) = 2^c$, hence the above estimate for $\mathrm{card}\, TIM_{\mathbb{P}}(x)$ cannot be improved.

Furthermore we show that if $J \subset A_p(G)$ is a closed ideal such that $Z(J) = F$ satisfies (a) or (b) of the above theorem, then the Banach algebra A_p/J is not Arens regular.

Some more notation: If $\mathbb{P} \subset PM_p(G)$, let $W_{\mathbb{P}}(x) = C(\lambda\delta_x) + E_{\mathbb{P}}(x)$, $x \in G$. If $\phi \in PM_p = A_p^*$, then $\mathrm{supp}\, \phi$ is the set of $x \in G$ such that for any open $V \subset G$ with $x \in V$ there is some $v \in A_p$ such that $\mathrm{supp}\, v \subset V$ and $(\phi, v) \neq 0$, [Hz]. For the definition of WRNP, see [Sa1] or [Gr2], p. 156. These, as well as other notations can be found in any of [Gr3], [Gr4] or [Hz].

The main results

Proposition 1. *Every nonmetrisable locally compact group G contains an infinite compact abelian metrisable subgroup.*

Proof. G contains a compact normal infinite subgroup N (see [HR] (8.7)). By E.I. Zelmanov's impressive result ([Ze], Theorem 2) N contains an infinite abelian subgroup whose closure A is a compact abelian infinite subgroup. Now by Rudin's Theorem 7 ([Ru3], p. 203) A contains an infinite compact metrisable subgroup. □

Corollary 2. *Let G be a nondiscrete locally compact group. Then G contains a separable metric nondiscrete closed subgroup.*

Proof. If G is metrisable, let $x_n \in G$ be distinct and such that $x_n \to e$. Let H be the closed subgroup generated by $\{x_n\}$. □

Proposition 3. *Let $\mathbb{P} \subset \mathbb{R}$ be norm closed $A_p(G)$ submodules of $PM_p(G)$ such that $a \in \sigma(\mathbb{P})$. Then any $\psi \in TIM_{\mathbb{P}}(a)$ has an extension $\psi_0 \in TIM_{\mathbb{R}}(a)$.*

Proof. Let $\psi_1 \in \mathbb{R}^*$ be a norm preserving extension of ψ, by Hahn-Banach. Then $1 = \|\psi_1\| = \|\psi\| = (\psi, \lambda\delta_a) = (\psi_1, \lambda\delta_a)$ since $\lambda\delta_a \in \mathbb{P} \subset \mathbb{R}$. Let now $S = S(a) = \{u \in A_p;\ 1 = \|u\| = u(a)\}$. Then pointwise multiplication in A_p renders $S(a)$ into an abelian semigroup as readily checked. Let $M \in \ell^\infty(S)^*$ be a translation invariant mean. Define $\psi_0 \in \mathbb{R}^*$ by $(\psi_0, \phi) = M(\psi_1, u \cdot \phi)$, where $(\psi_1, u \cdot \phi)$ is considered as a bounded function on S (i.e. in $\ell^\infty(S)$). It is now routine to check that $(\psi_0, v \cdot \phi) = (\psi_0, \phi)$ for all $v \in S(a)$ and $\phi \in \mathbb{R}$. Now, by [Gr3], Proposition 1, $\psi_0 \in TIM_{\mathbb{R}}(a)$ is the required extension. □

Corollary 4. $\operatorname{card} TIM_{\mathbb{R}}(a) \geq \operatorname{card} TIM_{\mathbb{P}}(a)$. *Furthermore, if $TIM_{\mathbb{P}}(a)$ fails to have the WRNP, then so does $TIM_{\mathbb{R}}(a)$.*

Proof. Let $i : \mathbb{P} \to \mathbb{R}$ be the canonical imbedding. If $TIM_{\mathbb{R}}(a)$ has the WRNP, then by E. Saab's Corollary 2 ([Sa1], p. 310), $i^* TIM_{\mathbb{R}}(a) = TIM_{\mathbb{P}}(a)$ also has the WRNP, which cannot be. □

The following is folklore.

Proposition 5. *Let X, Y be Banach spaces and $W_1 \subset W_2 \subset X$ closed subspaces and $Y_0 \subset Y$ a finite dimensional subspace. Let $q : Y \to Y/Y_0$, $p_j : X \to X/W_j$ be the canonical maps.* (a) *If p_2 is not weakly compact, then p_1 is not weakly compact;* (b) *If $t : X \to Y$ is a nonweakly compact operator, then $qt : X \to Y/Y_0$ is not weakly compact.*

Proof. (a) Let $p_3 : X/W_1 \to X/W_2$ be the canonical map. If p_1 is weakly compact, then so is $p_2 = p_3 p_1$. (b) There is a closed subspace Y_1, such that $Y = Y_0 \oplus Y_1$, thus $Y/Y_0 \approx Y_1$. Thus q $[I - q]$ is the projection on Y_1 $[Y_0]$ resp. and $I - q$ is weakly compact. If qt is weakly compact, so is $t = qt + (I - q)t$ which cannot be. □

Theorem 6. *Let G be any locally compact group and $\mathbb{P} \subset PM_p(G)$ a w^* closed A_p submodule with $F = \sigma(\mathbb{P})$. Assume that for some nondiscrete closed subgroup H and $a, b \in G$, $int_{aHb} F \neq \emptyset$. If $x \in \operatorname{int}_{aHb} F$, then $\operatorname{card} TIM_{\mathbb{Q}}(x) \geq 2^c$ and $TIM_{\mathbb{Q}}(x)$ does not have the WRNP, for any norm closed A_p module $\mathbb{Q}$ such that $\mathbb{P}_c \subset \mathbb{Q} \subset PM_p(G)$.*

Remarks. (1) If F is metrisable, then our Corollary 7 of [Gr4] implies part of the above theorem. The above is an improvement in that F need not be metrisable. In particular one can take for an arbitrary nondiscrete locally compact group $G = H = F$ and get that $\mathbb{P} = PM_p(G)$ satisfies card $TIM_{\mathbb{P}}(x) \geq 2^c$ and $TIM_{\mathbb{P}}(x)$ does not have the WRNP (a fortiori is not weakly compact in $PM_p(G)^*$) for any $x \in G$. If $p \neq 2$ this is a new result. The C^* algebra methods of Z. Hu in [Hu] do not seem to work in this case.

(2) One cannot improve the cardinality estimate of Theorem 6. For assume that $1 < p < \infty$, and G is an arbitrary nondiscrete locally compact group. Then by Corollary 2 G contains a closed separable metrisable nondiscrete subgroup H. By Herz's theorem [Hz], p. 92 the restriction map $r : A_p(G) \to A_p(H)$ is onto and $\|r\| \leq 1$. Hence $r^* PM_p(H) = \mathbb{P}$ is norm closed and by Theorem 4.14 in [Ru1], $\mathbb{P}$ is w^* closed. $\mathbb{P}$ is an A_p module, since $(v \cdot r^*\phi, u) = (r^*[(rv) \cdot \phi], u)$ and $\sigma(\mathbb{P}) \subset H$, see [Hz]. But $\sigma(\mathbb{P}) = H$ since if $x \in H$ and $\phi = \lambda\delta_x \in PM_p(H)$, then $r^*\phi = \lambda\delta_x \in \mathbb{P}$. Now $A_p(H)$ is norm separable since H is separable metric. Since r^* is an isometry into (see [Hz], p. 91), card $PM_p(H) =$ card $\mathbb{P} \leq c$. Hence card $\mathbb{P}^* \leq 2^c$. We can apply now our Theorem 6 to $\mathbb{P} \subset PM_p(G)$, $\sigma(\mathbb{P}) = H = F$ and $a = b = e$ and get that $2^c \geq \operatorname{card} \mathbb{P}^* \geq \operatorname{card} TIM_{\mathbb{P}}(x) \geq 2^c$ for all $x \in H$. Thus card $TIM_{\mathbb{P}}(x) = 2^c$.

Proof. Let $V \subset G$ be open such that $\operatorname{cl} V$ is compact and $x \in V \cap aHb \subset F$. Let, by Corollary 2, H_0 be a nondiscrete separable closed metrisable subgroup of G. Then $x = ahb$ for some $h \in H$. Thus $x \in V \cap ahH_0b \subset V \cap aHb \subset F$. Let $F_0 = \operatorname{cl}(V \cap ahH_0b)$, a compact metrisable subset of F. If $c = ah$, then $x \in \operatorname{int}_{cH_0b} F_0$ since $x \in V \cap cH_0b \subset F_0$. Let $\mathbb{P}_0 = w^* \operatorname{cllin}\{\lambda\delta_x; x \in F_0\}$, $\mathbb{P}_1 = w^* \operatorname{cllin}\{\lambda\delta_x; x \in F\}$ where cl denotes closure and lin, linear span. Then $\mathbb{P}_0 \subset (\mathbb{P}_1)_c \subset \mathbb{P}_c$, since F_0 is compact and since $\mathbb{P}_1$ is the smallest w^* closed A_p module with $\sigma(\mathbb{P}_1) = F$, [Hz].

Apply now our Theorem 3 of [Gr4] to $\mathbb{P}_0$, the metrisable set F_0 and the closed nondiscrete group H_0. Then $\sigma(\mathbb{P}_0) = F_0$, $x \in \operatorname{int}_{cH_0b} F_0$, thus $x \in D_1(J_0)$ (see [Gr4]) where $J_0 = \{u \in A_p;\ (\phi, u) = 0, \text{ for } \phi \in \mathbb{P}_0\}$, a closed ideal such that $\mathbb{P}_0 = (A_p/J_0)^*$. By Theorem 4 of [Gr4] we get that there is some onto operator $t : \mathbb{P}_0 \to \ell^\infty$ (thus t^* is a w^*-w^* and norm isomorphism into) such that $t^*\mathcal{F} \subset TIM_{\mathbb{P}_0}(x)$ where $\mathcal{F} = \{\eta \in \ell^{\infty*};\ 1 = (\eta, 1) = \|\eta\| \text{ and } \eta = 0 \text{ on } c_0\}$.

Let now $\beta\mathbb{N}$ be the Stone-Čech compactification of the positive integers $\mathbb{N}$. Then $\beta\mathbb{N} \sim \mathbb{N} \subset \mathcal{F}$ is a w^* perfect set of cardinality 2^c (see [Ru3], p. 204). Thus $\Gamma = t^*(\beta\mathbb{N} \sim \mathbb{N})$ is a w^* perfect subset of $TIM_{\mathbb{P}_0}(x)$ and card $TIM_{\mathbb{P}_0}(x) \geq 2^c$.

But Γ is isomorphic to a canonical ℓ^1 basis, i.e. there is some $d > 0$ such that $\sum_1^n |\alpha_i| \geq \|\sum_1^n \alpha_i t^*\varphi_i\| \geq d \sum_1^n |\alpha_i|$ for all $n \geq 1$, $\alpha_i \in C$ and distinct $\varphi_1, \dots, \varphi_n \in \beta\mathbb{N} \sim \mathbb{N}$. To prove this it is enough (since $\|t^*\| \leq 1$ and t^* is a norm isomorphism into) to show that $\sum_1^n |\alpha_i| = \|\sum_1^n \alpha_i\varphi_i\|$. Now $\ell^\infty = C(\beta\mathbb{N})$, hence there is some $f \in C(\beta\mathbb{N})$ such that $\|f\| = 1$ and $f(\varphi_i) = \bar{\alpha}_i/|\alpha_i|$. Thus $(f, \sum_1^n \alpha_i\varphi_i) = \sum_1^n |\alpha_i| \leq \|\sum_1^n \alpha_i\varphi_i\| \leq \sum_1^n |\alpha_i|$. It follows that the w^* compact set $TIM_{\mathbb{P}_0}(x)$ contains a w^* perfect set which is isomorphic to a canonical ℓ^1 basis. Hence by our Lemma 1.2 on p. 157 in [Gr2], $TIM_{\mathbb{P}_0}(x)$ does not have the WRNP. To get the result about $TIM_{\mathbb{Q}}(x)$ apply Corollary 4. □

With a view to future applications, we have under the assumptions of Theorem 6:

Corollary 6′. *Let $W \subset W_{\mathbb{Q}}(x)$ be any closed subspace. Then the canonical map $q : \mathbb{Q} \to \mathbb{Q}/W$ is not a weakly compact operator, for any $x \in \operatorname{int}_{aHb} F$.*

Proof. By Proposition 5(a) we can assume that $W = W_{\mathbb{Q}}(x)$. Since $\mathbb{Q}/W_{\mathbb{Q}}(x) = (\mathbb{Q}/E_{\mathbb{Q}}(x))/C(\lambda\delta_x)$ and by Proposition 5(b) we need only show that the canonical map $q : \mathbb{Q} \to \mathbb{Q}/E_{\mathbb{Q}}(x)$ is not weakly compact. If q is weakly compact, then so is $q^* : (\mathbb{Q}/E_{\mathbb{Q}}(x))^* \to \mathbb{Q}^*$. But then $\{\psi \in \mathbb{Q}^*; \|\psi\| \leq 1 \text{ and } \psi = 0 \text{ on } E_{\mathbb{Q}}(x)\}$ and a fortiori $TIM_{\mathbb{Q}}(x)$ is a weakly relatively compact subset of $\mathbb{Q}^*$. But $TIM_{\mathbb{Q}}(x)$ is w^*, hence weakly, closed. Thus $TIM_{\mathbb{Q}}(x)$ is weakly compact and a fortiori has the WRNP. This cannot be by Theorem 6. □

We do not know if the next theorem holds for $p \neq 2$.

Theorem 7. *Let G be a locally compact group, and $\mathbb{P} \subset PM_2(G)$ a w^* closed $A_2(G)$ module with $F = \sigma(\mathbb{P})$, $a, b \in G$. Assume that R (or T) is a closed subgroup of G and $S \subset R$ (or T) is a symmetric set such that $aSb \subset F$. If $x \in aSb$, then $\operatorname{card} TIM_{\mathbb{Q}}(x) \geq 2^c$ and $TIM_{\mathbb{Q}}(x)$ does not have the WRNP for any norm closed $A_2(G)$ module $\mathbb{Q}$ such that $\mathbb{P}_c \subset \mathbb{Q} \subset PM_2(G)$.*

Remarks. If F is metrisable, then Corollary 6 of [Gr4] implies the fact that $\operatorname{card} TIM_{\mathbb{Q}}(x) \geq 2^c$.

Proof. S is a compact subset of R, since the map $t : \prod_1^\infty D_i \to S$, $D_i = \{0, 1\}$, $t\varepsilon = \sum_1^\infty \varepsilon_i t_i$ is continuous. Let $F_0 = aSb$, a compact metrisable subset of aRb, hence of G. Let $\mathbb{P}_0 = w^* \text{ cl lin } \{\lambda\delta_x; x \in F_0\}$. Then since $\sigma(\mathbb{P}_0) = F_0$, $\mathbb{P}_0 \subset \mathbb{P}_c$. We can apply Corollary 2′ of [Gr4] with F_0 instead of F and get that $x \in D_1(J_0)$, where $J_0 = \{u \in A(G); (\phi, u) = 0 \text{ for } \phi \in \mathbb{P}_0\}$. Hence by [Gr4], Theorem 4, there is an onto operator $t : \mathbb{P}_0/W_{\mathbb{P}_0}(x) \to \ell^\infty$ such that $t^*\mathcal{F} \subset TIM_{\mathbb{P}_0}(x)$. The proof of Theorem 6 shows that card $TIM_{\mathbb{P}_0}(x) \geq 2^c$ and $TIM_{\mathbb{P}_0}(x)$ does not have the WRNP. Apply now Corollary 4 to $\mathbb{Q}$. □

Remarks. (1) Theorem 7 holds true if $S = \bigcup_{\alpha\in I}(x_\alpha + S_\alpha)$, where $x_\alpha \in R$, S_α or $-S_\alpha$ are ultrathin symmetric and I is any index set. Symmetric sets are such. This holds since Corolllary 2′ of [Gr4] holds for such sets S.

(2) If $F = G$ thus $\mathbb{P} = PM_2(G)$ and $x = e$, a much better and definitive result on card $TIM_{\mathbb{P}}(e)$ has been obtained by Zhiguo Hu [Hu].

(3) One cannot improve the cardinality estimate of Theorem 7. Indeed, R (or T) is a closed subgroup of the otherwise arbitrary group G. Thus R (or T) is a set of synthesis for G (see [Hz]). Thus $PM_2(R)$ ($PM_2(T)$) can be identified with $\mathbb{P}_1 = \{\phi \in PM_2(G); \text{supp } \phi \subset R\}$, a w^* closed $A_2(G)$ submodule of $PM(G)$ with $F = \sigma(\mathbb{P}_1) = R$ (or T), see [Hz]. Thus $\mathbb{P}_1 \approx L^\infty(R)$ (or $\mathbb{P}_1 \approx \ell^\infty$). Now let $F = aSb$ and $\mathbb{P} = w^* \text{ cl lin } \{\lambda\delta_x; x \in aSb\}$. Then $\mathbb{P} \subset \mathbb{P}_1$ and card $\mathbb{P} \leq \operatorname{card} \mathbb{P}_1 = c$. But Theorem 7 implies $2^c \leq \text{card } TIM_{\mathbb{P}}(x) \leq \text{card } \mathbb{P}^* \leq \text{card } \mathbb{P}_1^* = 2^c$ for all $x \in aSb$.

Corollary 7′. *With assumptions as in Theorem 7 let $W \subset W_{\mathbb{Q}}(x)$ be any closed subspace. Then the canonical map $q : \mathbb{Q} \to \mathbb{Q}/W$ is not a weakly compact operator, for any $x \in aSb$.*

Proof. See the proof of Corollary 6′. □

Corollary 8. *Let $J \subset A_p(G)$ be a closed ideal, $F = Z(J) = \{x \in G; v(x) = 0$ for $v \in J\}$, $a, b \in G$. Assume one of the following:*

(a) *For some nondiscrete closed subgroup $H \subset G$, $int_{aHb}F \neq \emptyset$ or*

(b) *$p = 2$, G contains R (or T) as a closed subgroup, $S \subset R$ (or T) is a symmetric set such that $aSb \subset F$.*

Then A_p/J is a Banach algebra which is not Arens regular.

Proof. Let $\mathbb{P} = (A/J)^*$. Then $W = WAP(\mathbb{P}) \subset W_{\mathbb{P}}(x)$ for $x \in \text{int}_{aHb}$ ($x \in aSb$ resp.), by [Gr4], Proposition 5. But then $q : \mathbb{P} \to \mathbb{P}/W$ is not weakly compact, thus clearly $\mathbb{P}/W \neq \{0\}$. Hence A/J is not Arens regular. □

Remarks. (1) As shown by J.P. Kahane, there exist perfect sets $F \subset G = T$ (and even continuous curves F (in $Lip\beta, \beta < 1$) in $G = R^2$) such that $A(G)/I_F = A(F) = C(F)$, an Arens regular Banach algebra, where $I_F = \{v \in A(G); v = 0 \text{ on } F\}$. If $\mathbb{P} = A(F)^*$, one has in this case, card $TIM_{\mathbb{P}}(x) = 1$ for all $x \in F$ (see [Gr3], p. 56–57).

(2) If J satisfies the conditions of Corollary 8 and G is second countable, then A_p/J is even an extremely non-Arens regular (ENAR) Banach algebra, i.e. there is a closed subspace of $\mathbb{P}/WAP(\mathbb{P})$ which has $\mathbb{P}$ as a quotient. We do not know if this is the case if G is not second countable.

The abelian case

If G is abelian (l.c.a.), the above results have implications on translation invariant (tr. inv.) subspaces $P \subset L^\infty(\widehat{G})$. Denote $L^\infty(\widehat{G}) = L^\infty$, $L^1(\widehat{G}) = L^1$, UC the uniformly continuous functions in L^∞, $UC_P = UC \cap P$, $\overline{P} = \{\overline{f}; f \in P\}$ and $\sigma(P) = \overline{P} \cap G$, where $G \subset L^\infty$ (are the continuous characters on $\widehat{G}$). Let $M_P(x) = \{\psi \in P^*; 1 = \|\psi\| = (\psi, \overline{x})\}$. Let $TIM_P(x) = \{\psi \in M_P(x); (\psi, f) = (\psi, (\overline{x}h) * f)$ if $f \in P$, $0 \leq h \in L^1$, $\int h d\chi = 1\}$ and $IM_P(x) = \{\psi \in M_P(x);\ (\psi, f) = x(\chi)(\psi, f_\chi)$ for $\chi \in \widehat{G}$, $f \in P\}$, where $f_\chi(\gamma) = f(\chi\gamma)$. Recall that by Proposition 9 in [Gr4], $IM_Q(x) \supset TIM_Q(x)$ with equality if $Q \subset UC$.

Corollary 9. *Let G be l.c.a., P [Q] be w^* [norm] closed tr. inv. subspaces of L^∞ such that $UC_P \subset Q \subset L^\infty$, $F = \sigma(P)$, $a \in G$. Assume that either*

(a) *for some nondiscrete closed subgroup $H \subset G$, $int_{aH} F \neq \emptyset$ or*

(b) *G contains R (or T) as a closed subgroup, and $S \subset R$ (or T) is a symmetric set such that $aS \subset F$.*

Then for any $x \in \text{int}_{aH} F$ [$x \in aS$] $\text{card}\, IM_Q(x) \geq \text{card}\, TIM_Q(x) \geq 2^c$ and both $IM_Q(x)$ and $TIM_Q(x)$ do not have the WRNP.

Proof. By [Sa1], p. 308 it is enough to prove the result for $TIM_Q(x)$. But by [Gr4] $\mathcal{F}^{**} : L^{\infty *} \to PM(G)^*$ is a norm and w^*–w^* linear homeomorphism such that $\mathcal{F}^{**}TIM_Q(x) = TIM_{\mathbb{Q}}(x)$, where $\mathcal{F}^*\mathbb{Q} = Q$ and $\mathcal{F} : L^1 \to A(G)$ is Fourier transform (see [Gr4]). Then as in [Gr4] and by the above Theorem 6, [7], card $TIM_{\mathbb{Q}} \geq 2^c$ and $TIM_{\mathbb{Q}}(x)$ does not have the WRNP. Hence again by [Sa1], p. 308 the same holds for $TIM_Q(x)$. □

Remarks. (1) The cardinality part of the above result is implied by Corollaries 10, 11 in [Gr4], in case F is metrisable.

(2) Let $F \subset R = G$, $\chi_t(x) = e^{itx}$ and $P_*(F) = w^*$ cllin $\{\chi_t; t \in F\} \subset L^\infty(\widehat{G})$.

(a) Assume that F is the Cantor 1/3 set; thus for $t_n = 2/3^n$, $F = \{\sum_1^\infty \varepsilon_i t_i;\ \varepsilon_i = 0, 1\}$, a symmetric set. Then for $t \in F$ and $P = P_*(F)$, $TIM_P(t) = \{\psi \in P^*;\ 1 = \|\psi\| = \psi(\overline{\chi}_t),\ (\psi, f_x) = \overline{\chi_t(x)}(\psi, f)$ for all $f \in P$, $x \in R\}$, where $f_x(y) = f(x+y)$, since F is compact and by [Gr4], Proposition 9. Then, since $\text{card}\, L^\infty(R)^* = 2^c \geq \text{card}\, TIM_P(t)$, Corollary 9 yields that card $TIM_P(t) = 2^c$ and $TIM_P(t)$ does not have the WRNP for any $t \in F$. Note that if $t = 0$, then $TIM_P(0) = IM_P(0)$, the set of honest to goodness invariant means on P.

(b) Assume that $F \subset R$ is a compact perfect Helson S set with $0 \in F$. There exist such by [He] (or see [Ru1]). In this case, with $P = P_*(F)$ we have that $\operatorname{card} TIM_P(t) = 1$, and $TIM_P(t)$ contains *one* element, hence certainly has the WRNP, for all $t \in F$. In particular for $t = 0$, card $IM_P(0) = 1$, for the set of honest to goodness invariant means on P (see [Gr3]). The same is the case if $F \subset R$ is any scattered compact set, by Loomis' lemma [Lo].

Question: Does there exist a perfect set $F \subset R$ such that for $P = P_*(F)$, there is some t_0 in F for which $\operatorname{card} TIM_P(t_0) = c$?

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