

ON THE SET OF ALL CONTINUOUS FUNCTIONS WITH UNIFORMLY CONVERGENT FOURIER SERIES

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ABSTRACT. In this article we calculate the exact location in the Borel hierarchy of UCF , the set of all continuous functions on the unit circle with uniformly convergent Fourier series. It turns out to be complete $F_{\sigma\delta}$. Also we prove that any $G_{\delta\sigma}$ set that includes UCF must contain a continuous function with divergent Fourier series.

INTRODUCTION

There are many criteria for uniform convergence of a Fourier series on the unit circle. One can find those tests in [Zy]. In the present paper, we study UCF from the point of view of descriptive set theory. In [Ke], it was a conjecture that UCF is complete $F_{\sigma\delta}$ (i.e., $F_{\sigma\delta}$ but not $G_{\delta\sigma}$). A lot of natural complete $F_{\sigma\delta}$ sets have been found. For example, the collection of reals that are normal or simply normal to base n [KL]; $C^\infty(\mathbb{T})$, the class of infinitely differentiable functions (viewed as a 2π -periodic function on \mathbb{R}); and UC_X , the class of convergent sequences in a separable Banach space X , are complete $F_{\sigma\delta}$ [Ke]. It turns out that UCF is complete $F_{\sigma\delta}$. We give two different proofs for it. Ajtai and Kechris [AK] have shown that ECF , the set of all continuous functions with everywhere convergent Fourier series, is complete CA , i.e., coanalytic non-Borel. We show that there is no $G_{\delta\sigma}$ set A such that $UCF \subseteq A \subseteq ECF$. Hence any $G_{\delta\sigma}$ set that includes UCF must contain a continuous function with divergent Fourier series. From this point of view, although there are many natural complete $F_{\sigma\delta}$ sets, we can claim that UCF is a very interesting set in analysis.

DEFINITIONS AND BACKGROUND

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of positive integers and $\mathbb{N}^{\mathbb{N}}$ the Polish space with the usual product topology and \mathbb{N} discrete. Let X be a Polish space. A subset A of X is CA if there is a Borel function from $\mathbb{N}^{\mathbb{N}}$ to X such that $f(\mathbb{N}^{\mathbb{N}}) = X - A$. A CA ($F_{\sigma\delta}$) subset A of X is called complete CA ($F_{\sigma\delta}$) if for any CA ($F_{\sigma\delta}$) subset B of $\mathbb{N}^{\mathbb{N}}$, there is a Borel (continuous) function f from $\mathbb{N}^{\mathbb{N}}$ to X such that the preimage of A of f is B , i.e., $B = f^{-1}(A)$. From the definition, it is easy to see that no

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complete CA ($F_{\sigma\delta}$) set is Borel ($G_{\delta\sigma}$). In particular, if an $F_{\sigma\delta}$ subset A of a Polish space is complete $F_{\sigma\delta}$ and the continuous preimage of an $F_{\sigma\delta}$ subset B of a Polish space, then B is also complete $F_{\sigma\delta}$.

Let \mathbb{R} be the set of real numbers. Let \mathbb{T} denote the unit circle and I the unit interval. Let E be \mathbb{T} or I . We denote by $C(E)$ the Polish space of continuous functions on E with the uniform metric

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in E\}.$$

$C(\mathbb{T})$ can also be considered as the space of all continuous 2π -periodic functions on \mathbb{R} , viewing \mathbb{T} as $\mathbb{R}/2\pi\mathbb{Z}$. Let UC denote the set of all sequences of continuous functions on I that are uniformly convergent, i.e.,

$$UC = \{(f_n) \in C(I)^{\mathbb{N}} : (f_n) \text{ converges uniformly} \}.$$

To each $f \in C(\mathbb{T})$, we associate its Fourier series

$$S[f] \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx},$$

where $\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t)e^{-int} dt$. Let

$$S_n(f, t) = \sum_{k=-n}^n \hat{f}(k)e^{ikt}$$

be the n th partial sum of the Fourier series of f . We say the Fourier series of f converges at a point $t \in \mathbb{T}$ if the sequence $(S_n(f, t))$ converges. Similarly, we define the uniform convergence of the Fourier series of f . Let ECF denote the set of all continuous functions with everywhere convergent Fourier series. According to a standard theorem [Ka], the Fourier series of f at t converges to $f(t)$ if it converges. Hence we have

$$\begin{aligned} ECF &= \{f \in C(\mathbb{T}) : \forall t \in [0, 2\pi] \left((S_n(f, t)) \text{ converges} \right)\} \\ &= \{f \in C(\mathbb{T}) : \forall t \in [0, 2\pi] \left(f(t) = \lim_{n \rightarrow \infty} S_n(f, t) \right)\}. \end{aligned}$$

We denote by NCF the complement of ECF . Let UCF denote the set all continuous functions with uniformly convergent Fourier series, i.e.,

$$UCF = \{f \in C(\mathbb{T}) : \text{the Fourier series of } f \text{ converges uniformly} \}.$$

RESULTS

Theorem ([AK]). *ECF is complete CA.*

(See [AK].)

Proposition 1. *UCF and UC are $F_{\sigma\delta}$.*

Proof. Let \mathbb{Q} be the set of all rational numbers. We consider \mathbb{T} as $[0, 2\pi]$ with 0 and 2π identified. By the definition of *UCF*,

$$\begin{aligned} f \in UCF &\iff S_N(f) \text{ converges uniformly} \\ &\iff \forall a \in \mathbb{N} \exists b \in \mathbb{N} \forall c, d \in \mathbb{N} \forall e \in \mathbb{Q} \left(|S_{b+c}(f, e) - S_{b+d}(f, e)| \leq \frac{1}{a} \right) \\ &\iff f \in \bigcap_{a \in \mathbb{N}} \bigcup_{b \in \mathbb{N}} \bigcap_{c, d \in \mathbb{N}} \bigcap_{e \in \mathbb{Q} \cap [0, 2\pi]} V(a, b, c, d, e), \end{aligned}$$

where $V(a, b, c, d, e)$ is the collection of $f \in C(\mathbb{T})$ such that $|S_{b+c}(f, e) - S_{b+d}(f, e)| \leq 1/a$, which is closed, since the function $f \mapsto \hat{f}(n)$ is continuous. Hence *UCF* is $F_{\sigma\delta}$. Similarly, so is *UC*. We are done. \square

Lemma 2. *The set $C_3 = \{\alpha \in \mathbb{N}^{\mathbb{N}} : \lim_{n \rightarrow \infty} \alpha(n) = \infty\}$ is complete $F_{\sigma\delta}$.*

(See [Ke, p. 180].) This set will be used to prove our main theorem.

Proposition 3. *UC is complete $F_{\sigma\delta}$.*

Proof. We define the function F from $\mathbb{N}^{\mathbb{N}}$ to $C(I)^{\mathbb{N}}$ as follows: for each $\beta \in \mathbb{N}^{\mathbb{N}}$,

$$F(\beta) = \left(\frac{1}{\beta(n)} \right).$$

Then it is easy to see that

$$\beta \in C_3 \iff F(\beta) \text{ converges to } 0 \iff F(\beta) \text{ converges uniformly,}$$

since $F(\beta)$ is a sequence of constant functions. Clearly, F is continuous. Hence *UC* is the continuous preimage of C_3 . By Proposition 1 and Lemma 2, *UC* is complete $F_{\sigma\delta}$. \square

Theorem 4. *There is a continuous function H from $\mathbb{N}^{\mathbb{N}}$ to $C(\mathbb{T})$ such that for all A with $UCF \subseteq A \subseteq ECF$,*

$$\beta \in C_3 \iff H(\beta) \in A,$$

and

$$\beta \notin C_3 \iff H(\beta) \in NCF.$$

In particular, UCF is complete $F_{\sigma\delta}$.

By this theorem, we have the following corollary.

Corollary 5. *There is no $G_{\delta\sigma}$ set A such that*

$$UCF \subseteq A \subseteq ECF,$$

i.e., any $G_{\delta\sigma}$ set that includes UCF must contain a continuous function with divergent Fourier series.

Proof. Suppose a $G_{\delta\sigma}$ set A satisfies $UCF \subseteq A \subseteq ECF$. Then by Theorem 4, we obtain $H^{-1}(A) = C_3$. Since A is $G_{\delta\sigma}$, so is C_3 . By Lemma 2, this contradicts our assumption. \square

It is a basic fact of descriptive set theory [Ke] that any Borel set is coanalytic. Since ECF is complete CA by Theorem [AK], it is a very natural guess that the complement of C_3 can be reducible to $ECF - UCF$. In fact, we have the following theorem.

Theorem 6. *There is a continuous function \tilde{H} from $\mathbb{N}^{\mathbb{N}}$ to $C(\mathbb{T})$ such that*

$$\beta \in C_3 \iff \tilde{H}(\beta) \in UCF,$$

and

$$\beta \notin C_3 \iff \tilde{H}(\beta) \in ECF - UCF.$$

In particular, UCF is complete $F_{\sigma\delta}$.

In order to prove Theorem 4 and Theorem 6, we need the following criterion due to Dini and Lipschitz [Zy, p. 63]. Let f be defined in a closed interval J , and let

$$\omega(\delta) = \omega(\delta; f) = \sup\{|f(x) - f(y)| : x, y \in J \text{ and } |x - y| \leq \delta\}.$$

The function $\omega(\delta)$ is called the *modulus of continuity* of f .

The Dini-Lipschitz test. *If f is continuous and its modulus of continuity $\omega(\delta)$ satisfies the condition $\omega(\delta) \log \delta \rightarrow 0$, then the Fourier series of f converges uniformly.*

We introduce the Féjer polynomials, for given $0 < n < N \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$Q(x, N, n) = 2 \sin Nx \sum_{k=1}^n \frac{\sin kx}{k},$$

$$R(x, N, n) = 2 \cos Nx \sum_{k=1}^n \frac{\sin kx}{k}.$$

These two polynomials were used in [Zy] to prove that there exists a continuous function whose Fourier series diverges at a point.

Lemma 7. *There are positive numbers $C_1, C_2 > 0$ such that*

$$|Q| < C_1 \text{ and } |R| < C_2,$$

i.e., these polynomials are uniformly bounded in x, N, n .

Since

$$\sum_{k=1}^n \frac{\sin kx}{k}$$

is uniformly bounded in n and x , Lemma 7 follows. By Lemma 7, we immediately have the following.

Proposition 8. *Let (N_k) and (n_k) be any two sequences of positive integers, with $n_k < N_k$, and let (α_k) be a sequence of real numbers such that $\alpha_1 + \alpha_2 + \alpha_3 + \dots < \infty$. Then the series $\sum \alpha_k Q(x, N_k, n_k)$, $\sum \alpha_k R(x, N_k, n_k)$ converge to continuous functions.*

Proof of Theorem 4. We fix A with $UCF \subseteq A \subseteq ECF$. Let $\alpha_k = 2^{-k}$, $n_k = N_k/2 = 2^{2^k}$ ($k = 1, 2, 3, \dots$). We define H from $\mathbb{N}^{\mathbb{N}}$ to $C(\mathbb{T})$ as follows: for all $\beta \in \mathbb{N}^{\mathbb{N}}$,

$$H(\beta) = \sum \alpha_k \frac{1}{\beta(k)} Q(x, N_k, n_k).$$

Claim 1. H is continuous and well-defined.

Proof. By Proposition 8, H is well-defined. By Lemma 7, it is easy to see that H is continuous.

We divide the rest of proof into two parts so that we have more intuition.

Case 1. $\lim_{n \rightarrow \infty} \beta(k) \neq \infty$.

We want to show that $H(\beta) \in NCF$. For each $k \in \mathbb{N}$ the inequality

$$\begin{aligned} & |S_{N_k+n_k}(H(\beta), 0) - S_{N_k}(H(\beta), 0)| = \left| \sum_{|l| \leq N_k+n_k} \widehat{H(\beta)}(l) - \sum_{|l| \leq N_k} \widehat{H(\beta)}(l) \right| \\ (1) \quad & = \alpha_k \frac{1}{\beta(k)} \left(1 + \frac{1}{2} + \dots + \frac{1}{n_k} \right) > \alpha_k \frac{1}{\beta(k)} \log n_k = 2^{-k} \frac{1}{\beta(k)} \log 2^{2^k} \\ & = \frac{1}{\beta(k)} \log 2 \end{aligned}$$

holds. Since $\lim_{n \rightarrow \infty} \beta(k) \neq \infty$, there exists a $p \in \mathbb{N}$ such that for infinitely many k 's, $\beta(k) = p$. Hence the Fourier series of $H(\beta)$ does not converge, since in (1) we have $1/p \log 2$ for infinitely many k 's. So we derive $H(\beta) \in NCF$.

Case 2. $\lim_{n \rightarrow \infty} \beta(k) = \infty$.

We show that $H(\beta) \in UCF$. We will demonstrate that $\omega(\delta; H(\beta)) \log \delta \rightarrow 0$ as $\delta \rightarrow 0$. Then by the Dini-Lipschitz test, this shows that the Fourier series of $H(\beta)$ converges uniformly. We take any $0 < \delta \leq 1/2$ and define $\nu = \nu(\delta)$ as the largest integer k satisfying $2^{2^k} \leq 1/\delta$. By Lemma 7, we have the following inequality:

$$\begin{aligned} & \left| \sum_{k=\nu+1}^{\infty} \alpha_k \frac{1}{\beta(k)} Q(x + \delta, N_k, n_k) - \sum_{k=\nu+1}^{\infty} \alpha_k \frac{1}{\beta(k)} Q(x, N_k, n_k) \right| \\ (2) \quad & \leq 2C \sum_{k=\nu+1}^{\infty} \alpha_k \frac{1}{\beta(k)} \leq 2C \sup\{ \frac{1}{\beta(k)} : k > \nu \} \sum_{k=\nu+1}^{\infty} \alpha_k \\ & = 4C \sup\{ \frac{1}{\beta(k)} : k > \nu \} 2^{-\nu-1} \leq 4C \sup\{ \frac{1}{\beta(k)} : k > \nu \} \frac{\log 2}{|\log \delta|}. \end{aligned}$$

Now we calculate the rest of $H(\beta)$. We clearly have

$$Q'(x, N, n) = NR(x, N, n) + 2 \sin Nx \sum_{k=1}^n \cos kx, \quad |Q'| \leq NC + 2n = nC_1,$$

for $N = 2n$ and $C_1 = 2C + 2$. By the mean value theorem, we have the following inequality:

$$\begin{aligned}
 (3) \quad & \left| \sum_{k \leq \nu} \alpha_k \frac{1}{\beta(k)} Q(x + \delta, N_k, n_k) - \sum_{k \leq \nu} \alpha_k \frac{1}{\beta(k)} Q(x, N_k, n_k) \right| \\
 & \leq C_1 \delta \left(2^{-1} 2^{2^1} \frac{1}{\beta(1)} + \dots + 2^{-\nu} 2^{2^\nu} \frac{1}{\beta(\nu)} \right) \\
 & \leq C_1 2^{2^{-\nu}} \sum_{k \leq \nu} 2^{-k} 2^{2^k} \frac{1}{\beta(k)} \leq C_1 \frac{1}{|\log \delta|} 2^{2^{-\nu}} \sum_{k \leq \nu} 2^{2^k - k} \frac{1}{\beta(k)}.
 \end{aligned}$$

By (2) and (3), we have the following:

$$(4) \quad |\omega(\delta; H(\beta)) \log \delta| \leq \max\{4C \sup\{\frac{1}{\beta(k)} : k > \nu\} \log 2, C_1 2^{2^{-\nu}} \sum_{k \leq \nu} 2^{2^k - k} \frac{1}{\beta(k)}\}.$$

Now if $\delta \rightarrow 0$, then $\nu \rightarrow \infty$. So it suffices to show that the right part of (4) goes to 0 as $\nu \rightarrow \infty$. Since $\beta(\nu) \rightarrow \infty$ as $\nu \rightarrow \infty$, $\sup\{1/\beta(k) : k > \nu\}$ goes to 0. We need to show that the rest goes to zero as ν diverges to infinity. This requires the following small claim.

Claim 2. $\sum_{k \leq \nu} 2^{2^k - k} \leq 2^{2^\nu - \nu + 1}$.

Proof. Use induction on ν . For $\nu = 1$, $2^{2^1 - 1} = 2 \leq 2^{2^1 - 1 + 1} = 2^2$. Suppose it is true for ν . By the induction assumption, $\sum_{k \leq \nu} 2^{2^k - k} + 2^{2^{\nu+1} - (\nu+1)} \leq 2^{2^\nu - \nu + 1} + 2^{2^{\nu+1} - (\nu+1)}$. It is enough to show that $2^{2^\nu - \nu + 1} + 2^{2^{\nu+1} - (\nu+1)} \leq 2^{2^{\nu+1} + 1}$. Letting $\theta = 2^{2^\nu}$, one can verify this inequality.

Fix ϵ . Take N_0 such that $1/\beta(k) < \epsilon$ for all $k \geq N_0$. For this N_0 , we choose $N > N_0$ so that $2^{-2^\nu + \nu} \sum_{k \leq N_0} 2^{2^k - k} < \epsilon$ for all $\nu \geq N$. Then for all $\nu \geq N$, by claim 2, the following inequality is valid:

$$\begin{aligned}
 & 2C_1 2^{-2^\nu} 2^\nu \sum_{k \leq \nu} 2^{2^k - k} \\
 & < 2C_1 \left(2^{-2^\nu + \nu} \sum_{k \leq N_0} 2^{2^k - k} \frac{1}{\beta(k)} + 2^{-2^\nu + \nu} \sum_{N_0 < k \leq \nu} 2^{2^k - k} \frac{1}{\beta(k)} \right) \\
 & < 2C_1 \left(\epsilon + 2^{-2^\nu + \nu} \epsilon \sum_{N_0 < k \leq \nu} 2^{2^k - k} \right) < 2\epsilon C_1 \left(1 + \frac{2^{2^\nu - \nu + 4}}{2^{2^\nu - \nu}} \right) \\
 & = 34\epsilon C_1.
 \end{aligned}$$

Hence the right side of (4) converges to zero as ν goes to the infinity, i.e., as $\delta \rightarrow 0$. So we derive $H(\beta) \in UCF$.

By case 1 and case 2, we obtain

$$\begin{aligned}
 \beta \notin C_3 & \Rightarrow H(\beta) \in NCF \text{ and} \\
 \beta \in C_3 & \Rightarrow H(\beta) \in UCF,
 \end{aligned}$$

respectively. Since NCF and A are disjoint, we have the following:

$$\begin{aligned}
 \beta \notin C_3 & \iff H(\beta) \in NCF \text{ and} \\
 \beta \in C_3 & \iff H(\beta) \in A.
 \end{aligned}$$

We have shown the first assertion of the theorem. In particular, C_3 is the preimage of UCF . Hence by Lemma 2, the second assertion follows. We have completed the proof of Theorem 4. \square

Proof of Theorem 6. Instead of Q , we use R . With N_k, n_k , and α_k as in the proof of Theorem 4, we define \tilde{H} from $\mathbb{N}^{\mathbb{N}}$ to $C(\mathbb{T})$ as follows: for each $\beta \in \mathbb{N}^{\mathbb{N}}$,

$$\tilde{H}(\beta) = \sum \alpha_k \frac{1}{\beta(k)} R(x, N_k, n_k).$$

The same proof as before will demonstrate that this function is well-defined and continuous and that if $\lim_{n \rightarrow \infty} \beta(n) = \infty$, then the Fourier series of $\tilde{H}(\beta)$ converges uniformly. So it suffices to show that if $\lim_{n \rightarrow \infty} \beta(n) \neq \infty$, then $\tilde{H}(\beta) \in ECF - UCF$. Suppose $\lim_{n \rightarrow \infty} \beta(n) \neq \infty$. The representation of $\tilde{H}(\beta)$ as Fourier series is $\sum a_v \sin vx$. We see that $\sum a_v \sin vx$ converges uniformly for $\delta \leq |x| \leq \pi$ for any $\delta > 0$, since the partial sums of $R(x, N_k, n_k)$ are uniformly bounded in k and x , for $\delta \leq |x| \leq \pi$. The series $\sum a_v \sin vx$ contains sines only, and hence it converges for $x = 0$, and so everywhere. Now we will show that $\sum a_v \sin vx$ does not converge uniformly. It is easy to see that

$$\sum_{v=1}^{3n_k} a_v \sin vx - \sum_{v=1}^{2n_k} a_v \sin vx = \sum_{v=2n_k+1}^{3n_k} a_v \sin vx = 2^{-k} \frac{1}{\beta(k)} \sum_{v=1}^{n_k} \frac{\sin(2n_k + v)x}{v}.$$

So if we let $x = \pi/4n_k$, then we have

$$\left| \sum_{v=1}^{3n_k} a_v \sin vx - \sum_{v=1}^{2n_k} a_v \sin vx \right| = 2^{-k} \frac{1}{\beta(k)} \sum_{v=1}^{n_k} \frac{\sin(2n_k + v)x}{v} \geq 2^{-k} \frac{1}{\beta(k)} \sin \frac{\pi}{4} \sum_{v=1}^{n_k} \frac{1}{v},$$

since, for all v in the interval $1 \leq v \leq n_k$,

$$\frac{3}{4}\pi \geq \frac{\pi}{4n_k}(2n_k + v) \geq \frac{\pi}{2}.$$

So finally,

$$(5) \quad \left| \sum_{v=1}^{3n_k} a_v \sin vx - \sum_{v=1}^{2n_k} a_v \sin vx \right| \geq 2^{-k} \frac{1}{\beta(k)} \sin \frac{\pi}{4} \sum_{v=1}^{n_k} \frac{1}{v} \geq 2^{-k} \frac{1}{\beta(k)} \log n_k \sin \frac{\pi}{4} \\ = \frac{\log 2}{\sqrt{2}} \frac{1}{\beta(k)}.$$

Hence $\sum a_v \sin vx$ does not converge uniformly, since in (5) the same value appears for infinitely many k 's. Hence, as in the proof of Theorem 4, we finish the proof of Theorem 6. \square

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