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ON THE COMMUTANT OF HYPONORMAL OPERATORS

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ABSTRACT. Let T be a pure hyponormal operator with compact self-commutator. We show that the unit ball of the commutant of T^* is compact in the strong operator topology.

Let H be a separable complex Hilbert space and let $L(H)$ denote the algebra of all bounded linear operators on H. An operator $T \in L(H)$ is hyponormal if $T^*T - TT^* \geq 0$ and essentially normal if $T^*T - TT^*$ is compact. Moreover T is pure if it has no nonzero reducing subspace on which it is normal. The main result of this paper is the following:

Theorem 1. *Let $T \in L(H)$ be a pure hyponormal essentially normal operator. Then the closed unit ball of the commutant algebra $\{T^*\}'$ is compact in the strong operator topology.*

The proof of this theorem relies on the following two results. The first one is a particular case of a more general result obtained by V. Lomonosov.

Theorem 2 (cf. [L, Lemma 1]). *Let $T \in L(H)$ be an essentially normal operator. Suppose there exists a sequence $\{T_n\}$ in the commutant of T such that $T_n \to 0$ weakly but $T_n \not\to 0$ strongly. Then there exist a normal operator $N \in L(H)$ and a nonzero operator $X \in L(H)$ such that $NX = XT$.*

This theorem has been used in [L] to show that for a transitive essentially normal operator $T \in L(H)$ either $\{T\}'$ or $\{T^*\}'$ has strongly compact ball. The second result we need is an asymmetric Fuglede-Putnam theorem due to J.G. Stampfli and B.L. Wadhwa.

Theorem 3 (cf. [SW, Theorem 1]). *Let $T \in L(H)$ be a hyponormal operator and suppose there exist a normal operator $N \in L(H)$ and $X \in L(H)$ with dense range, such that $TX = XN$. Then T is normal.*

Proof of Theorem 1. Suppose the conclusion is false. Now, recall that the strong topology is metrizable on bounded subsets of $L(H)$ and that the unit ball of $L(H)$ is metrizable and compact in the weak operator topology. Therefore there exists a sequence of operators $\{T_n\}$ in $\{T^*\}'$ such that $T_n \to 0$ weakly but $T_n \not\to 0$ strongly. An application of Theorem 2 above yields a normal operator $N \in L(H)$ and a nonzero operator $X \in L(H)$ such that $NX = XT^*$. Therefore $TX^* = X^*N^*$

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and the restriction S of T to $\overline{ImX^*}$ is still hyponormal. Applying Theorem 3 we infer that S is normal, which contradicts our initial assumption that T is pure hyponormal. The proof is complete.

The following is an easy consequence of Theorem 1.

Corollary 1. *Under the assumptions of Theorem 1, the weak*, weak and strong operator topologies agree on bounded subsets of* $\{T^*\}'$.

For a compact set $K \subset C$ let $Rat(K)$ denote the algebra of all rational functions with poles off K. An operator $T \in L(H)$ is multicyclic if there exists a finite family of vectors $\{x_1, ..., x_m\}$ in H such that the space $\{\sum_{k=1}^m f_k(T)x_k; f_1, \ldots, f_m \in Rat(\sigma(T))\}$ is dense in H.

Corollary 2. *Let* $T \in L(H)$ *be a multicyclic pure hyponormal operator. Then the weak*, weak and strong operator topologies agree on bounded subsets of* $\{T^*\}'$.

Proof. Since T is multicyclic hyponormal, it is also essentially normal by a celebrated theorem of Berger and Shaw [BS]. Now, an application of Theorem 1 yields the conclusion.

References

[BS] C.A. Berger and B.I. Shaw, *Selfcommutators of multicyclic hyponormal operators are always trace-class*, Bull. Amer. Math. Soc. **79** (1973), 1193-1199. MR **51:**11168

[L] V. Lomonosov, *A construction of an intertwining operator*, Funktsional. Anal. i Prilozhen. **14** (1980), 67–68; English transl. in Functional Anal. Appl. **14** (1980), 54–55. MR **81k:**47032

[SW] J.G. Stampfli and B.L. Wadhwa, *An asymmetric Putnam-Fuglede theorem for dominant operators*, Indiana Univ. Math. J. **25** (1976), 359-365. MR **53:**14197

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